A FAMILY RESOLVENT COCYCLE AND HIGHER SPECTRAL FLOW

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ABSTRACT. In this paper, we introduce a family resolvent cocycle and express the Chern Character of Dai-Zhang higher spectral flow as a pairing of a family resolvent cocycle and the odd Chern character of a unitary matrix, which generalize the odd index formula of Carey et al. to the family case.

1. Introduction

In [8], Connes and Moscovici established an odd local index formula in the noncommutative geometry framework, which may be thought of as a far reaching generalisation of the classical index theorem for Toeplitz operators. Motivated by the work due to Coburn et al. [6], Carey et al. introduced the resolvent cocycle and extended the Connes-Moscovici local index formula to the type II setting in [3] and [4]. In [10], Higson introduced the residue cocycle and gave another proof of the Connes-Moscovici local index formula. In [5], Carey et al. established the relations between the resolvent cocycle and the residue cocycle.

On the other hand, Dai and Zhang defined a higher spectral flow as a $K$-group element and showed that this higher spectral flow can be computed analytically by $\hat{\eta}$-forms and is related to the family index in the same way as the spectral flow is related to the index [9]. In [13], Perrot gave a bivariant generalization of the Connes-Moscovici local index formula. In [11], Benameur and Carey defined an bivariant JLO cocycle for a smooth fibration of closed manifolds and a family of generalised Dirac operators along the fibres. Then they decomposed the Chern Character of the Dai-Zhang higher spectral flow as a pairing of the bivariant JLO cocycle and the Chern character of an idempotent matrix. Motivated by [3] and [11], in this paper, we introduce a family resolvent cocycle and express the Dai-Zhang higher spectral flow as a pairing of a family resolvent cocycle and the Chern Character of a unitary which generalize the odd index formula of Carey et al. to the family case.
This paper is organized as follows: In Section 2, we recall the definitions of the Bismut superconnection and the higher spectral flow. In Section 3, we obtain some norm and trace estimates which will be used in the next Sections. In Section 4, we introduced a family resolvent cocycle and rewrite the formula of the Chern Character of the higher spectral flow. In Section 5, we prove our main theorem and compute the Chern Character of the higher spectral flow by a family resolvent cocycle.

2. Higher spectral flow

Firstly we recall the definition of the Bismut superconnection. Let $M_0$ be a $(q' + p)$ dimensional compact connected manifold, and $B_0$ be a $q'$ dimensional compact connected manifold. We assume that $\pi: M_0 \to B_0$ is a submersion of $M_0$ onto $B_0$, which defines a fibration of $M_0$ with fibre $Z$. For $y \in B_0$, $\pi^{-1}(y)$ is then a submanifold $Z_y$ of $M_0$. $T_Z$ denotes the $p$-dimensional vector bundle on $M_0$ whose fibre $T_xZ$ is the tangent space at $x$ to the fibre $Z_{\pi x}$. We assume that $M_0$ and $B_0$ are oriented.

We fix a connection for this fibration which amounts to a splitting of the tangent bundle $TM_0$ into the horizontal bundle $T^HM_0$ and the vertical bundle $T_Z$, i.e., $TM_0 = T^HM_0 \oplus T_Z$. Vector $X \in TB_0$ will be identified with their horizontal lifts $X \in T^HM_0$, and $T^HM_0$ is isomorphic to $T_{\pi(x)}B_0$ via $\pi_*$. Recall that $B_0$ is Riemannian, so we can lift the Euclidean scalar product $g_{B_0}$ of $TB_0$ to $T^HM_0$. And we assume that $T_Z$ is endowed with a scalar product $g_Z$. Thus we can introduce in $TM$ a new scalar product $g_{B_0} \oplus g_Z$, and denote by $\nabla^L$ the Levi-Civita connection on $TM$ with respect to this metric. Let $\nabla_{B_0}$ denote the Levi-Civita connection on $TB_0$ and we still denote by $\nabla_{B_0}$ the pullback connection on $T^HM_0$. Let $\nabla^Z = P_Z(\nabla^L)$ where $P_Z$ denotes the projection to $T_Z$. Let $\nabla^\oplus = \nabla_{B_0} \oplus \nabla^Z$ and $\nabla^L = \nabla^L - \nabla^\oplus$ and $T$ is the torsion tensor of $\nabla^\oplus$. Let $SO(T_Z)$ be the $SO(n)$ bundle of oriented orthonormal frames in $T_Z$. Now we assume that bundle $T_Z$ is spin. Let $S(T_Z)$ be the associated spinors bundle and $\nabla^Z$ can be lifted to give a connection $\nabla^S$ on $S(T_Z)$. Let $D$ be the tangent Dirac operator. Let $E$ be the vector bundle $\pi^*(\wedge^*B_0) \otimes S(T_Z)$. This bundle carries a natural action $m_0$ of the degenerate Clifford module $C_0(M_0)$.

The Clifford action of a horizontal cotangent vector $e^*_i \in \Gamma(M_0, T^*_H M_0)$ is given by exterior multiplication $m_0(e^*_i) = \varepsilon(e^*_i)$ acting on the first factor $\wedge^*H M_0$ in $E$, while the Clifford action of a vertical cotangent vector simply equals its Clifford action on $S(T_Z)$. Define the connection

\begin{align*}
(2.1) & \quad \nabla^{E, \oplus} := \pi^*\nabla_{B_0} \oplus 1 + 1 \otimes \nabla^S, \\
(2.2) & \quad \omega(X)(Y, Z) := g(\nabla_X Y, Z) - g(\nabla^S_Y X, Z), \\
(2.3) & \quad \nabla^{E, 0}_X := \nabla^{E, \oplus}_X + \frac{1}{2} m_0(\omega(X)),
\end{align*}

for $X, Y, Z \in TM_0$. 

Let \( \{e_i\}_{i=1}^{q} \) be an oriented orthonormal basis of \( TB_0 \) and \( \{f_j\}_{j=1}^{p} \) be an oriented orthonormal basis of \( TZ \), let \( c(e_i^*) \) and \( c(f_j^*) \) denote the Clifford action. Then the Bismut superconnection acting on \( \Gamma(M, \pi^*(B_0) \otimes S(TZ)) \) is defined by

\[
B = \sum_{i=1}^{q} c(e_i^*) \nabla_{e_i} E,0 + \sum_{j=1}^{p} c(f_j^*) \nabla_{f_j} E,0.
\]

By Theorem 10.17 in [1], we have

\[
F = B^2 = - \sum_{j=1}^{p} (\nabla_{f_j} E,0)^2 + \sum_{j=1}^{p} \nabla_{f_j} E,0 \nabla_{f_j} E,0 + \frac{1}{4} r_Z = D^2 + F_{[+]},
\]

where \( D \) is a family of generalized Dirac operators over \( B_0 \), \( F_{[+]} \) is an operator with coefficients in \( \Omega_{\geq 1}(B_0) \) and \( r_Z \) is the scalar curvature of fibres.

Higher spectral flow, for a family of fiberwise self-adjoint elliptic operators \( D = (D_b)_{b \in B_0} \) introduced in [9], is only well defined under the assumption that the \( K^1 \) class defined by the family is trivial. Assume that the index bundle of \( D_0 \) vanishes and let \( Q_0, Q_1 \) be spectral sections of \( D_0, D_1 \) respectively. If we consider the total family \( D = \{D_{b,a}\} \) parametrized by \( B_0 \times I \), then there is also a total spectral section \( P = \{P_{b,a}\} \). Let \( P_u \) be the restriction of \( P \) over \( B \times \{u\} \). The (higher) spectral flow \( sf\{((D_0, Q_0), (D_1, Q_1))\} \) between the pairs \( (D_0, Q_0), (D_1, Q_1) \) is an element in \( K(B_0) \) defined by

\[
sf\{((D_0, Q_0), (D_1, Q_1))\} = [Q_1 - P_1] - [Q_0 - P_0] \in K(B_0).
\]

It is easy to check that \( sf\{((D_0, Q_0), (D_1, Q_1))\} \) does not depend on the choice of the global spectral section \( P \). In this paper we are mainly interested in the affine path \( D_t := D + tU^{-1}[D, U] \) where \( D \) is a family of generalized Dirac operators over \( B_0 \) whose index class in \( K_1(B_0) \) is trivial, and \( U \) is a given element of \( GL_N(C^\infty(M_0)) \). In this case, the endpoints are conjugate and we consider the higher spectral flow with respect to the spectral sections \( P_0 = P \) and \( P_1 = U^{-1}PU \), where \( P \) is a fixed spectral section for \( D \). It turns out that the higher spectral flow does not depend on \( P \) either and is an invariant of the principal symbol of \( D \) and of the homotopy class of \( U \). We denote it \( sf(D, U) \). Indeed, Dai and Zhang proved the following proposition (for related notations, see [9]).

**Proposition 2.1** ([9]). \( sf(D, U) \) is well defined and

\[
\text{Ind}(TU) = -sf(D, U).
\]

The next proposition is an easy rephrasing of a result of Dai and Zhang:

**Proposition 2.2** ([9]). Let \( B \) be the Bismut superconnection associated with \( \sigma D \), then the cohomology class of the differential form \( \frac{1}{i\pi^2} \int_0^1 Tr_{\sigma}(B_t e^{-B_t^2})dt \) coincides with the Chern character of the higher spectral flow, i.e.,

\[
Ch(sf(D, U)) = \frac{-1}{i\pi^2} \int_0^1 Tr_{\sigma}(B_t e^{-B_t^2})dt.
\]
3. Norm and trace estimates

Throughout this section, let $D : \text{dom} D \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}$. In a number of estimates, we will also consider a bounded self-adjoint operator $A$. The operators $A$ that are of interest satisfy $s^2 + sA + D^2 \geq 0$ for all real $s \geq 0$. However it is also convenient at times to assume that $||A||$ is relatively small: $||A|| < \sqrt{2}$, for example. This can be achieved by scaling $A$: see Observation 2 of Section 5 in [3].

**Lemma 3.1** (Lemma 5.1 in [3]). Let $D$ be an unbounded self-adjoint operator,

(a) For $\lambda = a + iv \in \mathbb{C}, 0 < a < 1/2, s \geq 0$ we have the estimate

\[
(\lambda - (1 + D^2 + s^2))^{-1} \leq (v^2 + (1 + a^2)^{-1/2} \leq \frac{1}{1 - a}.
\]

(b) If $A$ is bounded, self-adjoint and $s^2 + sA + D^2 \geq 0$ we have

\[
(\lambda - (1 + D^2 + s^2 + sA))^{-1} \leq (v^2 + (1 - a)^{-1/2} \leq \frac{1}{1 - a}.
\]

(c) If $A$ is bounded, self-adjoint and $c = ||A|| < \sqrt{2}$ we have

\[
(\lambda - (1 + D^2 + s^2 + sA))^{-1} \leq (v^2 + (1 + s^2 - a - sc)^{-1/2} \leq \frac{1}{1/2 - a}.
\]

Let $Q = (1 + s^2 + F)$, where $F$ is $B^2$ as defined in Section 2 and $s \in [0, \infty)$. Define

\[
(\lambda - Q)^{-1} = \left(\lambda - (1 + s^2 + D^2)\right)^{-1}
\]

\[
+ \sum_{k \geq 0} \left[\left(\lambda - (1 + s^2 + D^2)\right)^{-1} F_{[+]} \right]^k \left(\lambda - (1 + s^2 + D^2)\right)^{-1}.
\]

Then for $z \in \mathbb{C}$ and $\text{Re}(z) > p/2$, we write $Q^{-z}$ using Cauchy’s formula (see [12])

\[
Q^{-z} = \frac{1}{2\pi i} \int_{l} \lambda^{-z}(\lambda - Q)^{-1} d\lambda,
\]

where $l$ is a vertical line $\lambda = a + iv$ parametrized by $v \in \mathbb{R}$ with $0 < a < 1/2$ fixed. Let $T^{(n)} = [F, \ldots [F, [F, T]]]$, then:

**Lemma 3.2** (Compare with Lemma 6.9 in [3]). Let $m, n, k$ be non-negative integers and $T \in \mathcal{O}P^m$. Then

\[
(\lambda - Q)^{-n}T = T(\lambda - Q)^{-n} + nT^{(1)}(\lambda - Q)^{-(n+1)} + \frac{n(n+1)}{2}T^{(2)}(\lambda - Q)^{-(n+2)} \]

\[+ \cdots + \left( \binom{n+k-1}{k} T^{(k)}(\lambda - Q)^{-(n+k)} + P(\lambda) \right)
\]

\[
= \sum_{j=0}^{k} \left( \binom{n+j-1}{j} T^{(j)}(\lambda - Q)^{-(n+j)} + P(\lambda) \right),
\]
where the remainder $P(\lambda)$ has order $-(2n + k - m + 1)$ and is given by

$$P(\lambda) = \sum_{j=1}^{k} \binom{j + k - 1}{k} (\lambda - Q)^{j-n-1}T^{(k+1)}(\lambda - Q)^{-j-k}. \tag{3.7}$$

**Corollary 3.3** (Compare with Corollary on Page 484 in [3]). Let $n,M$ be positive integers and $A \in OP^k$, let $R = (\lambda - Q)^{-1}$. Then

$$R^nAR^{-n} = \sum_{j=0}^{M} \binom{n + j - 1}{j} A^{(j)}R^j + P, \tag{3.8}$$

where

$$P = \sum_{j=0}^{n} \binom{j + M - 1}{M} R^{n+1-j}A^{(M+1)}R^{M+j-n} \tag{3.9}$$

has order $k - M - 1$.

We recall Lemma 6.10 in [3], let $OP^k$ denote the set of order $\leq k$ vertical pseudodifferential operators along the fiber.

**Lemma 3.4** (Compare with Lemma 6.10 in [3]). Let $k,n$ be non-negative integers, and suppose $\lambda \in \mathbb{C}, a = \text{Re}(\lambda)$ with $0 < a < 1/2$. For $A \in OP^k$ and $R_a(\lambda) = (\lambda - (1 + D^2 + s^2))^{-1}$, we have

$$\|R_a(\lambda)^{n/2+k/2}AR_a(\lambda)^{-n/2}\| \leq C_{n,k} \text{ and } \|R_a(\lambda)^{-n/2}AR_a(\lambda)^{n/2+k/2}\| \leq C_{n,k}, \tag{3.10}$$

where $C_{n,k}$ is constant independent of $s$ and $\lambda$ (square roots use the principal branch of log).

Then for the family case, we have:

**Lemma 3.5** (Compare with Lemma 6.11 in [3]). Let $A_j \in OP^{n_j}$ for $j = 1,\ldots, m$ and let $0 < a = \text{Re}(\lambda) < 1/2$ as above. We consider the operator

$$R^F_s(\lambda)A_1R^F_s(\lambda)A_2 \cdots R^F_s(\lambda)A_m\tilde{R}^F_s(\lambda), \tag{3.11}$$

where $R^F_s(\lambda) = (\lambda - (1 + s^2 + F))^{-1}$, $\tilde{R}^F_s(\lambda) = (\lambda - (1 + u^2 + sX + F))^{-1}$ and $X$ is self-adjoint, bounded and $s^2 + sX + D^2 \geq 0$. Then for all $M \geq 0$

$$R^F_s(\lambda)A_1R^F_s(\lambda)A_2 \cdots A_m\tilde{R}^F_s(\lambda) = \sum_{|k|=0}^{M} C(k)A^{(k_1)}_1 \cdots A^{(k_m)}_mR^F_s(\lambda)^{m+|k|}\tilde{R}^F_s(\lambda) + P_{M,m,F}, \tag{3.12}$$

where $P_{M,m,F}$ is of order (at most) $-2m - M + 3 + |n|$, and $k, n$ are multi-indices with $|k| = k_1 + \cdots + k_m$ and $|n| = n_1 + \cdots + n_m$. The constant $C(k)$ is given by

$$C(k) = \frac{(|k| + m)!}{k_1!k_2!\cdots k_m!(k_1 + 1)(k_1 + k_2 + 2)\cdots(|k| + m)} = (|k| + m)!\alpha(k). \tag{3.13}$$
Lemma 3.6. With the assumptions and notation of the last lemma including the assumption that \( A_i \in OP^{n_i} \) for each \( i \), there is a positive constant \( C \) such that
\[
\| (\lambda - (1 + D^2 + s^2))^{m+M/2+1/2-n/2} P_{M,m,F} \| \leq C
\]
independent of \( s \) and \( \lambda \) (though it depends on \( M, m \) and the \( A_i \)).

Proof. The remainder \( P_{M,m,F} \) in the previous lemma obtained after applying the pseudodifferential expansion has terms of two kinds. The first kind we consider are the bookkeeping terms at the end of the proof of the last lemma. They are of the form
\[
P = A^{(k_1)}_1 \cdots A^{(k_m)}_m R^{m+|k|}_F \tilde{R}_F
\]
with \(|k| > M\).

Next we prove the family term
\[
R^{-(m+M/2+1/2)+|n|/2} A^{(k_1)}_i \cdots A^{(k_m)}_m R^{m+|k|}_F \tilde{R}_F
\]
is uniformly bounded. We know that
\[
R_F = R + \sum_{d \geq 1} R(F_{i+1})^d.
\]
In order to estimate (3.15), we only need estimate the following term. By Lemma 3.4 we have
\[
\| R^{-(m+M/2+1/2)+|n|/2} A^{(k_1)}_i \cdots A^{(k_m)}_m \left[ R^{l_1} R(F_{i+1})^d R^{(m+|k|-l_1-1)} \right] \| \leq C_0 \quad \text{for each } l_1 \geq 1,
\]
where \( 1 \leq j \leq i + k_1 + k_2 + \cdots + k_{i-1} \) and \( 0 \leq k_1, k_2, \ldots, k_{i-1} \leq M \). Then this term is bounded.

On the other hand, we will prove \( \tilde{R}_F \) is bounded. By Observation 2 on page 497 in [5], without loss of generality, we assume \( \| X \| \leq \sqrt{2} \), then
\[
\tilde{R}_F^s(\lambda) = (\lambda - (1 + s^2 + sX + F))^{-1} = (\lambda - (1 + s^2 + sX + D^2)^{-1} - F_{i+1})^{-1} = (\lambda - (1 + s^2 + sX + D^2)^{-1} + \sum_{k \geq 1} (\lambda - (1 + s^2 + sX + D^2))^{-1})^k.
\]
By Lemma 3.1(c), we obtain

\[ \|(\lambda - (1 + s^2 + sX + D^2))^{-1}\| \leq \frac{1}{1/2 - a}. \]  

Since \( \|X\| \leq \sqrt{T} \), we have \( (1 + s^2 + sX + D^2)^{-1} \leq (1 + s^2 + s\|X\| + D^2)^{-1} \), then

\[ \|F_+^{\varphi}(\lambda - (1 + s^2 + sX + D^2))^{-1}\| \]

\[ = \|F_+^{\varphi}(1 + s^2 + sX + D^2)^{-1}\frac{(1 + s^2 + sX + D^2)}{\lambda - (1 + s^2 + sX + D^2)}\| \]

\[ \leq \|F_+^{\varphi}(1 + s^2 + sX + D^2)^{-1}\| \sup_{x \in [0, +\infty]} \left| \frac{1}{\lambda - (1 + x)} \right| \]

\[ \leq \|F_+^{\varphi}(1 + D^2)^{-\frac{1}{2}}\| \|(1 + D^2)^{\frac{1}{2}}(1 + s^2 + sX + D^2)^{-1}\| \]

\[ \times \frac{1 + x}{\sqrt{(1 + x - a)^2 + v^2}} \]

\[ \leq \tilde{C}_0 \| (1 + D^2)^{1/2} \| \frac{1 + x}{1 + x - a} \]

\[ \leq \tilde{C}_1. \]  

Combining these assertions, we see

\[ \|\tilde{R}^F(\lambda)\| \leq \frac{C}{\frac{1}{2} - a}. \]  

The other terms are the ones \( P_1, P_2, \ldots, P_m \) obtained in the proof of the last lemma. Recall:

\[ P_1 = R_F A_1^{(M+1)} R_F^{M+1} R_F A_2 R_F \cdots R_F A_m \tilde{R}_F, \]

while a typical summand of \( P_2 \) is an integer multiple of:

\[ A_1^{(k_1)} R_F^{s+k_1-\frac{1}{2}} A_2^{(M+1)} R_F^{M+j} R_F A_3 R_F \cdots R_F A_m \tilde{R}_F, \]

where \( 1 \leq j \leq 2 + k_1 \) and \( 0 \leq k_1 \leq M \). We work with the typical summand of \( P_1 \) above, and let

\[ B = A_1^{(k_1)} A_2^{(k_2)} \cdots A_l^{(k_l)} \]

which has order \((k_1 + k_2 + \cdots + k_l) + (n_1 + n_2 + \cdots + n_l - 1) = |k| + |n|_{l-1} \), where we have used the notation \( |n|_j = n_1 + n_2 + \cdots + n_j \). We will also use the notation \( |n|_{j+1} = n_1 + n_2 + \cdots + n_{j+1} \). We need to show that

\[ R^{-(m+M/2+1/2)+|n|/2} BR_F^{s+1+|k|} A_i^{(M+1)} R_F^{m+j} R_F A_{i+1} R_F \cdots R_F A_m \tilde{R}_F \]

is bounded independent of \( s \) and \( \lambda \). So, we calculate

\[ R^{-(m+M/2+1/2)+|n|/2} BR_F^{s+1+|k|} A_i^{(M+1)} R_F^{M+j} R_F A_{i+1} R_F \cdots R_F A_m \tilde{R}_F \]

\[ = \left( R^{-(m+M/2+1/2)-|n|/2} BR^{(|k|+|n|_{l-1})/2} R^{(m+M/2+1/2-|n|/2)} \right) R^{(|k|/2}} \]
Definition 4.1. \( \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes L^2(M_0, S(TZ) \otimes \pi^* \Lambda^* (T^* B_0) \) acted by \( \mathcal{A} = M^2 \otimes M^2 \otimes M_k \otimes k(C^\infty(M_0)) \), where \( M^2 \) denotes the set of the \( 2 \times 2 \) order matrix. Introduce the two dimensional Clifford algebra in the form

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and define the grading on \( \mathcal{H} \) by \( \Gamma = \sigma_2 \otimes \sigma_3 \otimes 1 \in \mathcal{A} \).

Let \( u \in \mathcal{A} \) be unitary and we introduce the following even operators (i.e., they commute with \( \Gamma \)):

\[
q = \sigma_3 \otimes \begin{pmatrix} 0 & -iu^{-1} \\ iu & 0 \end{pmatrix}, \quad \tilde{B} = \sigma_2 \otimes 1_2 \otimes B,
\]

\[
\tilde{B}q + q\tilde{B} = \sigma_1 \otimes \begin{pmatrix} 0 & [B,u^{-1}] \\ -[B,u] & 0 \end{pmatrix}.
\]
Define $\text{Str}^{\text{even}}(a) = \frac{1}{2} \text{Tr}^{\text{even}}(\Gamma(a))$, then

\begin{equation}
\tilde{B}_{r,s}^2 = \tilde{B}_s^2 + s(1-2r)\sigma_1 \otimes \begin{pmatrix} 0 & [B,u^{-1}] \\ [B,u] & 0 \end{pmatrix} + s^2.
\end{equation}

From Lemma 3.5 and Lemma 3.6, we have:

**Lemma 4.2** (Compare with Lemma 7.1 in [3]). Let $\dim Z = p$ and $\pi : M_0 \to B_0$ be a fibration. Let $m$ be a nonnegative integer, and for $j = 0, \ldots, m$ let $A_j \in \text{OP}^0$. Define $\tilde{F} = \sigma_2 \otimes 1_2 \otimes F$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \Gamma(T^*B_0) \otimes S(Z)$. Let $l$ be the vertical line $v \mapsto \lambda = a + iv$ for $v \in \mathbb{R}$ and $0 \leq a \leq 1$, $R_{\tilde{F}}^\pm(\lambda) = (\lambda - (1 + s^2 + \tilde{F}))^{-1}$ and $R_{\tilde{F}}^{\pm}(\lambda) = \lambda - (1 + s^2 + \tilde{F} + s[B,q])^{-1}$. Then for $r \in \mathbb{C}$ and $\text{Re}(r) > 0$ the operator

\begin{equation}
B(s) = \frac{1}{2\pi i} \int \lambda^{-p/2-r} A_0 R_{\tilde{F}}^\pm(\lambda) A_1 R_{\tilde{F}}^\pm(\lambda) A_2 \cdots R_{\tilde{F}}^\pm(\lambda) A_m \tilde{R}_{\tilde{F}}^\pm(\lambda) d\lambda,
\end{equation}

is trace class for $m > p/2$ and the function $s^m \|B(s)\|_1$ is integrable on $[0, \infty)$ when

\begin{equation}
p + \varepsilon < 1 + m \text{ and } 1 + \varepsilon < m + 2 \text{Re}(r).
\end{equation}

**Lemma 4.3** (Compare with Lemma 7.2 in [3]). Let $m$ be a nonnegative integer, and for $j = 0, \ldots, m$ let $A_j \in \text{OP}^0$, $k_j \geq 0$. Let $l$ be the vertical line $v \mapsto \lambda = a + iv$ for $v \in \mathbb{R}$ and $0 \leq a \leq 1$, $R_{\tilde{F}}^\pm(\lambda) = (\lambda - (1 + s^2 + \tilde{F}))^{-1}$. Then for $\text{Re}(r) > 0$ the operator

\begin{equation}
B(s) = \frac{1}{2\pi i} \int \lambda^{-p/2-r} A_0 R_{\tilde{F}}^\pm(\lambda) A_1 R_{\tilde{F}}^\pm(\lambda) A_2 \cdots R_{\tilde{F}}^\pm(\lambda) A_m R_{\tilde{F}}^\pm(\lambda) d\lambda,
\end{equation}

is trace class for $\text{Re}(r) + m - \frac{|k|}{2} > 0$ and the function $s^\alpha \times \|B(s)\|_1$ is integrable on $[0, \infty)$ when

\begin{equation}
1 + \alpha + |k| - 2m < 2(\text{Re}(r) - \varepsilon).
\end{equation}

**Lemma 4.4** (Compare with Lemma 7.3 in [3]). With the notation as set out at the beginning of this section and with $R_{\tilde{F}}^\pm(\lambda) = (\lambda - (1 + s^2 + \tilde{F}))^{-1}$, $R_{\tilde{F}}^{\pm}(\lambda) = (\lambda - (1 + s^2 + \tilde{F} + s[B,q]))^{-1}$, we have for $\text{Re}(r) > 0$ and any positive integer $M > p - 1$

\begin{align}
\text{Str}^{\text{even}} \left( q(1 + s^2 + \tilde{F} + s[B,q])^{-1} \tilde{F}^{-r} \right) \\
= \sum_{m=1,\text{odd}}^{M} s^m \text{Str}^{\text{even}} \left( \frac{1}{2\pi i} \int \lambda^{-p/2-r} q(R_{\tilde{F}}^\pm(\lambda) \{ \tilde{B},q \})^m R_{\tilde{F}}^{\pm}(\lambda) d\lambda \right) \\
+ s^{M+1} \text{Str}^{\text{even}} \left( \frac{1}{2\pi i} \int \lambda^{-p/2-r} q(R_{\tilde{F}}^\pm(\lambda) \{ \tilde{B},q \})^{M+1} R_{\tilde{F}}^{\pm}(\lambda) d\lambda \right),
\end{align}

where $\text{Str}^{\text{even}}$ denotes taking supertrace with value in $\Omega^{\text{even}}(B_0)$. 

Let \( \tilde{B}_r = (1 - r) \tilde{B} - rq \tilde{B} q \) and \( \tilde{B}_{r,s} = \tilde{B}_r + sq \), then
\[
(4.9) \quad \tilde{B}_r \equiv \tilde{B}_{r,0} = \sigma_2 \otimes \begin{pmatrix} B + ru^{-1}[B, u] & 0 \\ 0 & B + ru[B, u^{-1}] \end{pmatrix}.
\]

**Theorem 4.5.** Let \( n > p = \dim Z \) and \( l \leq \dim B_0 \). Then
\[(4.10) \quad Ch\left(sf(D, u^* Du)\right)[l] = \frac{\Gamma(n/2)}{\Gamma((n + l - 1)/2)\sqrt{\pi}} \int_0^1 Tr_{l}^{\text{even}} \left( u^* [B, u](1 + (B + ru[B, u])^2)^{-\frac{1}{2}} \right) dr
\]
\[
= \frac{\Gamma(n/2)}{2\Gamma((n + l - 1)/2)\sqrt{\pi}} \int_0^1 Str_{l}^{\text{even}} \left( \frac{dB_r}{dr}(1 + B_r^2)^{-\frac{1}{2}} \right) dr.
\]

**Proof.** From (4.19), (4.29) and (4.41) in [9], we have
\[(4.11) \quad Ch\left(sf(D, u^* Du)\right) = -\frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\pi}} \int_0^1 Tr_{l}^{\text{even}} \left( \frac{\partial B_t(r)}{dr} \right) \exp(-B_r^2(r))
\]
where \( B_t(r) = B_t(0) + ru[B_t(0), u^{-1}] \) and \( B_t(0) = \sqrt{t} \psi_t B, \psi_t : dy_j \rightarrow \frac{dy_j}{\sqrt{t}} \).

Taking the \( l \)-degree component of \( \Omega(B_0) \) on the last Eq., we obtain
\[(4.12) \quad Ch\left(sf(D, u^* Du)\right) = -\frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\pi}} \int_0^1 Tr_{l}^{[l]} \left( \sqrt{t} u^{-1}[B, u] \exp(-t(B + ru^{-1}[B, u])^2) \right) dr.
\]

By the definition of the Gamma function, we have
\[(4.13) \quad 1 = \frac{1}{\Gamma(n/2)} \int_0^1 t^{n/2 - 1} e^{-t} dt.
\]

Combining the above equations, and by Lemma 1 in [12] and the following Lemma 4.6, we have
\[(4.14) \quad Ch\left(sf(D, u^* Du)\right) = -\frac{1}{\Gamma(n/2)} \frac{1}{\sqrt{\pi}} \int_0^1 t^{n/2 - 1} e^{-t} \sqrt{\pi} \int_0^1 Tr_{l}^{[l]} \left( u^{-1}[B, u] e^{-t(B + ru^{-1}[B, u])^2} \right) dr dt
\]
\[
= -\frac{1}{\Gamma(n/2)} \frac{1}{\sqrt{\pi}} \int_0^1 \int_0^1 t^{n/2 - 1} u^{-1}[B, u] e^{-t(B + ru^{-1}[B, u])^2} dr dt
\]
\[
= -\frac{1}{\Gamma(n/2)} \frac{1}{\sqrt{\pi}} \int_0^1 Tr_{l}^{[l]} \left( u^{-1}[B, u](1 + (B + ru^{-1}[B, u])^2)^{-\frac{1}{2}} \right) dr.
\]

Then similarly to the computations in [3, p. 73], we get Theorem 4.5. \( \square \)
Lemma 4.6. The following expression

\[ (4.15) \quad \int_0^{+\infty} \int_0^1 t^{n-1} e^{-t} \sqrt{Tr}^l \left( u^{-1}[B,u] e^{-t(B+ru^{-1}[B,u])^2} \right) dr dt \]

is uniformly convergent.

Proof. Since \( n > \dim Z \geq 1 \), then \( \frac{n-3}{2} \geq -\frac{3}{2} \). By Theorem 4.8 in [9], we have when \( t \to 0 \),

\[ (4.16) \quad t^{n-1} e^{-t} T_r^l \left( \sqrt{tr}^{-1}[B,u] e^{-t(B+ru^{-1}[B,u])^2} \right) \to O(t^{-\frac{1}{2}}). \]

So the above integral is convergent at \( t = 0 \).

Let \( (B + ru^{-1}[B,u])^2 = D_u^2 + F_{u}^l \), where \( F_{u}^l \) is a first order differential operator with coefficient in \( \Omega_{\geq 1}(B_0) \). By the Duhamel’s principle and the Hölder equality, we have

\[ \begin{align*}
\| T_r^l \left( u^{-1}[B,u] e^{-t(B+ru^{-1}[B,u])^2} \right) \|_1 & \leq \sum_{k \geq 0} t^k \int_{\Delta_k} \| u^{-1}[B,u] e^{-t \sigma D_u^2} F_{r}^l \| \| e^{-t \sigma D_u^2} \|_1 \| \| e^{-t \sigma D_u^2} \|_1 \| \| (e + D_u^2)^{-\frac{1}{2}} F_{r}^l \| \| e^{-t \sigma D_u^2} \|_1 \| \| e^{-t \sigma D_u^2} \|_1 d\Delta_k. \\
& \leq \sum_{k \geq 0} t^k \int_{\Delta_k} \| u^{-1}[B,u] \| \| e^{-t \sigma D_u^2} \|_1 \| \| e^{-t \sigma D_u^2} \|_1 \| \| e^{-t \sigma D_u^2} \|_1 \| \| e^{-t \sigma D_u^2} \|_1 d\Delta_k. \\
& \leq \sum_{k \geq 0} t^k \int_{\Delta_k} L_0 (t r e^{-t \sigma D_u^2}) e^{-\frac{1}{2} (k-1) t} \| \| e^{-t \sigma D_u^2} \|_1 \| \| e^{-t \sigma D_u^2} \|_1 \| \| e^{-t \sigma D_u^2} \|_1 d\Delta_k.
\end{align*} \]

By

\[ (4.18) \quad \sup_{x \in [0, +\infty)} \left( e^{-c x} (\varepsilon + x)^{\frac{1}{2}} \right) = e^{-\frac{1}{2} c x} \left( \frac{1}{2c} \right)^{\frac{1}{2}}, \]

then

\[ (4.19) \quad t^{\frac{n-3}{2}} e^{-t} \| T_r^l \left( u^{-1}[B,u] e^{-t(B+ru^{-1}[B,u])^2} \right) \|_1 \leq t^k \int_{\Delta_k} L_0 (t r e^{-t \sigma D_u^2}) e^{-\frac{1}{2} (k-1) t} \| \| e^{-t \sigma D_u^2} \|_1 \| \| e^{-t \sigma D_u^2} \|_1 \| \| e^{-t \sigma D_u^2} \|_1 d\Delta_k, \]

where \( L_0 \) is constant. By Theorem 8 in [2], then

\[ (4.20) \quad t^{\frac{n-3}{2}} e^{-t} T_r^l \left( \sqrt{tr}^{-1}[B,u] e^{-t(B+ru^{-1}[B,u])^2} \right) \]

exponentially decays and is bounded independently of \( u \), and is uniformly convergent, when \( t \to +\infty. \)\( \square \)
Define
\begin{equation}
C_{p/2+r,l} = \frac{\Gamma(p/2 + r + l/2 - 1/2)\sqrt{\pi}}{\Gamma(p/2 + r)}.
\end{equation}

And then, the main result of this section is the following lemma. By the Mellin transform, we may prove the family case of Lemmas 5.6, 5.8 in [3], by the associated Lemmas 5.6 and 5.12 in [11], similar to the proof of Lemma 7.4 in [3], we get:

**Lemma 4.7.** Let \( N = [(p + q')/2] + 1 \) be the least positive integer strictly greater than \((p + q')/2\). Then there is a \( \delta', 0 < \delta < 1 \) such that up to an exact form on \( B_0 \),

\[
\text{Ch} \left( s f(D, u^* Du) \right) C_{p/2+r,l} = \int_{0}^{t} \text{Str}^{even} \left( s \left( 1 + s^2 + \tilde{F} + s \{ \tilde{B}, q \} \right)^{-\frac{\delta}{2} - 1} \right) \left[ l \right] ds
\]

\[
= \frac{1}{2\pi i} \sum_{m=1, odd}^{2N-1} s^m \text{Str}^{even} \left( \int \lambda^{-p/2 - r} q(R_{\bar{s}}(\lambda) \{ \tilde{B}, q \})^m \right) \times R_{\bar{s}}(\lambda) \left[ l \right] ds + \text{holo},
\]

where \( \text{holo} \) is a function of \( r \) holomorphic for \( \text{Re}(r) > -(p + q')/2 + \delta'/2 \).

4.2. A resolvent cocycle

At this point it is interesting to perform the supertrace, so that we have an expression which only depends on our original spectral triple \((A, \mathcal{H}, D)\).

**Definition 4.8.** For \( m \geq 0 \), operators \( A_0, \ldots, A_m \), \( A_j \in OP^{k_j} \) and \( 2\text{Re}(r) > k_0 + \cdots + k_m - 2m \) define

\[
\langle A_0, \ldots, A_m \rangle_{m,s,r} = \text{Tr}^{even} \left( \frac{1}{2\pi i} \int \lambda^{-p/2 - r} A_0 R_{\bar{s}}^F(\lambda) A_1 \cdots A_m R_{\bar{s}}^F(\lambda) d\lambda \right).
\]

The conditions on the orders and on \( r \) are sufficient for the trace to be well-defined, then:

**Lemma 4.9** (Compare with Lemma 7.6 in [3]). For any integers \( m \geq 0, k \geq 1 \) and operators \( A_0, \ldots, A_m \) with \( A_j \in OP^{k_j} \), and \( 2\text{Re}(r) > k + \sum k_j - 2m \), we may choose \( r \) with \( \text{Re}(r) \) sufficiently large such that

\[
k \int_{0}^{\infty} s^{k-1} \langle A_0, \ldots, A_m \rangle_{m,s,r} ds
\]

\[
= -2 \sum_{j=0}^{m} \int_{0}^{\infty} s^{k+1} \langle A_0, \ldots, A_j, 1, A_{j+1}, \ldots, A_m \rangle_{m+1,s,r} ds.
\]
Lemma 4.10 (Compare with Lemma 7.7 in [3]). For any integers $m \geq 0$, $k \geq 1$ and $\text{Re}(r)$ sufficiently large, and $A_j \in \mathcal{O}P_k^j$, $j = 0, \ldots, m$, we have
\begin{equation}
(4.25) \quad \int_0^\infty s^k (A_0, \ldots, A_m)_{m,s,r} ds = \int_0^\infty s^k (A_m, A_0, \ldots, A_{m-1})_{m,s,r} ds.
\end{equation}

Lemma 4.11. For operators $A_0, \ldots, A_m, A_j \in \mathcal{O}P_k^j$, $k_j \geq 0$, and $\text{Re}(r)$ sufficiently large we have
\begin{equation}
(4.26) \quad -\langle A_0, \ldots, [F, A_j], \ldots, A_m \rangle_{m,s,r}
= \langle A_0, \ldots, A_{j-1} A_j, \ldots, A_m \rangle_{m-1,s,r} - \langle A_0, \ldots, A_j A_{j+1}, \ldots, A_m \rangle_{m-1,s,r},
\end{equation}
and for $k \geq 1$
\begin{equation}
(4.27) \quad \int_0^\infty s^k \left[ (BA_0, A_1, \ldots, A_m)_{m,s,r} - \langle A_0, A_1, \ldots, A_mB \rangle_{m,s,r} \right] ds
= B_{Bo} \int_0^\infty s^k (A_0, A_1, \ldots, A_m)_{m,s,r} ds.
\end{equation}

Proof. The first identity follows from observing that
\begin{equation}
(4.28) \quad -[F, A_j] = R_s(\lambda)^{-1} A_j - A_j R_s(\lambda)^{-1},
\end{equation}
then
\begin{equation}
(4.29) \quad A_0 R_s^F(\lambda) A_1 R_s^F(\lambda) \cdots (R_s^F(\lambda)^{-1} A_j - A_j R_s^F(\lambda)^{-1})
R_s^F(\lambda) A_{j+1} R_s^F(\lambda) \cdots A_m R_s^F(\lambda)
= A_0 R_s^F(\lambda) A_1 R_s^F(\lambda) \cdots A_{j-1} A_j R_s^F(\lambda) \cdots A_m R_s^F(\lambda)
- A_0 R_s^F(\lambda) A_1 R_s^F(\lambda) \cdots A_j A_{j+1} R_s^F(\lambda) \cdots A_m R_s^F(\lambda).
\end{equation}

We have the equalities
\begin{equation}
(4.30) \quad \int_0^\infty s^k \langle BA_0, A_1, \ldots, A_m \rangle_{m,s,r} ds
= \int_0^\infty s^k \text{Tr} \left( \frac{1}{2\pi i} \int_1 \lambda^{-p/2-r} BA_0 R_s^F(\lambda) \cdots A_m R_s^F(\lambda) \right) d\lambda,
\end{equation}
\begin{equation}
(4.31) \quad \int_0^\infty s^k \langle A_0, A_1, \ldots, A_mB \rangle_{m,s,r} ds
= \int_0^\infty s^k \text{Tr} \left( \frac{1}{2\pi i} \int_1 \lambda^{-p/2-r} A_0 R_s^F(\lambda) \cdots A_mB R_s^F(\lambda) \right) d\lambda ds.
\end{equation}

Combining the above equations, we obtain
\begin{align*}
&\int_0^\infty s^k \left[ (BA_0, A_1, \ldots, A_m)_{m,s,r} - \langle A_0, A_1, \ldots, A_mB \rangle_{m,s,r} \right] ds \\
&= \frac{1}{2\pi i} \int_0^\infty s^k \text{Tr} \left[ B, \int_1 \lambda^{-p/2-r} A_0 R_s^F(\lambda) \cdots A_mB R_s^F(\lambda) d\lambda \right] ds \\
&= B_{Bo} \int_0^\infty s^k \text{Tr} \left( \int_1 \lambda^{-p/2-r} A_0 R_s^F(\lambda) \cdots A_mB R_s^F(\lambda) d\lambda \right) ds.
\end{align*}
(4.32) \[ d_{B_0} \int_0^\infty s^k(A_0, A_1, \ldots, A_m)_{m,s,r} ds. \]  

Suspecting that the higher spectral flow is given by pairing a cocycle with the Chern character of a unitary, we remove the normalisation coming from \( Ch_m(u) \) from our resolvent formula to define a cocycle. The factor of \( \sqrt{2\pi i} \) is for compatibility with the Kasparov product [7].

**Definition 4.12.** Let \( C(m) \) denote the constant \( \frac{-2\sqrt{\pi}}{\Gamma((m+1)/2)} \). Then, for \( Re(\lambda) > -m/2 + 1/2 \) and \( da = [B, a] \), we define \( \phi^r_m : A^{m+1} \to C \) with \( A = C^\infty(M_0) \) by

\[ (4.33) \quad \phi_m^r(a_0, \ldots, a_m) = C(m) \int_0^\infty s^m(a_0, da_1, \ldots, da_m)_{m,s,r} ds. \]

By Lemma 4.9, the condition on \( r \) ensures that the integral converges. We note that this constant \( C(m) \) is distinct from \( C(k) \) which takes a multi-index \( k \) as its argument.

**Proposition 4.13.** For \( p \geq 1 \), the collection of functionals \( \phi^r = \{ \phi^r_m \}_{m=1}^{2N-1}, m = 1, 3, \ldots, 2N - 1 \) odd, such that

\[ (B\phi^r_{m+2} + b\phi^r_m)(a_0, \ldots, a_m) \]

(4.34) \[ = -C(m) d_{B_0} \int_0^\infty s^m(a_0, [B, a_1], \ldots, [B, a_{m+1}])_{m+1,s,r} ds, \]

(4.35) \[ (B\phi^r_1 + d_{B_0}\phi^r_0)(a_0) = 0, \]

where \( a_i \in A \). Moreover, there is a \( \delta' \), \( 0 < \delta' < 1 \), then \( b\phi^r_{(2N-1)}(a_0, \ldots, a_{2N}) \) is a holomorphic function of \( r \) for \( Re(r) > -(p + q')/2 + \delta'/2 \).

**Proof.** We start with the computation of the coboundaries of the \( \phi^r_m \). Recall the definition \( B : C^{k+1}(A) \to C^k(A) \),

\[ (4.36) \quad Bc(a_0, \ldots, a_m) = \sum_{j=0}^k c(1, a_j, \ldots, a_{m+1}, a_0, \ldots, a_{j-1}). \]

By \( \phi^r_{m+2} \), Lemma 4.9 and Lemma 4.11, we get

\[ (4.37) \]

\[ (B\phi^r_{m+2})(a_0, \ldots, a_{m+1}) \]

\[ = \sum_{j=0}^{m+1} (-1)^j (m+1)^j \phi^r_{m+2}(1, a_j, \ldots, a_{m+1}, a_0, \ldots, a_{j-1}) \]

\[ = \sum_{j=0}^{m+1} \langle C(m+2)(-1)^j (m+1)^j \int_0^\infty s^{m+2}(1, [B, a_j], \ldots, [B, a_{m+1}], [B, a_0], \ldots, [B, a_{j-1}])_{m+2,s,r} ds \]

\[ = \sum_{j=0}^{m+1} \langle C(m+2) \int_0^\infty s^{m+2}([B, a_0], \ldots, [B, a_{j-1}], 1, [B, a_j], \ldots, [B, a_{m+1}])_{m+2,s,r} ds \]
Observe that for $\phi$, we have

\[
\begin{align*}
&= \mathcal{C}(m + 2) \frac{m + 1}{2} \int_0^\infty s^m \langle [B, a_0], \ldots, [B, a_{m+1}] \rangle_{m+1,s,r} ds \\
&= \mathcal{C}(m) \int_0^\infty s^m \langle [B a_0 - a_0 B], \ldots, [B, a_{m+1}] \rangle_{m+1,s,r} ds \\
&= \mathcal{C}(m) \int_0^\infty s^m \left( (B a_0, [B, a_1], \ldots, [B, a_{m+1}])_{m+1,s,r} \\
&\quad - \langle a_0, [B, a_1], \ldots, [B, a_{m+1}] B \rangle_{m+1,s,r} \right) ds \\
&= \mathcal{C}(m) \int_0^\infty s^m \sum_{j=1}^{m+1} (-1)^j \langle a_0, [B, a_1], \ldots, [B^2, a_j], \ldots, [B, a_{m+1}] \rangle_{m+1,s,r} ds \\
&= \mathcal{C}(m) dB_0 \int_0^\infty s^m \langle a_0, [B, a_1], \ldots, [B, a_{m+1}] \rangle_{m+1,s,r} ds.
\end{align*}
\]

We now compute the Hochschild coboundary of $\phi^c$. From the definitions $\phi^c$ and Lemma 4.11, we have

\[
\begin{align*}
(B \phi^c_1)(a_0) &= \frac{\mathcal{C}(1)}{2\pi i} \int_0^\infty s \text{Tr} \left( \int_l \lambda^{-p/2-r} R^F_s(\lambda) [B, a_0] R^F_s(\lambda) d\lambda \right) ds \\
&= \frac{\mathcal{C}(1)}{2\pi i} \int_0^\infty s \text{Tr} \left( \int_l \lambda^{-p/2-r} [B, R^F_s(\lambda) a_0 R^F_s(\lambda)] d\lambda \right) ds \\
&= \frac{\mathcal{C}(1)}{2\pi i} \int_0^\infty s \text{Tr} \left( [B, \int_l \lambda^{-p/2-r} R^F_s(\lambda) a_0 R^F_s(\lambda) d\lambda] \right) ds \\
&= dB_0 \frac{\mathcal{C}(1)}{2\pi i} \int_0^\infty s \int_l \lambda^{-p/2-r} R^F_s(\lambda) a_0 R^F_s(\lambda) ds \\
&\quad \left( B^2 \phi^c_1 \right) \left( a_0, [B, a_1], \ldots, [B, a_{m+1}] \right) ds \\
&= dB_0 \mathcal{C}(1) \int_0^\infty s (1, a_0)_{1,s,r} ds.
\end{align*}
\]

(4.38)

We now compute the Hochschild coboundary of $\phi^c_2$. From the definitions $\phi^c_2$ and Lemma 4.11, we have

\[
\begin{align*}
(B \phi^c_2)(a_0, \ldots, a_{m+1}) &= \frac{\mathcal{C}(1)}{2\pi i} \int_0^\infty s^m \langle [a_0 a_1, a_2, \ldots, a_{m+1}] \rangle_{m+1,s,r} ds \\
&= \mathcal{C}(m) \int_0^\infty s^m \langle [a_0 a_1, [B, a_2], \ldots, [B, a_{m+1}] \rangle_{m,s,r} ds.
\end{align*}
\]

(4.39)
\[
\sum_{j=1}^{k} (-1)^j (a_0, [B, a_1], \ldots, a_j | B, a_{j+1}) + [B, a_j]a_{j+1}, \ldots, [B, a_{m+1}])_{m,s,r}
\]

\[
+ \langle a_{m+1}a_0, [B, a_1], \ldots, [B, a_m] \rangle_{m,s,r} \right) ds
\]

\[
= \mathcal{C}(m) \int_{0}^{\infty} s^m \left( \langle a_0, [B, a_1], \ldots, [B, a_{m+1}] \rangle_{m,s,r} \right) ds
\]

\[
- \langle a_0, [B, a_1], \ldots, [B, a_{m+1}] \rangle_{m,s,r} \right) ds
\]

\[
- \mathcal{C}(m) \int_{0}^{\infty} s^m \left( \langle a_0, [B, a_1], a_2, \ldots, [B, a_{m+1}] \rangle_{m,s,r} \right) ds
\]

\[
\vdots
\]

\[
- \mathcal{C}(m) \int_{0}^{\infty} s^m \left( \langle a_0, [B, a_1], \ldots, a_m [a_{m+1}] \rangle_{m,s,r} \right) ds
\]

\[
- \langle a_{m+1}a_0, [B, a_1], \ldots, [B, a_m] \rangle_{m,s,r} \right) ds
\]

\[
\sum_{j=1}^{m+1} (-1)^j \mathcal{C}(m) \int_{0}^{\infty} s^m \left( \langle a_0, [B, a_1], \ldots, [B^2, a_j], \ldots, [B, a_{m+1}] \rangle_{m+1,s,r} \right) ds.
\]

For \(m = 1, 3, 5, \ldots, 2N - 3\), by Eqs. (4.37)-(4.39) we obtain

\[
(B\phi_{m+2}^r + b\phi_m^r) (a_0, \ldots, a_m)
\]

\[
(4.40)
\]

\[
= - \mathcal{C}(m) d_{B_0} \int_{0}^{\infty} s^m \langle a_0, [B, a_1], \ldots, [B, a_{m+1}] \rangle_{m+1,s,r} ds.
\]

So we just need to check the claim that \(b\phi_{2N-1}^r\) is holomorphic for Re(\(r\)) > \(-(p + q')/2 + \delta'\) for some suitable \(\delta'\). The proof that (4.40) is holomorphic for \(m = 2N - 1\) is similar to the analyticity proof in Lemma 4.7.

\[\square\]

5. A family residue cocycle

In this section, we derive a new expression stated as Theorem 5.1 from the resolvent expansion of the higher spectral flow formula Theorem 4.5. This leaves us with a formula for higher spectral flow that involves an integral over the parameter \(s\). By integrating out the \(s\) dependence in the formula of Theorem 5.1, we find a higher spectral flow formula which involves a sum of zeta functions. One immediately recognises that individual terms in this formula may be obtained from our resolvent cocycle by using the pseudodifferential calculus. Thus, from the resolvent cocycle we derive the residue cocycle in the final section. Our final formula for the higher spectral flow follows immediately by evaluating the residue cocycle on \(Ch_*(u^r)\).
The aim of this section is to establish the family higher spectral flow formula which is summarised in the following result. We compute the Chern Character of the higher spectral flow from $D$ to $u^*Du$, where $u \in A$ is unitary with $[D,u]$ bounded. Next we compute the residue of the gamma function $\Gamma(x)$. For the meromorphic function $\Gamma(x)$ which has simple poles at nonnegative integers, then

$$\text{res}(\Gamma(x), -n) = \frac{(-1)^n}{n!}. \tag{5.1}$$

The ‘constant’ $C_{p/2+r,l}$ has simple poles at $r = \frac{1-(p+l)}{2}$, and $p = \dim Z$ is odd, $l$ even. Therefore, $\Gamma(\frac{p}{2} + z)$ is holomorphic at $z = \frac{1-(p+l)}{2}$. For the simple poles $z = \frac{1-(p+l)}{2}$, by (5.1) we have

$$\text{res}_{z = \frac{1-(p+l)}{2}} C_{\frac{p}{2} + z, l} = \text{res}_{z = \frac{1-(p+l)}{2}} \left( \Gamma\left(\frac{p+l}{2} + z - \frac{1}{2}\right) \right) \frac{\sqrt{\pi}}{\Gamma\left(\frac{p+l}{2}\right)} \tag{5.2}$$

where we have used $\Gamma(x+1) = x\Gamma(x)$ and

$$\Gamma\left(\frac{1-l}{2}\right) = \frac{(-1)^{\frac{1}{2}+l}}{1 \times 3 \times \cdots \times (l-1)} \Gamma\left(\frac{1}{2}\right). \tag{5.3}$$

Set $A_l = \frac{(-1)^{\frac{1}{2}+l}}{1 \times 3 \times \cdots \times (l-1)} C_{\frac{p}{2} + z, l}$, then we have

$$A_l \times \text{res}_{z = \frac{1-(p+l)}{2}} C_{\frac{p}{2} + z, l} = 1. \tag{5.4}$$

With these preliminaries, we can state the main result of the paper.

**Theorem 5.1.** In the cohomology of $B_0$, the following equality holds

$$\text{Ch}(\text{sf}(D, u^*Du)) \quad \tag{5.5}$$

$$\quad = \frac{1}{\sqrt{2\pi t}} \sum_{i=0}^{\dim B_0} A_l \left( \text{res}_{s = \frac{1-(p+l)}{2}} \left( \sum_{m=1, \text{odd}}^{2N-1} \langle \phi_m^r, \text{Ch}_m(u) \rangle \right) \right)^{[l]}.$$

**Proof.** By Lemma 4.7, we have

$$C_{p/2+r,l}(\text{Ch}(\text{sf}(D, u^*Du)))^{[l]} \quad \tag{5.6}$$

$$= \int_0^\infty \text{Str}^{\text{even}} \left( q(1 + \bar{F} + s^2 + s[\bar{B},q])^{-p/2-r} \right)^{[l]} ds.$$

Then we take residues at $r = (1-(p+l))/2$ of both sides, by Lemma 3.1 in [3], we have

$$\text{Ch}(\text{sf}(D, u^*Du))^{[l]} \quad \tag{5.7}$$
Then we have the pseudodifferential expansion of the higher spectral flow.

**Proposition 5.2.** There is a \( \delta' \), \( 0 < \delta' < 1 \) and \( 0 \leq l \leq \dim B_0 \), \( l \) even, such that up to an exact form on \( B_0 \),

\[
(5.8) \quad Ch\left(sf(D, u^* Du)\right)|^{[l]} \mathcal{C}_{p/2+r,l}^s = \int_0^\infty Str^{even}\left(q(1 + \tilde{F} + s^2 + s(\tilde{B}, q))^2\right)^{[l]} ds
\]

\[
= \frac{1}{2\pi i} \sum_{m=1,odd}^{2N-1} \int_0^\infty s^m Str^{even}\left(\int_{\lambda}^{p/2-r} q\left(\tilde{B}, q\right)M R^s_{\lambda}(\tilde{B}, q)\right) d\lambda + \text{holo} \]

\[
= \frac{1}{2\pi i} \sum_{m=1,odd}^{2N-1} \left( \left( \sum_{m=1,odd}^{2N-1} \langle \phi_m^r, Ch_m u \rangle \right) + \frac{1}{2N-1} \right)
\]

\[
+ d_{B_0} res_{\lambda=1} \mathcal{C}_{p/2+r,l}^s \int_0^\infty s^m (1, [B, u^{-1}], [B, u], \ldots, [B, u^{-1}]) ds^{[l-1]} .
\]

\[\square\]
where \( \text{holo} \) is a function of \( r \) holomorphic for \( \text{Re}(r) > -(p + q')/2 + \delta' /2 \).

Consequently the sum of functions on the right-hand side has an analytic continuation to a deleted neighbourhood of \( r = (1 - p - l)/2 \) (given by the left-hand side) with at worst a simple pole at \( r = (1 - p - l)/2 \).

**Proposition 5.3.** There is a \( \delta' \), \( 0 < \delta' < 1 \) and \( 0 \leq l < \text{dim } B \) such that

\[
\text{Ch}(sf(D, u^* Du))^{[l]} = \sum_{m=1, \text{odd}}^{2N-1} 2N-1-m \sum_{|k|=0}^{2N-1-m} \mathcal{C}_k \frac{\Gamma(p/2 + r + m + |k|)}{\Gamma(p/2 + r)(m + |k|)!} \int_0^\infty s^m \text{Str}^{\text{even}} \left( q \{ \tilde{B}, q \}^{(k_1)} \ldots \{ \tilde{B}, q \}^{(k_m)}(1 + \tilde{F})^{-((k_1+m)/2)} \right) ^{[l]} ds + \text{holo},
\]

where \( \text{holo} \) is a function of \( r \) holomorphic for \( \text{Re}(r) > -(p + q')/2 + \delta' /2 \).

Consequently the sum of functions on the right-hand side has an analytic continuation to a deleted neighbourhood of \( r = (1 - p - l)/2 \) (given by the left-hand side) with at worst a simple pole at \( r = (1 - p - l)/2 \).

We denote the analytic continuation of a function analytic in a right half-plane to a deleted neighbourhood of the critical point by putting the function in boldface. Thus we define the functionals for each integer \( j \geq 0 \):

\[
\text{Str}^{\text{even}}_{j,l} \left( q \{ \tilde{B}, q \}^{(k_1)} \ldots \{ \tilde{B}, q \}^{(k_m)}(1 + \tilde{F})^{-((k_1+m)/2)} \right) ^{[l]} = r \text{es}_{r=(1-(p+l))/2} (r - (1 - p)/2)^j \text{Str}^{\text{even}} \left( q \{ \tilde{B}, q \}^{(k_1)} \ldots \{ \tilde{B}, q \}^{(k_m)} \right) ^{[l]} \times (1 + \tilde{F})^{-((p/2+r+|k|)+(m-1)/2)} \right) ^{[l]},
\]

(5.10)

Recall that the \( \sigma_{h,j} \)s are the symmetric functions of the half-integers \( 1/2, 3/2, \ldots, h-1/2 \). Now we start from the formula of Proposition 5.3 and take residues at \( r = (1 - (p + l))/2 \) of both sides and multiply \( A_l \),

(5.11)

\[
\text{Ch}(sf(D, u^* Du))^{[l]} = A_l \sum_{m=1, \text{odd}}^{2N-1} 2N-1-m \sum_{|k|=0}^{2N-1-m} \mathcal{C}_k (-1)^{|k|+m} r \text{es}_{r=(1-(p+l))/2} \frac{\Gamma((m+1)/2)\Gamma(p/2+r+|k|)+(m-1)/2)}{\Gamma((p/2+r+|k|)+(m-1)/2)},
\]
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\textbf{References}


\begin{align*}
&\times Str_{\text{even}}^{(k_1)}\ldots\{\tilde{B},q\}^{(k_m)}(1+\tilde{F})^{-((p/2+r+|k|+1)/2)}[l]\\
&= A_l \sum_{m=1,\text{odd}}^{2N-1} \sum_{|k|=0}^{2N-1-m} (-1)^{|k|+m} \frac{\Gamma((m+1)/2)\alpha(k)}{2} \sum_{j=0}^{h} \sigma_{h,j} \\
&\times Str_{\text{even}}^{(k_1)}\ldots\{\tilde{B},q\}^{(k_m)}(1+\tilde{F})^{-((|k|+m/2))}[l] \\
&= A_l \sum_{m=1,\text{odd}}^{2N-1} \sum_{|k|=0}^{2N-1-m} (-1)^{|k|+(m+1)/2} \frac{\Gamma((m+1)/2)\alpha(k)}{2} \sum_{j=0}^{h} \sigma_{h,j} \\
&\times Str_{\text{even}}^{(k_1)}\ldots\{u[\tilde{B},u^*]^{(k_1)}\ldots[\tilde{B},u^*]^{(k_m)} - u^*[\tilde{B},u]^{(k_1)}\ldots[\tilde{B},u]^{(k_m)} \\
&\times (1+\tilde{F})^{-((|k|+m/2))}[l].
\end{align*}

To understand this formula in terms of cyclic (co)homology and Chern characters, we show that our higher spectral flow formula is obtained by pairing a cyclic cocycle with the Chern character of a unitary. We note that our resolvent cocycle \(\phi^\ast\) pairs with normalised chains, so that by Lemma 3.1,

\begin{equation}
\phi^\ast(Ch_{\ast}(u)) = -\phi^\ast(Ch_{\ast}(u^*))
\end{equation}

is modulo functions holomorphic for \(Re(r) > (1 - (p + q'))/2 - \delta\), then:

\textbf{Theorem 5.4.} Assume that \((A, H, B)\) is a family spectral triple associated the fibration. For \(m\) odd, define functionals \(\phi_m\) by

\begin{equation}
\phi_m(a_0, \ldots, a_m) := \sum_{l=0}^{\dim B_0} \phi_l(a_0, \ldots, a_m) = \sqrt{2\pi i} \sum_{l=0}^{\dim B_0} 2N-1-m \sum_{|k|=0}^{h} (-1)^{|k|/2} \frac{\Gamma((m+1)/2)\alpha(k)}{2} \sum_{j=0}^{h} \sigma_{h,j} \\
\times \sigma_{h,j} Str_{\text{even}}^{(k_1)}\ldots\{a_0[B,a_1]^{(k_1)}\ldots[B,a_m]^{(k_m)}(1+F)^{-|k|+m/2}},
\end{equation}

where \(h = |k|+(m+1)/2\). Then \(\phi = (\phi_m)\) is a \((b,B)\)-cocycle in the cohomology of \(B_0\) and

\begin{equation}
Ch_{\ast}(sf(D,u^*Du)) = -\frac{1}{\sqrt{2\pi i}} \sum_{|k|=0}^{2N-1-m} \langle \phi_m, Ch_{\ast}(u) \rangle.
\end{equation}


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