REducing Subspaces of a ClasS of MultipliCation Operators

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Abstract. Let $M_{z^N}(N \in \mathbb{Z}_d^+)$ be a bounded multiplication operator on a class of Hilbert spaces with orthogonal basis $\{z^n : n \in \mathbb{Z}_d^+\}$. In this paper, we prove that each reducing subspace of $M_{z^N}$ is the direct sum of some minimal reducing subspaces. For the case that $d = 2$, we find all the minimal reducing subspaces of $M_{z^N}(N = (N_1, N_2), N_1 \neq N_2)$ on weighted Bergman space $A_2^\alpha(\mathbb{D}_2)(\alpha > -1)$ and Hardy space $H^2(\mathbb{D}_2)$, and characterize the structure of $\mathcal{V}^*(z^N)$, the commutant algebra of the von Neumann algebra generated by $M_{z^N}$.

1. Introduction

Let $T$ be a bounded linear operator on a Hilbert space. If $M$ is a closed subspace satisfying $TM \subseteq M$, then $M$ is called an invariant subspace of $T$. In addition, if $M$ also is invariant subspace of $T^*$, then $M$ is called a reducing subspace of $T$. Combining the methods in analysis, algebra and geometry, the reducing subspaces of multiplication operators with Blaschke products are characterized. The details can be found in the book [3] and its references.

On the polydisk, the research begins with some special functions. The reducing subspaces of $M_{z_1^{N_1}z_2^{N_2}}$ are described in [4, 5, 6]. For $p(z_1, z_2) = \alpha z_1^n + \beta z_2^n$ or $z_1^{N_1}z_2^{N_2}$, the reducing subspaces of Toeplitz operator $T_p$ are described in [1, 2, 7]. A reducing subspace $M$ is called minimal if there is no nonzero reducing subspace $N$ such that $N$ is a proper subspace of $M$. For $N_1 \neq N_2$, the results in [6] shows that $M_{z_1^{N_1}z_2^{N_2}}$ has more minimal reducing subspaces on unweighted Bergman space than on the weighted Bergman space in several cases. It is prove that all $L_{n,m} = \text{span}\{z_1^{n+1}z_2^{m+l} : l \in \mathbb{Z}_d^+\}$ are the only minimal reducing subspaces of $M_{z_1^{N_1}z_2^{N_2}}$ on $A_2^\alpha(\mathbb{D}_2)$ with $\alpha > -1$ and $\alpha \neq 0$. While on the unweighted Bergman space $A^2(\mathbb{D}_2)$, $L_{n,m}^* = \text{span}\{a z_1^{n+hN_1}z_2^{m+hN_2} + b z_1^{n+(m+hN_2)}z_2^{n+hN_1} : h = 0, 1, 2, \ldots\} (a, b \in \mathbb{C})$ are all the minimal reducing

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subspaces of \( M_{z_1, z_2} \), if \( \rho_1(m) = \frac{(m+1)N_1}{N_2} - 1 \) and \( \rho_2(n) = \frac{(n+1)N_2}{N_1} - 1 \) are nonnegative integers.

Denote by \( \mathbb{Z}_+ \) and \( \mathbb{N} \) the set of all the nonnegative integers and all the positive integers, respectively. For \( d \in \mathbb{N} \), write \( m = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}_+^d \), \( m! = m_1!m_2! \cdots m_d! \) and \( |m| = m_1 + m_2 + \cdots + m_d \).

Let \( \mathcal{H} \) be a Hilbert space with the orthogonal basis \( \{z^m\}_{m \in \mathbb{Z}_+^d} \), and satisfy that the multiplication operator \( M_g \) is bounded for each polynomial \( g \). This kind of space contains a lot of classical spaces, such as weighted Bergman spaces over polydisk \( A_d^2(\mathbb{D}) \), weighted Bergman spaces over unit ball \( A_d^2(\mathbb{B}_d) \), Hardy space over unit ball \( H^2(\mathbb{B}_d) \), and so on. Recall that \( \mathbb{B}_d = \{z = (z_1, z_2, \ldots, z_d) : \sum_{i=1}^d |z_i|^2 < 1\} \) and \( S = \{z = (z_1, z_2, \ldots, z_d) : \sum_{i=1}^d |z_i|^2 = 1\} \). Denote by \( d\sigma \) the Haar measure on \( S \), and by \( H(\mathbb{B}_d) \) all the analytic functions on \( \mathbb{B}_d \). The Hardy space \( H^2(\mathbb{B}_d) \) is defined by

\[
H^2(\mathbb{B}_d) = \{f \in H(\mathbb{B}_d) : \lim_{r \to 1^-} \int_S |f(rz)|^2 d\sigma < +\infty\}.
\]

Let \( dA(z) \) denote the normalized area measure over \( \mathbb{B}_d \), and let \( dA_0(z) = C_n (1 - |z|^2)^n dA(z) \), where \( C_n \) is a constant such that \( dA_n \) is normalized. The weighted Bergman space \( A_d^2(\mathbb{B}_d) \) is the Hilbert space of all holomorphic functions over \( \mathbb{B}_d \), which are square integrable with respect to \( dA_n(z) \).

Guo and Huang [3] point that \( \mathcal{M} \) is a nonzero reducing subspace for \( M_{z^N} = M_{z_1, z_2, z_3, \ldots, z_n} \) on the Hilbert space \( \mathcal{H} \) if and only if

\[
\mathcal{M} = \bigoplus_n [\mathcal{M}_n],
\]

where \( [\mathcal{M}_n] \) is the closure of the linear span of \( \{z^{kN} \mathcal{M}_n\} (k \geq 0) \) and \( \mathcal{M}_n \) is a closed linear subspace of \( E_n = \text{span}\{z^m : M_{z_1}^h M_{z_2}^h z^m = M_{z_1}^h M_{z_2}^h z^n, \forall h \in \mathbb{Z}_+\} \), where \( n = \{m = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}_+^d : 0 \leq m_i < N_i \text{ for some } i\} \).

In this paper, we continue to consider the reducing subspaces of \( M_{z^N} \) on \( \mathcal{H} \), and prove that every \( [\mathcal{M}_n] \) is the direct sum of some minimal reducing subspaces. In particular, on \( A_d^2(\mathbb{B}_2) \) and \( H^2(\mathbb{B}_2) \), we describe all the minimal reducing subspaces of \( M_{z_1, z_2} \) with \( N_1 \neq N_2 \), and characterize the commutant algebra \( \mathcal{V}^*(z_1^{N_1} z_2^{N_2}) \).

2. The results in general Hilbert space

Let \( \mathcal{H} \) be the Hilbert space defined in above section, and \( \Omega = \{n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}_+^d : 0 \leq n_i < N_i \text{ for some } i\} \).

Define an equivalence on \( \Omega \) by

\[
q \sim n \iff \frac{\gamma_q + hN}{\gamma_q} = \frac{\gamma_n + hN}{\gamma_n}, \forall h \geq 1,
\]

where \( \gamma_m = \|z^m\|^2 \). For \( n \in \Omega \), set \( \mathfrak{S}_n := \{q \in \Omega : q \sim n\} \) and \( \mathcal{H}_n := \text{span}\{z^J : J \in \mathfrak{S}_n\} \). Then \( \cup_{n \in F} \mathfrak{S}_n = \Omega \) and \( \oplus_{n \in F} \mathcal{H}_n = \text{span}\{z^J : J \in \Omega\} \).
where $F$ is the partition of $\Omega$ by the equivalence $\sim$. Let $P_m$ be the orthogonal projection from $H$ onto $H_m$. Denote by $M$ the multiplication operator $M_z$. It is easy to check that
\[
M^*(z^{m+hN}) = \frac{\gamma_{m+hN}}{\gamma_{m+(h-1)N}} z^{m+(h-1)N}
\]
\[
M^h M^h z^m = \frac{\gamma_{m+hN}}{\gamma_m} z^m
\]
for any $m \in \mathbb{N}$ and $h \in \mathbb{N}$. For $n \in \Omega$, denote by $\tilde{P}_n$ the orthogonal projection from $H$ onto $\text{span}\{z^j : \frac{\gamma_{j+hN}}{\gamma_j} \approx \frac{\gamma_{j+bN}}{\gamma_j}, J \in \mathbb{Z}_+, \forall h \in \mathbb{Z}_+\}$. By the spectrum decomposition, we see that $\tilde{P}_n$ is in the von Neumann algebra generated by $M_z$. For every reducing subspace $M$ of $M$, denote by $P_M$ the orthogonal projection from $H$ onto $M$. Therefore, $\tilde{P}_n P_M = P_M \tilde{P}_n$. Since
\[
(P_M z^m, z^j) = (P_M z^m, M z^{j-N}) = (P_M M^* z^m, z^{j-N}) = 0
\]
for $l \notin \Omega$ and $m \in \Omega$, we have $P_M z^m \in \text{span}\{z^j : J \in \Omega\}$ and $P_M z^j \perp \{z^j : J \in \Omega\}$. Therefore, $P_M P_M = P_M P_M$.

In the following, we prove that each nonzero reducing subspace for $M_z$ always contains a minimal reducing subspace, and every reducing subspace is the direct sum of several minimal reducing subspaces.

**Theorem 2.1.** Suppose $M$ be a nonzero reducing subspace of $M$ on $H$. Then
\[
M = \bigoplus_{n \in F}[M_n] = \bigoplus_{n \in F} \bigoplus_{j=1}^{q_n} [e_{nj}],
\]
where $\{e_{nj}\}_{j=1}^{q_n}$ (1 $\leq q_n \leq +\infty$) is the orthogonal basis of $M_n \neq \{0\}$.

**Proof.** (1) Choose a nonzero function $g$ in $M$. Let $h_0$ be the minimal nonnegative integer such that
\[
P_{\Omega} M^* h_0 g \neq 0,
\]
where $P_{\Omega}$ is the orthogonal projection from $H$ onto $\text{span}\{z^j : J \in \Omega\}$. Clearly, there exists $n \in \Omega$, such that $f = P_n P_{\Omega} M^* h_0 g \neq 0$. In this case, $f = P_n P_M M^* h_0 g = P_M P_n M^* h_0 g = \sum_{J \in \Omega_b} h_j z^j$. Then $f \in M \cap H_n$. By $f \in H_n$, we obtain that
\[
M^q (f z^{hN}) = \begin{cases} 
\gamma_{n+hN} z_j^{(h-q)N} & \text{if } h \geq q \geq 0 \\
0 & \text{if } q > h \geq 0.
\end{cases}
\]
Moreover, $M^q (f z^{hN}) = f z^{(h-q)N}$ for $h, q \geq 0$; $f z^{h_1N} \perp f z^{h_2N}$ with $h_1 \neq h_2$, since
\[
(f z^{h_1N}, f z^{h_2N}) = (M^{h_1} f, M^{h_2} f)
\]
\[
= \begin{cases} 
\gamma_{n+(h_1)N} (f z^{(h_1-h_2)N}, M^* f) & \text{if } h_1 > h_2 \geq 0 \\
\gamma_{n+(h_2)N} (f z^{(h_2-h_1)N}, M^* f) & \text{if } h_2 > h_1 \geq 0
\end{cases}
\]
Thus, we conclude that \([f] = \overline{\text{span}}\{f^h N : h \in \mathbb{Z}_+\} = \bigoplus_{h=0}^{+\infty} \text{span}\{f^h N\} \subset \mathcal{M}\) is a reducing subspace of \(M\). It is easy to see that \([f_i] = [f]\) for each \(f_i \in [f]\). Thus \([f]\) is minimal.

(2) Denote by \(\mathcal{M}_n = P_n M\). Notice that \(P_n M \perp P_m M\) for \(m \notin \mathbb{Z}_n\). If \(P_n M \neq \{0\}\), choose an orthogonal basis \(\{e_{n_j}\}_{j=1}^{\infty} (1 \leq q_n \leq +\infty)\) of \(P_n M\). Notice that \([e_{n_j}] \perp [e_{m_i}]\) for \((n, j) \neq (m, i)\), since

\[
\langle e_{n_j} z^{h_1 N}, e_{m_i} z^{h_2 N} \rangle = \langle M^{h_1} e_{n_j}, M^{h_2} e_{m_i} \rangle = \begin{cases} 
\frac{\gamma_{n+h_1 N}}{\gamma_n} \langle e_{n_j} z^{(h_1-h_2-1) N}, M^{*} e_{m_i} \rangle, & \text{if } h_1 > h_2 \geq 0 \\
\frac{\gamma_{m+h_2 N}}{\gamma_m} \langle M^{*} e_{n_j}, e_{m_i} z^{(h_2-h_1-1) N} \rangle, & \text{if } h_2 > h_1 \geq 0 \\
\frac{\gamma_{m+h_1 N} + \gamma_{n+h_2 N}}{\gamma_m}, & \text{if } h_2 = h_1 = h \geq 0
\end{cases}
\]

\[= 0.
\]

By the result in (1), we know that \([e_{n_j}] = \bigoplus_{h=0}^{+\infty} \mathbb{C} e_{n_j} z^h N\) is a minimal reducing subspace of \(M\). Thus \([P_n M] = \bigoplus_{h=0}^{+\infty} z^h N P_n M = \bigoplus_{h=0}^{+\infty} \bigoplus_{j=1}^{q_n} \mathbb{C} e_{n_j} z^h N = \bigoplus_{j=1}^{q_n} [e_{n_j}]\). So we finish the proof. □

Put \(V^*(z^N)\) the commutant algebra of the von Neumann algebra generated by \(M_2^N\). Then \(V^*(z^N)\) is a von Neumann algebra and is the norm closed linear span of its projections. Recall that two reducing subspaces \(M_1\) and \(M_2\) of \(M_2^N\) are called unitarily equivalent if there exists a unitary operator \(U\) from \(M_1\) onto \(M_2\) and \(U\) commutes with \(M_2^N\). One can show that \(M_1\) is unitarily equivalent to \(M_2\) if and only if \(P_{M_1}\) and \(P_{M_2}\) are equivalent in \(V^*(z^N)\), that is, there is a partial isometry \(V\) in \(V^*(z^N)\) such that

\[V^* V = P_{M_1}, \ V V^* = P_{M_2}.
\]

**Proposition 2.2.** Let \(n, m \in \Omega\) and \(e_{n_j}, e_{m_i}\) be defined as in Theorem 2.1. Then the following statements hold.

(i) \(L_n = [z^n]\) and \(L_m = [z^m]\) are unitarily equivalent if and only if \(n \sim m\);

(ii) \([e_{n_j}]\) and \([e_{m_i}]\) are unitarily equivalent if and only if \(n \sim m\).

**Proof.** (i) On the one hand, assume that \(L_n\) and \(L_m\) are unitarily equivalent, then there is a partial isometry \(U \in V^*(z^N)\) such that \(U|_{L_n}\) is a unitary operator from \(L_n\) onto \(L_m\). Obviously, \(UM^* M(z^{n+h} N) = M^* MU(z^{n+h} N)\). It follows that

\[
\frac{\gamma_{n+h+1} N}{\gamma_n} U(z^{n+h} N) = \frac{\gamma_{m+h+1} N}{\gamma_m} U(z^{n+h} N).
\]

Since \(U(z^{n+h} N) \neq 0\), we have \(\frac{\gamma_{n+h+1} N}{\gamma_n} = \frac{\gamma_{m+h+1} N}{\gamma_m}\) for \(h \geq 0\), i.e., \(n \sim m\).
On the other hand, if \( n \sim m \), let

\[
U(z^J) = \begin{cases} 
\sqrt{\frac{2\alpha}{\gamma_m}} z^{m+hN}, & \text{if } J = n + hN \\
0, & \text{if } J \neq n + hN
\end{cases}
\]

for \( h = 0, 1, 2, \ldots \). Then \( U \) is a partial isometry on \( \mathcal{H} \) and \( U|_{L_n} \) is a unitary operator from \( L_n \) onto \( L_m \). It is easy to check that \( U \in \mathcal{V}^*(z^N) \) by direct calculation.

(ii) Let \( P_{n_j} \) be the orthogonal projection from \( \mathcal{H} \) onto \( \{e_{n_j}\} \). Obviously, there is \( n_0 \sim n \) such that \( \langle e_{n_j}, z^{n_0} \rangle \neq 0 \), that is, \( P_{n_0}P_{n_j} \neq 0 \). Notice that \( P_{n_j} \) and \( P_{n_0} \) are all minimal projection in \( \mathcal{V}^*(z^N) \). As in [7], we have \( P_{n_j} \) is unitarily equivalent to \( P_{n_0} \). Similarly, there is \( m_0 \sim m \) such that \( P_{m_0} \) is unitarily equivalent to \( P_{m_0} \). Therefore, \( \{e_{n_j}\} \) is unitarily equivalent to \( \{e_{m_i}\} \) if and only if \( L_{n_0} \) is unitarily equivalent to \( L_{m_0} \). By (i), we get the desired result. \( \square \)

3. The results on \( A^2_\alpha(\mathbb{B}_2) \) and \( H^2(\mathbb{B}_2) \)

In this section, we consider the reducing subspaces of \( M_{e_{n_1}, e_{n_2}} \) with \( N_1, N_2 \geq 1 \) and \( N_1 \neq N_2 \) on the weighted Bergman space \( A^2_\alpha(\mathbb{B}_2) \) \((\alpha > -1)\) and the Hardy space \( H^2(\mathbb{B}_2) \). Let \( n \in \mathbb{Z}_+^2 \). Denote by \( (n + hN)! = \prod_{i=1}^{2} (n_i + hN_i)! \) and \( |n + hN| = \sum_{i=1}^{2} (n_i + hN_i) \). On \( A^2_\alpha(\mathbb{B}_2) \), we have

\[
\gamma_{n+hN} = \|z^{n+hN}\|_\alpha^2 = \Gamma(\alpha + 3)(n + hN)!/\Gamma(\alpha + 3 + |n + hN|)
\]

for \( \alpha > -1 \). Obviously, \( \{z^m/\sqrt{\gamma_m}\}_{m \in \mathbb{Z}_+^2} \) is an orthogonal basis of \( A^2_\alpha(\mathbb{B}_2) \).

Notice that on the Hardy space \( H^2(\mathbb{B}_2) \), \( \gamma_{n+hN} = \|z^{n+hN}\|^2 = (n + hN)!/(1 + |n + hN|)! = \Gamma(\alpha + 3)(n + hN)!/\Gamma(\alpha + 3 + |n + hN|) \) with \( \alpha = -1 \).

By Proposition 2.2, we know that the unitarily equivalent of reducing subspaces is converted to the equivalence of some numbers. So the relevant research on Bergman space \( A^2_\alpha(\mathbb{B}_2) \) and that on the Hardy space \( H^2(\mathbb{B}_2) \) are similar. In the following, define

\[
\gamma_{n+hN} = \Gamma(\alpha + 3)(n + hN)!/\Gamma(\alpha + 3 + |n + hN|)
\]

for \( \alpha \geq -1 \) and \( n \in \mathbb{Z}_+^2 \).

As in above section, define

\[
\Omega = \{(n_1, n_2) \in \mathbb{Z}_+^2 : 0 \leq n_i < N_i \text{ for some } i\},
\]

and

\[
q \sim n \Leftrightarrow \frac{\gamma_{q+hN}}{\gamma_q} = \frac{\gamma_{n+hN}}{\gamma_n}, \forall h \geq 1
\]

for \( q, n \in \Omega \). Since

\[
\lim_{h \to \infty} \frac{\gamma_{q+hN}}{\gamma_{n+hN}} = \lim_{h \to \infty} \frac{(q + hN)!\Gamma(\alpha + 3 + |n + hN|)}{(n + hN)!\Gamma(\alpha + 3 + |q + hN|)} = 1,
\]

and

\[
\lim_{h \to \infty} \frac{\gamma_{q+hN}}{\gamma_{n+hN}} = \lim_{h \to \infty} \frac{(q + hN)!\Gamma(\alpha + 3 + |n + hN|)}{(n + hN)!\Gamma(\alpha + 3 + |q + hN|)} = 1,
\]

for \( q, n \in \Omega \).
\[ q \sim n \text{ if and only if } \gamma_{q+hN} = \gamma_{n+hN}, \forall h \in \mathbb{Z}_+. \]

If \( m \in \mathbb{N}_n \), then
\[
\frac{\gamma_{m+hN}}{\gamma_{n+(h+1)N}} = \frac{\gamma_{m+hN}}{\gamma_{m+(h+1)N}}, \forall h \in \mathbb{Z}_+. \]

Since \( \Gamma(x + 1) = x\Gamma(x) \) for \( x > 0 \), we get
\[
\prod_{j=1}^{N_1+N_2} \left( \alpha + 2 + n_1 + hN_1 + n_2 + hN_2 + j \right)
\]
\[
\prod_{i=1}^2 \prod_{j=1}^{N_i} (m_i + hN_i + j)
\]
\[
\prod_{i=1}^2 \prod_{j=1}^{N_i} (m_i + hN_i + j)
\]

Let \( g(\lambda) = \prod_{j=1}^{N_1+N_2} \left( \alpha + 2 + n_1 + n_2 + \lambda(N_1 + N_2) + j \right) \prod_{i=1}^2 \prod_{j=1}^{N_i} (m_i + \lambda N_i + j) - \prod_{j=1}^{N_1+N_2} \left( \alpha + 2 + m_1 + m_2 + \lambda(N_1 + N_2) + j \right) \prod_{i=1}^2 \prod_{j=1}^{N_i} (n_i + \lambda N_i + j) \). Obviously, \( g \) is a polynomial over \( \mathbb{C} \) and \( g(h) = 0 \) for any \( h \in \mathbb{Z}_+ \). By fundamental theorem of algebra, \( g(\lambda) \equiv 0 \) for all \( \lambda \in \mathbb{C} \). Set
\[
E_1 = \{ \frac{m_1+j}{N_1} : j = 1, 2, \ldots, N_1 \}; \quad E_2 = \{ \frac{n_2+j}{N_2} : j = 1, 2, \ldots, N_2 \};
\]
\[
E_3 = \{ \frac{2+m_1+m_2+j}{N_1+N_2} : j = 1, 2, \ldots, N_1 + N_2 \};
\]
\[
F_1 = \{ \frac{m_1+j}{N_1} : j = 1, 2, \ldots, N_1 \}; \quad F_2 = \{ \frac{n_2+j}{N_2} : j = 1, 2, \ldots, N_2 \};
\]
\[
F_3 = \{ \frac{2+m_1+m_2+j}{N_1+N_2} : j = 1, 2, \ldots, N_1 + N_2 \}.
\]

Therefore,
\[
\bigcup_{i=1}^3 E_i = \bigcup_{i=1}^3 F_i.
\]

Denote by \( \delta = \text{GCD}(N_1, N_2) \), then \( N_i = \delta q_i \) for \( i = 1, 2 \) and \( \text{GCD}(q_1, q_2) = 1 \).

**Lemma 3.1.** Let \( \alpha \geq -1, n, m \in \Omega \) such that \( n \sim m \) and \( n \neq m \). Then
\[
n_1 + n_2 = m_1 + m_2 \text{ or } n_1 + n_2 = m_1 + m_2 \pm 1.
\]

**Proof.** Without lose of generality, assume \( n_1 + n_2 > m_1 + m_2 + 1 \) and \( n_1 > m_1 \).

Denote by \( \tilde{E}_i = \tilde{E}_1 \cup \tilde{F}_i \) and \( \tilde{F}_i = \tilde{F}_1 \setminus \tilde{E}_i \) for \( i = 1, 2, 3 \). Then \( \tilde{E}_i \cap \tilde{F}_i = \emptyset \) and
\[
\tilde{E}_1 \cup \tilde{E}_2 \cup \tilde{E}_3 = \tilde{F}_1 \cup \tilde{F}_2 \cup \tilde{F}_3.
\]

Clearly,
\[
\frac{2+\alpha+n_1+n_2}{N_1+N_2} + 1, \quad \frac{1+\alpha+n_1+n_2}{N_1+N_2} + 1 \in \tilde{E}_1 \cap \tilde{E}_2 \quad \text{and} \quad \frac{4+\alpha+m_1+m_2}{N_1+N_2}, \quad \frac{4+\alpha+m_1+m_2}{N_1+N_2} \in \tilde{F}_1 \cap \tilde{F}_2.
\]

Furthermore, for \( i, j \in \{1, 2\} \) we claim that
\[
\begin{align*}
(a) & \text{ if } \frac{2+\alpha+n_1+n_2}{N_1+N_2} + 1 \in \tilde{E}_i \text{, then } \frac{1+\alpha+n_1+n_2}{N_1+N_2} + 1 \in \tilde{E}_j \text{ for } j \neq i. \\
(b) & \text{ if } \frac{4+\alpha+m_1+m_2}{N_1+N_2} \in \tilde{F}_i \text{, then } \frac{4+\alpha+m_1+m_2}{N_1+N_2} \in \tilde{F}_j \text{ for } j \neq i.
\end{align*}
\]
In fact, if \( \frac{2+\alpha+n_1+n_2}{N_1+N_2} + 1, \frac{1+\alpha+n_1+n_2}{N_1+N_2} + 1 \in \tilde{E}_i \) for some \( i \in \{1, 2\} \), there are integers \( 1 \leq p_i, q_i \leq N_i \) such that
\[
\frac{2 + \alpha + n_1 + n_2}{N_1 + N_2} + 1 = \frac{n_i + p_i}{N_i},
\]
\[
\frac{1 + \alpha + n_1 + n_2}{N_1 + N_2} + 1 = \frac{n_i + q_i}{N_i}.
\]
Then \( 0 \neq \frac{1}{N_1+N_2} = \frac{p_i-q_i}{N_i} > \frac{p_i-q_i}{N_1+N_2} \geq \frac{1}{N_1+N_2} \), which is a contradiction. So \( (a) \) holds. Since the proof of \( (a) \) and \( (b) \) are similar, we omit the details of \( (b) \).

Next, we find the contradictions for three cases respectively.

(1) If \( m_2 > n_2 \), then \( \min \tilde{F}_2 > \max \tilde{E}_2 \). Since one of \( \frac{2+\alpha+n_1+n_2}{N_1+N_2} + 1 \) and \( \frac{1+\alpha+n_1+n_2}{N_1+N_2} + 1 \) is in \( \tilde{E}_2 \), \( \lambda > \max \tilde{E}_2 \geq \frac{1+\alpha+n_1+n_2}{N_1+N_2} + 1 > \max \tilde{F}_3 \) for \( \lambda \in \tilde{F}_2 \). It means that \( \tilde{F}_3 \cap \tilde{F}_2 = \emptyset \), which is contradict with \( (b) \).

(2) If \( m_2 = n_2 \), then \( E_2 = \tilde{F}_2 \). Equality \( (3) \) implies that \( \tilde{F}_3 = \tilde{F}_1 \), which is also contradict with \( (b) \).

(3) If \( m_2 < n_2 \), we consider the maximum of equality \( (3) \), we have
\[
\frac{2 + \alpha + n_1 + n_2}{N_1 + N_2} + 1 = \max \left\{ \frac{n_1}{N_1} + 1, \frac{n_2}{N_2} + 1 \right\}.
\]
If \( \frac{2+\alpha+n_1+n_2}{N_1+N_2} + 1 = \frac{n_1}{N_1} + 1 \in \tilde{E}_1 \), then \( \frac{2+\alpha+n_1+n_2}{N_1+N_2} = \frac{n_1}{N_1} \). Since \( \frac{1+\alpha+n_1+n_2}{N_1+N_2} + 1 \) \( \notin \tilde{E}_2 \), we have \( \frac{1+\alpha+n_1+n_2}{N_1+N_2} + 1 \notin \tilde{F}_3 \), which contradicts \( (a) \).

If \( \frac{2+\alpha+n_1+n_2}{N_1+N_2} + 1 = \frac{n_2}{N_2} + 1 \in \tilde{E}_2 \), by the symmetry of \( n_1 \) and \( n_2 \), we get \( \frac{1+\alpha+n_1+n_2}{N_1+N_2} + 1 \notin \tilde{E}_1 \), which also contradicts \( (a) \). So we finish the proof. \( \square \)

**Lemma 3.2.** Let \( \alpha \geq -1, n, m \in \Omega \) and \( n \neq m \). Suppose \( n_1 + n_2 = m_1 + m_2 \), then \( n \sim m \) if and only if \( n \in \Delta_1 \cup \Delta_1, \) \( \) where \( \Delta_1 = \{(kq_1, kq_2 - 1): 1 \leq k \leq \delta, k \in \mathbb{N}\} \) and \( \Delta_1 = \{(kq_1 - 1, kq_2): 1 \leq k \leq \delta, k \in \mathbb{N}\} \).

**Proof.** The sufficiency is easy to check, we only show the proof of necessity. If \( n_1 + n_2 = m_1 + m_2 \), then \( E_3 = \tilde{F}_3 \) and \( E_1 \cup E_2 = F_1 \cup F_2 \). Since \( n \neq m \), we have \( n_1 \neq m_1 \). Without lose of generality, let \( n_1 > m_1 \), then \( n_2 < m_2 \). Eq. \( E_1 \cup E_2 = F_1 \cup F_2 \) shows that
\[
\max \left\{ \frac{n_1}{N_1}, \frac{n_2}{N_2} \right\} = \max \left\{ \frac{m_1}{N_1}, \frac{m_2}{N_2} \right\};
\]
\[
\min \left\{ \frac{n_1+1}{N_1}, \frac{n_2+1}{N_2} \right\} = \min \left\{ \frac{m_1+1}{N_1}, \frac{m_2+1}{N_2} \right\}.
\]
Thus
\[
\begin{align*}
\frac{n_1}{N_1} &= \frac{m_1}{N_1} \quad \frac{n_2}{N_2} = \frac{m_2}{N_2}, \\
\frac{n_1+1}{N_1} &= \frac{m_1+1}{N_1} \quad \frac{n_2+1}{N_2} = \frac{m_2+1}{N_2}.
\end{align*}
\]
It follows that \( \frac{m_1-n_1+1}{N_1} = \frac{m_2-n_2+1}{N_2} \). Since \( n_1 - n_1 = n_2 - m_2 \) and \( N_1 \neq N_2 \), we get \( m_1 - n_1 + 1 = n_2 - m_2 + 1 = 0 \). Further, \( \frac{n_1}{N_1} = \frac{m_2}{N_2} \) implies that \( m_2 = n_2 + 1 = \frac{2}{q_1}n_1 \). Thus there exists \( k \) such that \( n_1 = kq_1 \). Then \( m_2 = kq_2 \).
If \( n_1 + n_2 = m_1 + m_2 \pm 1 \), there are three cases: (i) \( n_1 = m_1 \); (ii) \( n_2 = m_2 \); (iii) \( n_1 \neq m_1 \) and \( n_2 \neq m_2 \). We give the characterization of \( n \) and \( m \), respectively.

**Lemma 3.3.** Let \( \alpha \geq -1, n \in \Omega \). There is \( m \in \Omega \) such that \( m \sim n \) and \( m \neq n \). Then the following statements hold.

(i) If \( n_1 = m_1 \), then \( \alpha \in \mathbb{Q} \) and there is an integer \( 0 \leq i_0 < q_1 \) such that \( \frac{2 + \alpha + i_0}{q_1} q_2 \in \mathbb{Z}_+ \). In this case, \((n, m) \in \Delta_2 \cup \Delta_3 \), where \( \Delta_2 = \{(kq_1 - 2 - \alpha, kq_2) : k = \frac{2 + \alpha + i_0}{q_1} + i, 0 \leq i \leq \delta - 1 \} \) and \( \Delta_3 = \{(kq_1 - 2, kq_2 - 1) : k = \frac{2 + \alpha + i_0}{q_1} + i, 0 \leq i \leq \delta - 1 \} \).

(ii) If \( n_2 = m_2 \), then \( \alpha \in \mathbb{Q} \) and there is an integer \( 0 \leq j_0 < q_2 \) such that \( \frac{2 + \alpha + j_0}{q_2} q_1 \in \mathbb{Z}_+ \). In this case, \((n, m) \in \Delta_3 \cup \Delta_5 \), where \( \Delta_3 = \{(kq_1, kq_2 - 2 - \alpha) : k = \frac{2 + \alpha + j_0}{q_2} + j, 0 \leq j \leq \delta - 1 \} \) and \( \Delta_5 = \{(kq_1 - 1, kq_2 - 2 - \alpha) : k = \frac{2 + \alpha + j_0}{q_2} + j, 0 \leq j \leq \delta - 1 \} \).

(iii) If \( n_1 \neq m_1 \) and \( n_2 \neq m_2 \), then \( \alpha \in \mathbb{N} \) and \( q_1, q_2 \in \{1 + \alpha, 1\} \). Furthermore,

\[(a) \text{ if } q_1 = 1, \text{ then } q_2 = 1 + \alpha \text{ and } (n, m) \in \Delta_4 \cup \Delta_6, \text{ where } \Delta_4 = \{(kq_1, kq_2 - 2 - \alpha) : 2 \leq k \leq \delta + 1, k \in \mathbb{N} \} \text{ and } \Delta_6 = \{(kq_1 - 2, kq_2 - 1 - \alpha) : 2 \leq k \leq \delta + 1, k \in \mathbb{N} \}.

(b) if \( q_2 = 1 \), then \( q_1 = 1 + \alpha \) and \( (n, m) \in \Delta_5 \cup \Delta_7, \text{ where } \Delta_5 = \{(kq_1 - 2 - \alpha, kq_2) : 2 \leq k \leq \delta + 1, k \in \mathbb{N} \} \text{ and } \Delta_7 = \{(kq_1 - 1 - \alpha, kq_2 - 2) : 2 \leq k \leq \delta + 1, k \in \mathbb{N} \}.

**Proof.** By Lemma 3.1, we assume \( n_1 + n_2 = m_1 + m_2 + 1 \), or else exchanging \((n_1, n_2)\) and \((m_1, m_2)\). Therefore,

\begin{equation}
\tilde{E}_1 \sqcup \tilde{E}_2 \sqcup \left\{ \frac{2 + \alpha + n_1 + n_2}{N_1 + N_2} \right\} = \tilde{F}_1 \sqcup \tilde{F}_2 \sqcup \left\{ \frac{2 + \alpha + m_1 + m_2}{N_1 + N_2} + 1 \right\}.
\end{equation}

(i) By \( n_1 = m_1 \), we have \( n_2 = m_2 + 1 \). Eq. (2) implies that

\[\frac{2 + \alpha + n_1 + n_2}{N_1 + N_2} = n_2 \frac{N_2}{N_1} \]

that is \( n_2 = \frac{2 + \alpha + n_1}{N_1} \). So there exists \( k \geq 0 \) such that \( n_2 = kq_2, n_1 = kq_1 - 2 - \alpha \).

It follows that \( n = (kq_1 - 2 - \alpha, kq_2) \) and \( m = n + (0, -1) = (kq_1 - 2 - \alpha, kq_2 - 1) \in \Omega \). By \( n, m \in \Omega \), we have \( kq_2 = \frac{2 + \alpha + h}{q_1} q_2 \in \mathbb{N} \) for some nonnegative integer \( h = i_0 + i q_1 (0 \leq i_0 < q_1, 0 \leq i \leq \delta - 1) \). That is, \( k = \frac{2 + \alpha + h}{q_1} = \frac{2 + \alpha + i_0}{q_1} + i \).

Since \( 0 \leq i_0 < q_1 \), the choice of \( i_0 \) is unique. So we finish the proof of necessity. The sufficiency is easy to check. So (i) holds.

(ii) By the symmetry of \( n_1 \) and \( n_2 \), we have the statement (ii) holds.
(iii) First, if \( n_1 > m_1 \), \( n_2 \neq m_2 \) implies that \( n_2 + 1 \leq m_2 \) and \( n_1 \geq m_1 + 2 \).

Considering the maximum and minimum of Eq. (4), it is easy to see

\[
\begin{align*}
1 + \frac{n_1}{N_1} &= \max\{1 + \frac{2 + \alpha + n_1 + n_2}{N_1 + N_2}, 1 + \frac{m_2}{N_2}\}, \\
m_1 + 1 + \frac{1}{N_1} &= \min\{\frac{2 + \alpha + n_1 + n_2}{N_1 + N_2}, n_2 + 1 + \frac{1}{N_2}\}.
\end{align*}
\]

We claim that

\[
1 + \frac{n_1}{N_1} = \frac{2 + \alpha + n_1 + n_2}{N_1 + N_2} + 1.
\]

Or else, assume \( \frac{n_1}{N_1} + 1 = \frac{m_2}{N_2} + 1 > \frac{2 + \alpha + n_1 + n_2}{N_1 + N_2} + 1 \). Clearly

\[
\frac{2 + \alpha + m_2}{N_2} > \frac{m_2}{N_2} = \frac{n_1}{N_1} \geq \frac{n_1 + 1}{N_1}.
\]

Therefore,

\[
\frac{m_1 + 1}{N_1} < \frac{3 + \alpha + m_1 + m_2}{N_1 + N_2} = \frac{2 + \alpha + n_1 + n_2}{N_1 + N_2}.
\]

So (5) implies that

\[
\frac{n_2 + 1}{N_2} = \frac{m_1 + 1}{N_1}.
\]

Since \( n_1 + n_2 = m_1 + m_2 + 1 \), we get \( \frac{n_2 - n_1 - 1}{N_2} = \frac{n_1 - m_1 - 1}{N_1} = \frac{m_2 - n_2}{N_2} \). Let \( m_2 - n_2 = pq_1 \), then \( m_2 - n_2 - 1 = pq_2 \). That is, \( p \in \mathbb{N} \) and \( 1 = p(q_1 - q_2) \). Therefore, \( p = 1, q_1 = q_2 + 1 \), forcing \( N_1 \geq 2 \) and \( N_1 > N_2 \). Then \( 1 + \frac{n_2 - n_1}{N_1} > 1 + \frac{m_1 - 1}{N_1} \). The Eq. (4) shows that

\[
1 + \frac{n_1}{N_1} = \frac{2 + \alpha + n_1 + n_2}{N_1 + N_2}.
\]

If \( \frac{2 + \alpha + n_1 + n_2}{N_1 + N_2} \in \tilde{F}_2 \), then \( \text{Card} \tilde{F}_2 \geq \text{Card} \tilde{E}_2 \geq 2 \). By \( N_1 > N_2 \), we have \( \frac{n_2 + 1}{N_2} > \frac{m_1 + 2}{N_1} \). The equalities (7) and (8) show that \( \frac{m_1 + 2}{N_1} \notin \tilde{E}_1 \sqcup \tilde{E}_2 \sqcup \{\frac{2 + \alpha + n_1 + n_2}{N_1 + N_2}\} \), which is a contradiction.

If \( \frac{2 + \alpha + n_1 + n_2}{N_1 + N_2} \in \tilde{F}_1 \), then

\[
\frac{2 + \alpha + n_1 + n_2}{N_1 + N_2} = \frac{m_1 + 2}{N_1}.
\]

In fact, equality (8) implies that

\[
\{z \in \tilde{F}_1 : z < \frac{2 + \alpha + n_1 + n_2}{N_1 + N_2}\} = \{z \in \tilde{E}_2 : z < \frac{2 + \alpha + n_1 + n_2}{N_1 + N_2}\}.
\]

Since \( N_1 \neq N_2 \), we have \( \text{Card} \{z \in \tilde{F}_1 : z < \frac{2 + \alpha + n_1 + n_2}{N_1 + N_2}\} = 1 \). So (9) holds.

Combining (8) and (9), we get \( n_1 = m_1 + 3 \). It means that \( q_1 = m_2 - n_2 = 2 \) and \( q_2 = 1 \). By (7) and (9), we have \( \frac{2 + \alpha}{N_2} = \frac{1}{N_1} \), i.e., \( 2(2 + \alpha) = 1 \), which is contradict with \( \alpha > -1 \). So we get the claim.
By (6), there is
\begin{equation}
\frac{2 + \alpha + n_2}{N_2} = \frac{n_1}{N_1}.
\end{equation}

It follows that
\begin{equation}
\begin{cases}
n_1 = kq_1 \\
n_2 = kq_2 - 2 - \alpha
\end{cases}
\end{equation}
for some $k \geq 2 + \alpha$. Therefore,
\begin{equation}
\frac{2 + \alpha + n_1 + n_2}{N_1 + N_2} = \frac{k}{\delta} > \frac{kq_2 - (1 + \alpha)}{\delta q_2} = \frac{n_2 + 1}{N_2}.
\end{equation}

Then Eq. (5) deduces that
\begin{equation}
m_1 + 1 = \frac{n_2 + 1}{N_2}, \text{ i.e., } m_1 + 1 = kq_1 - \frac{(1 + \alpha)q_1}{q_2}.
\end{equation}

If $N_1 = 1$, then $N_2 > 1$. Since $m_2 - n_2 > 1$, we have $\frac{n_2 + 2}{N_2} \in \tilde{E}_2$, but $\frac{n_2 + 2}{N_2} \notin \tilde{F}_1 \cup \tilde{F}_2 \cup \tilde{E}_3$, which is a contradiction.

If $N_1 > 1$, then
\begin{equation}
\max\left\{\frac{1 + n_1 - 1}{N_1}, \frac{2 + \alpha + n_1 + n_2}{N_1 + N_2}\right\} = 1 + \frac{m_2}{N_2}.
\end{equation}

By Eq. (11), we have
\begin{equation}
\frac{2 + \alpha + n_1 + n_2}{N_1 + N_2} < \frac{n_1 - 1}{N_1} + 1. \text{ Therefore,}
\end{equation}
\begin{equation}
\frac{n_1 - 1}{N_1} = \frac{m_2}{N_2}.
\end{equation}

It follows that $m_2 = kq_2 - \frac{2}{q_1} \in \mathbb{Z}_+$. Combining $n_1 + n_2 = m_1 + m_2 + 1$ with (12), (14) and $N_1 \neq N_2$, we conclude that $\frac{2}{q_1} = 1 + \alpha$. Therefore, $q_2 = 1 + \alpha$, $q_1 = 1$, and $\alpha \in \mathbb{N}$. In this case, $(m_1, m_2) = (n_1 - 2, n_2 + 1)$ and $(n, m) \in \Delta_4$.

Next, if $n_1 < m_1$, we have $n_2 > m_2 + 1$ and $n_1 + 1 < m_1$. Since $n_1$ and $n_2$ are symmetric; $m_1$ and $m_2$ are symmetric, it is easy to check that $q_2 = 1$, $q_1 = 1 + \alpha$, and $(n, m) \in \Delta_5$. So (iii) holds. \qed

Remark 3.4. In above lemma, the number $k$ in condition (i) and (ii) is not always an integer. If $n$ and $m$ satisfy one of the conditions (i), (ii) and (iii), then $n \sim m$ and $n \neq m$.

Notice that $\Delta_1 \neq \emptyset$ and does not change with $\alpha$. However, $\{\Delta_i\} (i = 2, 3, 4, 5)$ heavily depend on the $\alpha$, and some of them may be empty. By careful computation, we know that each two of $\{\Delta_i, \tilde{\Delta}_i : i = 1, \ldots, 5\}$ are either equal or disjoint. Therefore, we assert that the Card of $\mathcal{S}_n$ heavily depend on the $\alpha$.

For the case that $\alpha = -1$, it is easy to see that $\Delta_4 = \Delta_5 = \emptyset$, $\Delta_1 = \Delta_2$ and $\tilde{\Delta}_2 = \tilde{\Delta}_3$. So we have the following result.
Theorem 3.6. ∗ is following results. Recall that $\delta_{N}$.

Theorem 3.7. $V$ if and only if $\text{Card} (\delta_{N}) = 2$ for some $1 \leq k \leq GCD(N_1, N_2)$.

Lemma 3.5. If $\alpha = -1$, then $\exists_n \neq \{n\}$ if and only if

$$3_n = \{(kq_1, kq_2 - 1), (kq_1 - 1, kq_2), (kq_1 - 1, kq_2 - 1)\}$$

for some $1 \leq k \leq GCD(N_1, N_2)$.

For the case that $\alpha > -1$, we have the following statements hold.

1° If $\alpha \in (-1, +\infty) \setminus \mathbb{Q}$, then $\Delta_i = \emptyset$ for $i = 2, 3, 4, 5$. Therefore, $\text{Card} 3_n \neq 1$ if and only if $\text{Card} 3_n = 2$ for $n \in \Delta_1 \cup \Delta_2$.

2° If $\alpha \in (\mathbb{Q} \cap (-1, +\infty)) \setminus \mathbb{Z}_+$, then $\Delta_1 = \Delta_5 = \emptyset$. Therefore, $\text{Card} 3_n \neq 1$ if and only if $\text{Card} 3_n = 2$, and $n \in \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$. Moreover, $\Delta_2$ and $\Delta_3$ are not non-empty sets at the same time. In fact, let $\alpha = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $p > 1$, $q > -p$ and $GCD(p, |q|) = 1$. By $\frac{2+\alpha+i_0}{q_1} = (2+\alpha)p+q \in \mathbb{Z}_+$. Since $GCD(p, |q|) = 1$, we have $GCD((2+i_0)p+q, p) = 1$. So $p \mid q_2$. Similar, $\frac{2+\alpha+i_0}{q_2} = q_1 \in \mathbb{Z}_+$ implies that $p \mid q_2$. Thus we get $p = 1$, which is a contradiction.

3° If $\alpha \in \mathbb{Z}_+$, then $\Delta_2$ and $\Delta_3$ are not empty.

1. If $N_2 \neq (1+\alpha)N_1$ and $N_1 \neq (1+\alpha)N_2$, then $\delta_4 = \delta_5 = \emptyset$. Therefore, $\text{Card} 3_n \neq 1$ if and only if $\text{Card} 3_n = 2$, for $n \in \Delta_1 \cup \Delta_3 \cup \Delta_2 \cup \Delta_4$.

2. If $N_2 = (1+\alpha)N_1$, $\alpha \neq 0$, then $\Delta_5 = \emptyset$, $\Delta_1 = \Delta_3$, $\Delta_1 = \Delta_4$ and $\Delta_3 = \Delta_4$. $\text{Card} 3_n \neq 1$ if and only if $\text{Card} 3_n = 2$ or $\text{Card} 3_n = 3$. Moreover, $\text{Card} 3_n = 2$ if and only if $n \in \Delta_2 \cup \Delta_3$; $\text{Card} 3_n = 3$ if and only if $n \in \Delta_1 \cup \Delta_1 \cup \Delta_3$. In this case, $n \sim n + (-1, 1) \sim n + (1, 0)$ for $n \in \Delta_1$.

3. If $N_1 = (1+\alpha)N_2$, $\alpha \neq 0$, then $\Delta_4 = \emptyset$, $\Delta_1 = \Delta_3$, $\Delta_2 = \Delta_5$ and $\Delta_1 = \Delta_2$. $\text{Card} 3_n \neq 1$ if and only if $\text{Card} 3_n = 2$ or $\text{Card} 3_n = 3$. Moreover, $\text{Card} 3_n = 2$ if and only if $n \in \Delta_3 \cup \Delta_3$; $\text{Card} 3_n = 3$ if and only if $n \in \Delta_1 \cup \Delta_1 \cup \Delta_2$. In this case, $n \sim n + (-1, 1) \sim n + (-1, 2)$ for $n \in \Delta_1$.

Combining above analysis and the results in section two, we have the following results. Recall that $\delta = GCD(N_1, N_2)$.

Theorem 3.6. On the Bergman space $A_{2}^{\alpha}(\mathbb{B}_2)$ with $\alpha \in (-1, +\infty) \setminus \mathbb{Q}$, $V^*(z^N)$ is $\ast$-isomorphic to

$$\bigoplus_{i=1}^{\infty} M_2(\mathbb{C}) \bigoplus_{i=1}^{\infty} \mathbb{C},$$

where $N = (N_1, N_2)$ and $N_1 \neq N_2$.

Theorem 3.7. On the Bergman space $A_{2}^{\alpha}(\mathbb{B}_2)$ with $\alpha \in (\mathbb{Q} \cap (-1, +\infty)) \setminus \mathbb{Z}_+$, $V^*(z^N)$ is $\ast$-isomorphic to

$$\bigoplus_{i=1}^{\infty} M_2(\mathbb{C}) \bigoplus_{i=1}^{\infty} \mathbb{C},$$
where \( s \in \{\delta, 2\delta\} \), where \( N = (N_1, N_2) \) and \( N_1 \neq N_2 \).

**Example 3.8.** Let \( \alpha = \frac{2}{3} \), \( N_1 = 6 \), \( N_2 = 9 \). Then \( \Delta_1 = \{(2, 2), (4, 5), (6, 8)\} \), \( \Delta_2 = \Delta_3 = \Delta_4 = \Delta_5 = \emptyset \). So on the Bergman space \( A^2_3(\mathbb{B}_2) \), \( V^*(z_1^0 z_2^0) \) is \( \ast \)-isomorphic to

\[
\bigoplus_{i=1}^{3} M_2(\mathbb{C}) \bigoplus_{i=1}^{+\infty} \mathbb{C}.
\]

**Example 3.9.** Let \( \alpha = \frac{2}{3} \), \( N_1 = 6 \), \( N_2 = 9 \). It is easy to check that \( (1) \Delta_1 = \{(2, 2), (4, 5), (6, 8)\} \); \( (2) \Delta_3 = \Delta_4 = \Delta_5 = \emptyset \); \( (3) \Delta_2 = \{(0, 4), (2, 7), (4, 10)\} \) with \( k = 1 + \frac{1}{2}, 2 + \frac{1}{2}, 3 + \frac{1}{2} \), respectively. Then on the Bergman space \( A^2_3(\mathbb{B}_2) \), \( V^*(z_1^0 z_2^0) \) is \( \ast \)-isomorphic to

\[
\bigoplus_{i=1}^{6} M_2(\mathbb{C}) \bigoplus_{i=1}^{+\infty} \mathbb{C}.
\]

**Theorem 3.10.** Let \( N = (N_1, N_2) \) and \( N_1 \neq N_2 \). On the Bergman space \( A^2_3(\mathbb{B}_2) \) with \( \alpha \in \mathbb{Z}_+ \), the following statements hold:

(i) if \( N_1 \neq (1 + \alpha)N_2 \) and \( N_2 \neq (1 + \alpha)N_1 \), then \( V^*(z^N) \) is \( \ast \)-isomorphic to

\[
\bigoplus_{i=1}^{3\delta} M_2(\mathbb{C}) \bigoplus_{i=1}^{+\infty} \mathbb{C};
\]

(ii) if \( N_1 = (1 + \alpha)N_2 \) or \( N_2 = (1 + \alpha)N_1 \), then \( V^*(z^N) \) is \( \ast \)-isomorphic to

\[
\bigoplus_{i=1}^{\delta} M_3(\mathbb{C}) \bigoplus_{i=1}^{\delta} M_2(\mathbb{C}) \bigoplus_{i=1}^{+\infty} \mathbb{C}.
\]

**Example 3.11.** If \( \alpha = 4 \), \( N_1 = 6 \), \( N_2 = 9 \), then \( \Delta_1 = \{(2, 2), (4, 5), (6, 8)\} \), \( \Delta_2 = \{(0, 9), (2, 12), (4, 15)\} \), \( \Delta_3 = \{(4, 0), (6, 3), (8, 6)\} \), \( \Delta_4 = \Delta_5 = \emptyset \). On the Bergman space \( A^2_3(\mathbb{B}_2) \), \( V^*(z_1^0 z_2^0) \) is \( \ast \)-isomorphic to

\[
\bigoplus_{i=1}^{9} M_2(\mathbb{C}) \bigoplus_{i=1}^{+\infty} \mathbb{C}.
\]

**Example 3.12.** If \( \alpha = 2 \), \( N_1 = 3 \), \( N_2 = 9 \), then \( \Delta_1 = \tilde{\Delta}_3 = \{(1, 2), (2, 5), (3, 8)\} \), \( \Delta_2 = \{(0, 12), (1, 15), (2, 18)\} \), \( \Delta_3 = \Delta_4 = \{(2, 2), (3, 5), (4, 8)\} \) and \( \Delta_5 = \emptyset \). On the Bergman space \( A^2_3(\mathbb{B}_2) \), \( V^*(z_1^0 z_2^0) \) is \( \ast \)-isomorphic to

\[
\bigoplus_{i=1}^{3} M_3(\mathbb{C}) \bigoplus_{i=1}^{3} M_2(\mathbb{C}) \bigoplus_{i=1}^{+\infty} \mathbb{C}.
\]

**Theorem 3.13.** On the Hardy space \( H^2(\mathbb{B}_2) \), \( V^*(z^N) \) is \( \ast \)-isomorphic to

\[
\bigoplus_{i=1}^{\delta} M_3(\mathbb{C}) \bigoplus_{i=1}^{+\infty} \mathbb{C},
\]
where $N = (N_1, N_2)$ and $N_1 \neq N_2$.

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**References**


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