STABILIZATION OF VISCOELASTIC WAVE EQUATION WITH VARIABLE COEFFICIENTS AND A DELAY TERM IN THE INTERNAL FEEDBACK

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ABSTRACT. In this paper, we consider the stabilization of the viscoelastic wave equation with variable coefficients in a bounded domain with smooth boundary, subject to linear dissipative internal feedback with a delay. Our stabilization result is mainly based on the use of the Riemannian geometry methods and Lyapunov functional techniques.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma$. It is assumed that $\Gamma$ consists of two parts $\Gamma_1$ and $\Gamma_2$ ($\Gamma = \Gamma_1 \cup \Gamma_2$) with $\Gamma_2 \neq \emptyset$, $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$. Let $\nu$ denote the outward normal vector field along the boundary and $\text{div}(X)$ denote the divergence of the vector field $X$ in the Euclidean metric. Let $A(x) = (a_{ij}(x))$ be a matrix function, with $a_{ij} = a_{ji}$ of class $C^1$ satisfying

$$\lambda \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i=1, j=1}^{n} a_{ij}(x)\xi_i \xi_j \leq \Lambda \sum_{i=1}^{n} \xi_i,$$

for positive constants $\lambda$ and $\Lambda$. Define

$$Au = -\text{div}(A(x)\nabla u) \text{ for } u \in H^1(\Omega).$$

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We consider the viscoelastic wave equation with variable coefficients and a delay in the dissipative internal feedback

\[
\begin{cases}
  u_{tt} + \mathcal{A}u - \int_0^t \beta(t-s)\mathcal{A}u(s)\,ds + [\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)] = 0, & x \in \Omega, t > 0, \\
  u(x,t)|_{\gamma_1} = 0, & t > 0, \\
  \frac{\partial u(x,t)}{\partial \nu}|_{\gamma_1} = 0, & t > 0, \\
  u(x,0) = u_0(x), & x \in \Omega, \\
  u_t(x,0) = u_1(x), & x \in \Omega, t \in (0, \tau), \\
  u_t(x,t-\tau) = f_0(x, t-\tau), & x \in \Omega, t \in (0, \tau)
\end{cases}
\] (1.1)

where \( \frac{\partial u(x,t)}{\partial \nu} \) is the co-normal derivative

\[
\frac{\partial u(x,t)}{\partial \nu} = \langle A(x)u, \nu \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the standard metric of the Euclidean space \( \mathbb{R}^n \). Moreover, \( \tau > 0 \) is a time delay, \( \mu_1, \mu_2 \) are real numbers with \( \mu_1 > 0, \mu_2 \neq 0 \), and the initial data \( u_0, u_1, f_0 \) are given functions belonging to suitable space. The purpose of this paper is to study the asymptotic stability for the solution of (1.1) with a delay term appearing in dissipative internal feedback.

Time delays arise in many applications because, in most instances, physical, chemical, biological, thermal, and economic phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [1, 12, 21] and references therein. In many cases, it was shown that delay is a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used, see [7]. The stability issue of systems with delay is, therefore, of theoretical and practical importance.

If \( A(x) = I \) is a constant matrix on \( \Omega \), that is to say the system (1.1) becomes the viscoelastic wave equation which has been considered by many authors during the past decades. For other related works, we refer the readers to [2, 3, 5, 6, 11, 13, 19, 20]. In [10], Kirane and Said-Houari considered the following linear viscoelastic wave equation

\[
\begin{cases}
  u_{tt} - \Delta u + \int_0^t \beta(t-s)\Delta u(s)\,ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & x \in \Omega, t > 0, \\
  u(x,t) = 0, & x \in \partial \Omega, t > 0, \\
  u(x,0) = u_0(x), & x \in \Omega, \\
  u_t(x,0) = u_1(x), & x \in \Omega, \\
  u_t(x,t-\tau) = f_0(x, t-\tau), & x \in \Omega, t \in (0, \tau).
\end{cases}
\] (1.2)

They proved the global existence of (1.2) by using the Faedo-Galerkin approximations together with some energy estimates and obtained the general decay
results of energy via suitable Lyapunov functionals for $\mu_2 \leq \mu_1$. Comparing with wave equation [14, 22], the presence of the viscoelastic damping such that the solution of (1.2) is still asymptotically stable even if $\mu_1 = \mu_2$.

For a general $A(x)$, the main tools here to cope with the system (1.1) are the differential geometrical methods which were introduced by [23] and extended in [4, 9, 15, 18, 24, 25] and many others. For a survey on the differential geometric methods, see [8, 26]. Recently, Z. H. Ning et al. [17] studied the wave equation

\[
\begin{cases}
  u_{tt} + Au + a(x)[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)] = 0, & x \in \Omega, t > 0, \\
  u(x,t) |_{\Gamma_2} = 0, & t > 0, \\
  \frac{\partial u(x,t)}{\partial n} |_{\Gamma_1} = 0, & t > 0, \\
  u(x,0) = u_0(x), & u_t(x,0) = u_1(x), & x \in \Omega, \\
  u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega, t \in (0, \tau)
\end{cases}
\]

and obtained the stabilization result based on the use of the Riemannian geometry methods, the energy-perturbed approach and the multiplier skills.

The main goal of the present paper is to obtain the stabilization of the viscoelastic wave equation with variable coefficients in a bounded domain with smooth boundary, subject to linear dissipative internal feedback with a delay. Our method of proof uses some ideas developed in [17] for the wave equation with Riemannian geometry methods and some estimates of the viscoelastic wave equation, enabling us to obtain suitable Lyapunov functionals, from which are derived the desired result.

This paper is organized as follows. In Section 2 we present some assumptions and state our main result. Section 3 is devoted to prove our main result.

2. Preliminaries and main result

First, let us introduce some notation used throughout this paper. Define

\[
g = A^{-1}(x) \quad \text{for} \quad x \in \mathbb{R}^n
\]

as a Riemannian metric on $\mathbb{R}^n$ and consider the couple $(\mathbb{R}^n, g)$ as a Riemannian manifold. For each $x \in \mathbb{R}^n$, the metric $g$ introduces an inner product and the norm on the tangent space on $\mathbb{R}_x^n = \mathbb{R}^n$ by

\[
\langle X, Y \rangle_g = \langle A^{-1}(x)X, Y \rangle, \quad |X|_g^2 = \langle X, X \rangle_g, \quad X, Y \in \mathbb{R}^n.
\]

If $f \in C^1(\mathbb{R}^n)$, we define the gradient $\nabla_g f$ of $f$ in the Riemannian metric $g$. It is easy to verify that $\nabla_g f = A(x) \nabla f$. On the other hand, we define the gradient $\nabla_g f$ of $f$ in the Riemannian metric $g$, via the Riesz representation theorem, by

\[
X(f) = \langle \nabla_g f, X \rangle_g,
\]

where $X$ is any vector field on $(\mathbb{R}^n, g)$.

For the relaxation function $\beta$, we assume
(G1) $\beta \in C^2[0, \infty)$ is a non-negative and non-increasing function satisfying

$$\beta(0) > 0, \quad 1 - \int_0^\infty \beta(s)ds = l > 0.$$ 

(G2) There exists a positive nonincreasing differentiable function $\zeta(t)$ such that

$$\beta'(t) \leq -\zeta(t)\beta(t), \quad \forall t \geq 0,$$

and

$$\int_0^\infty \zeta(t)dt = +\infty.$$

As in [16], let us introduce the function

$$z(x, \rho, t) = u_t(x, t - \tau \rho), \quad x \in \Omega, \; \rho \in (0, 1), \; t > 0.$$ 

Then, problem (1.1) is equivalent to

$$u_{tt} + Au - \int_0^t \beta(t - s)Au(s)ds + \mu_1 u_t(x, t) + \mu_2 z(x, t, 1, t) = 0, \quad x \in \Omega, \; t > 0,$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,$$

$$\tau z_t(x, \rho, t) + z_{\rho}(x, \rho, t) = 0, \quad x \in \Omega, \; \rho \in (0, 1), \; t > 0,$$

$$z(x, 0, t) = 0, \quad x \in \Gamma_2, \; t > 0,$$

$$z(x, 0, t) = u_t(x, t), \quad x \in \Gamma_1, \; t > 0,$$

$$z(x, \rho, 0) = f_0(x, -\rho \tau), \quad x \in \Omega, \; \rho \in (0, 1).$$

Let

$$H^{1}_{\Gamma_2}(\Omega) = \{ u \in H^1(\Omega) \mid u|_{\Gamma_2} = 0 \},$$

and

$$L^2(\Omega \times (0, 1)) = \left\{ u \mid \int_0^1 \int_\Omega u^2(x, -\rho \tau)dx d\rho < \infty \right\}.$$

We now state, without a proof, a well-posedness result, which can be established by a similar proof with [10].

**Theorem 2.1.** Assume that $|\mu_2| \leq \mu_1$ and (G1) are satisfied. Then for given $u_0 \in H^{1}_{\Gamma_2}(\Omega), \; u_1(x) \in L^2(\Omega)$ and $f_0 \in L^2(\Omega \times (0, 1))$ and $T > 0$, there exists a unique weak solution $(u, z)$ of the problem (2.4) on $(0, T)$ such that

$u \in C([0, T], H^{1}_{\Gamma_2}(\Omega)) \cap C^1([0, T], L^2(\Omega)), \; u_t \in L^2(0, T; H_0^{1}(\Omega)) \cap L^2((0, T) \times \Omega).$

Define the new energy functional as

$$E(t) = E(t, u, z) = \frac{1}{2} \int_\Omega u_t^2(t)dx + \frac{1}{2} (1 - \int_0^t \beta(s)ds) \int_\Omega |\nabla u|^2 dx,$$

$$+ \frac{1}{2} (\beta \circ \nabla u)(t) + \frac{\xi}{2} \int_0^1 \int_\Omega z^2(x, \rho, t)dx d\rho.$$ 

(2.5)
where
\[(\beta \circ \nabla g)w(t) = \int_0^t \beta(t - s) \int_\Omega |\nabla g w(t) - \nabla g w(s)|^2 dx ds,\]
and \(\xi\) is a positive constant satisfying
\[(2.6) \quad \tau |\mu_2| < \xi < \tau (2\mu_1 - |\mu_2|) \quad \text{for} \quad |\mu_2| < \mu_1\]
and
\[(2.7) \quad \xi = \tau \mu_1 \quad \text{for} \quad |\mu_2| = \mu_1 = \mu.\]

Our stability result is the following.

**Theorem 2.2.** Let \(u\) be the solution of (1.1). Assume that \(|\mu_2| \leq \mu_1\) and \(\beta\) satisfies (G1) and (G2). Then there exist two positive constants \(K\) and \(k\) such that the energy of problem (1.1) satisfies
\[(2.8) \quad E(t) \leq Ke^{-k\int_0^t \xi(s) ds}, \quad \forall t \geq 0.\]

**Remark 2.1.** Estimate (2.8) is also true for \(t \in [0, t_0]\) by virtue of the continuity and boundedness of \(E(t)\) and \(\xi(t)\).

### 3. Proof of Theorem 2.2

In this section, we show, using the Riemannian geometry methods and Lyapunov functionals that under the hypothesis \(|\mu_2| \leq \mu_1\), the energy of the solution of (1.1) decreases exponentially as \(t\) tends to infinity. We will discuss two cases, the case where \(|\mu_2| < \mu_1\) and the case \(|\mu_2| = \mu_1\). We will separate the two cases since the proofs are slightly different.

#### 3.1. Exponential stability for \(|\mu_2| < \mu_1\)

In this subsection, we will prove Theorem 2.2 for \(|\mu_2| < \mu_1\). We have the following lemmas.

**Lemma 3.1.** Let \(\beta\) satisfy (G1). Then for all regular solutions of problem (1.1), the energy functional defined by (2.5) is non-increasing and satisfies
\[(3.1) \quad E'(t) \leq -C \int_\Omega (u_t^2(x, t) + u_t^2(x, t - \tau)) dx + \frac{1}{2} (\beta' \circ \nabla g)u(t) - \frac{1}{2} \beta(t) \int_\Omega |\nabla g u(t)|^2 dx \leq 0\]
for some positive constant \(C\).

**Proof.** Differentiating (2.5), applying Green’s formula and by (2.4), we obtain
\[E'(t) = \int_\Omega u_t u_t dx + (1 - \int_0^t \beta(s) ds) \int_\Omega \nabla g u \cdot \nabla u_t dx - \frac{1}{2} \beta(t) \int_\Omega |\nabla g u|^2 dx\]
\[+ \int_0^t \beta(t - s) \int_\Omega [\nabla g u(t) - \nabla g u(s)] \cdot \nabla u_t(t) dxdx + \frac{1}{2} (\beta' \circ \nabla g)u(t)\]
\begin{equation}
\int_0^1 \int_\Omega z_t(x, \rho, t) dxd\rho
\end{equation}

\begin{equation}
= \int_\Omega [u_t u_{tt} + \nabla g u \cdot \nabla u_t - \int_0^t \beta(t-s)\nabla g u(s) \cdot \nabla u_t(s) ds] dx
\end{equation}

\begin{equation}
- \frac{1}{2} \beta(t) \int_\Omega |\nabla g u_t|^2 dx + \frac{1}{2} (\beta' \circ \nabla g u)(t) + \xi \int_0^1 \int_\Omega z_t(x, \rho, t) dxd\rho
\end{equation}

\begin{equation}
= \int_\Omega [- \mu_1 u_t^2(x, t) - \mu_2 u_t(x, t) u_t(x, t - \tau)] dx - \frac{1}{2} \beta(t) \int_\Omega |\nabla g u_t|^2 dx
\end{equation}

\begin{equation}
+ \frac{1}{2} (\beta' \circ \nabla g u)(t) + \xi \int_0^1 \int_\Omega z_t(x, \rho, t) dxd\rho.
\end{equation}

(3.2)

Noting the fact that
\begin{align*}
z(x, \rho, t) &= u_t(x, t - \rho \tau) = -\frac{1}{\tau} u_\rho(x, \rho \tau), \\
z_t(x, \rho, t) &= u_{tt}(x, t - \rho \tau) = \frac{1}{\tau^2} u_{\rho\rho}(x, t - \rho \tau)
\end{align*}

and by integrating by parts, we arrive at
\begin{equation}
\xi \int_0^1 \int_\Omega z_t(x, \rho, t) dxd\rho = \frac{\xi}{2\tau} \int_\Omega (u_t^2(x, t) - u_t^2(x, t - \tau)) dx.
\end{equation}

From (3.2) and (3.3), and Young’s inequality, we get
\begin{align*}
E'(t) &\leq (\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}) \int_\Omega u_t^2(x, t) dx - \left( \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \int_\Omega u_t^2(x, t - \tau) dx \\
&\quad - \frac{1}{2} \beta(t) \int_\Omega |\nabla g u_t|^2 dx + \frac{1}{2} (\beta' \circ \nabla g u)(t).
\end{align*}

Then, the inequality (3.1) follows directly from (2.8) and hypothesis (G1). \(\square\)

Now we are going to construct a Lyapunov functional \(L(t)\) equivalent to \(E(t)\). For this purpose, we first define the following functional
\begin{equation}
I(t) := \int_\Omega u_t u dx.
\end{equation}

Then, we have the following estimate.

**Lemma 3.2.** Under the assumption (G1), the functional \(I(t)\) satisfies, along the solution, the estimate
\begin{equation}
I'(t) \leq (1 + \frac{\mu_1}{4\delta_1}) ||u_t||^2 - \left( \frac{l}{2} - \delta_1 C(\mu_1 + |\mu_2|) \right) \int_\Omega |\nabla g u(t)|^2 dx
\end{equation}

\begin{equation}
+ \frac{\mu_2}{4\delta_1} \int_\Omega u_t^2(x, t - \tau) dx + \frac{1 - \frac{l}{2}}{2} (\beta \circ \nabla g u)(t)
\end{equation}

for any \(\delta_1 > 0\).
Proof. Differentiating and applying Green’s formula, we have

\[
I'(t) = \int_\Omega u_t^2 dx - \int_\Omega |\nabla u(t)|_G^2 dx + \int_\Omega \left( \int_0^t \beta(t-s) \nabla g u(s) ds \right)_G dx
\]

(3.6)

\[-\mu_1 \int_\Omega u_t(x,t) u dx - \mu_2 \int_\Omega u_t(x,t-\tau) u dx.
\]

The third term in the right-hand side of (3.6) can be estimated as follows:

\[
\int_\Omega \left( \int_0^t \beta(t-s) |\nabla g u(s)|_G ds \right)_G dx
\]

\[\leq \frac{1}{2} \int_\Omega |\nabla u^2|_G^2 dx + \frac{1}{2} \int_\Omega \left( \int_0^t \beta(t-s) |\nabla g u(s)|_G ds \right)_G^2 dx
\]

\[\leq \frac{1}{2} \int_\Omega |\nabla u^2|_G^2 dx
\]

(3.7)

\[+ \frac{1}{2} \int_\Omega \left( \int_0^t (\beta(t-s)|\nabla g u(s) - \nabla g u(t)|_G + |\nabla g u(t)|_G) ds \right)_G^2 dx.
\]

Now, using Youngs inequality and (G1), we obtain

\[\int_\Omega \left( \int_0^t (\beta(t-s)|\nabla g u(s) - \nabla g u(t)|_G + |\nabla g u(t)|_G) ds \right)_G^2 dx
\]

\[\leq \int_\Omega \left( (1+\eta) \left( \int_0^t \beta(t-s)|\nabla g u(t)|_G ds \right)^2 \right.
\]

\[\left. + (1+\frac{1}{\eta}) \left( \int_0^t \beta(t-s)|\nabla g u(s) - \nabla g u(t)|_G ds \right)^2 \right) dx
\]

(3.8)

\[\leq (1+\eta)(1-l)^2 \int_\Omega |\nabla g u(t)|_G^2 dx + (1+\frac{1}{\eta})(1-l)(\beta \circ \nabla g u)(t).
\]

Substituting (3.8) into (3.7), we have

\[
\int_\Omega \left( \int_0^t \beta(t-s) |\nabla g u(s)|_G ds \right)_G dx
\]

(3.9)

\[\leq \frac{1}{2} \left[ 1 + (1+\eta)(1-l)^2 \right] \int_\Omega |\nabla g u(t)|_G^2 dx + \frac{1}{2} (1+\frac{1}{\eta})(1-l)(\beta \circ \nabla g u)(t).
\]

Also, noting that \(\int_\Omega |u|^2 dx \leq c \int_\Omega |\nabla g u|^2 dx\) for any \(u \in H^1_\Gamma(\Omega)\), using Youngs and Poincaré inequalities gives

\[
-\mu_1 \int_\Omega u_t u dx \leq \mu_1 \delta_1 C \int_\Omega |\nabla g u(t)|_G^2 dx + \frac{\mu_1}{4\delta_1} \int_\Omega u_t^2 dx,
\]

(3.10)

and

\[
-\mu_2 \int_\Omega u_t(x,t-\tau) u dx \leq |\mu_2| \delta_1 C \int_\Omega |\nabla g u(t)|_G^2 dx + \frac{|\mu_2|}{4\delta_1} \int_\Omega u_t^2(x,t-\tau) dx.
\]

(3.11)
By inserting the estimates (3.9)-(3.10) into (3.6) and choosing \( \eta = l/(1-l) \), then (3.5) holds.

Now, let us introduce the following functional

\[
K(t) := \int_{\Omega} \int_0^1 e^{-\tau \rho z^2(x, \rho, t)}d\rho dx.
\]

Differentiating (3.12) with respect \( t \) and using (2.4), we have

\[
\frac{d}{dt} \left( \int_{\Omega} \int_0^1 e^{-\tau \rho z^2(x, \rho, t)}d\rho dx \right) = -\frac{2}{\tau} \int_{\Omega} \int_0^1 e^{-\tau \rho z^2(x, \rho, t)}z(x, \rho, t)d\rho dx
\]

\[
= -\frac{1}{\tau} \int_{\Omega} \int_0^1 \frac{\partial}{\partial \rho} (e^{-\tau \rho} z^2) d\rho dx - \int_{\Omega} \int_0^1 e^{-\tau \rho z^2(x, \rho, t)}d\rho dx
\]

\[
= \frac{1}{\tau} \int_{\Omega} u_t^2(x, t)dx - \frac{c}{\tau} \int_{\Omega} u_t^2(x, t-\tau)dx - K(t).
\]

Define the Lyapunov functional

\[
\mathcal{L}(t) := E(t) + \varepsilon I(t) + \varepsilon K(t),
\]

where \( \varepsilon \) is a positive real number which will be chosen later. It is straightforward to see that for \( \varepsilon > 0 \), \( \mathcal{L}(t) \) and \( E(t) \) are equivalent in the sense that there exist two positive constants \( \alpha_1 \) and \( \alpha_2 \) depending on \( \varepsilon \) such that for all \( t \geq 0 \)

\[
\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t).
\]

Now, we are ready to prove the general decay result for \( |\mu_2| \leq \mu_1 \).

**Proof of Theorem 2.2 for \( |\mu_2| < \mu_1 \).** Using the estimates (3.1), (3.5) and (3.13), we get

\[
\mathcal{L}(t) \leq -\left( C - \varepsilon(1 + \frac{\mu_1}{4\delta_1}) - \frac{\varepsilon}{\tau} \right) \int_{\Omega} u_t^2(x, t)dx
\]

\[
- \varepsilon \left( \frac{1}{2} - \delta_1 C(\mu_1 + |\mu_2|) \right) \int_{\Omega} |\nabla g u(t)|^2 dx
\]

\[
- \left( C + \frac{\varepsilon c}{\tau} - \frac{\varepsilon |\mu_2|}{4\delta_1} \right) \int_{\Omega} u_t^2(x, t-\tau)dx + \frac{\varepsilon(1-l)}{2} (\beta \circ \nabla g u)(t)
\]

\[
- \frac{1}{2} \beta(t) \int_{\Omega} |\nabla g u|^2 dx + \frac{1}{2}(\beta' \circ \nabla g u)(t) - \varepsilon K(t).
\]

By choosing \( \delta_1 \) and \( \varepsilon \) small enough, we can find two positive constants \( \gamma_1 \) and \( \gamma_2 \) such that

\[
\mathcal{L}'(t) \leq -\gamma_1 E(t) + \varepsilon \gamma_1 (\beta \circ \nabla g u)(t), \quad \forall t \geq 0.
\]

By multiplying (3.14) by \( \zeta(t) \), we have

\[
\zeta(t) \mathcal{L}'(t) \leq -\gamma_1 \zeta(t) E(t) + \gamma_2 \zeta(t)(\beta \circ \nabla g u)(t), \quad \forall t \geq 0.
\]
Recalling (G2) and using (3.1), we have
\[ \zeta(t) \mathcal{L}'(t) - \zeta'(t) \mathcal{L}(t) \leq -\gamma_1 \zeta(t) E(t) - 2\gamma_2 E'(t), \quad \forall t \geq 0, \]
which implies
\[ (\zeta(t) \mathcal{L}(t) + 2\gamma_2 E(t))' - \zeta'(t) \mathcal{L}(t) \leq -\gamma_1 \zeta(t) E(t), \quad \forall t \geq 0. \]
Using the fact that \( \zeta'(t) \leq 0 \) for any \( t \geq 0 \) and letting
\[ \mathcal{F}(t) = \zeta(t) \mathcal{L}(t) + 2\gamma_2 E(t), \quad \mathcal{F}(t) \sim E(t), \]
we obtain
\[ \mathcal{F}'(t) \leq -\gamma_1 \zeta(t) E(t), \quad \forall t \geq 0. \]
A simple integration of (3.16) over \( (0, t) \) leads to
\[ \mathcal{F}(t) \leq \mathcal{F}(0) e^{-\kappa \int_0^t \zeta(s) ds}, \quad \forall t \geq 0. \]
Consequently, (2.5) can be obtained by (3.15) and (3.17). The proof of Theorem 2.2 for \( |\mu_2| < \mu_1 \) is completed.

3.2. Exponential stability for \( |\mu_2| = \mu_1 \)

In this subsection, we assume that \( \mu_1 = |\mu_2| = \mu \). As we will see, we cannot directly perform the same proof as for the case \( |\mu_2| < \mu_1 \).

**Lemma 3.3.** Let \( \beta \) satisfy (G1). Then for all regular solutions of problem (1.1), the energy functional defined by (2.5) is non-increasing and satisfies
\[ E'(t) \leq 2(\beta' \circ \nabla g u)(t) - \beta(t) \int_\Omega |\nabla g u(t)|^2 dx \leq 0, \quad \forall t \geq 0. \]

*Proof.* The proof is an immediate consequence of Theorem 3.1 by choosing \( \xi = \tau \mu \).

Set
\[ \chi(t) = - \int_\Omega u_1 t \int_0^t \beta(t-s)(u(t)-u(s)) ds dx. \]
Then we have:

**Lemma 3.4.** Under the assumption (G1), the functional \( \chi(t) \) satisfies, along the solution, the estimate
\[
\chi'(t) \leq \left( \delta_2 + 2\delta_2(1-\delta^2) \right) \int_\Omega |\nabla g u|^2 dx + (\delta_3(1+\mu)
- \int_0^t \beta(s) ds \int_\Omega u_1^2(x,t) dx \\
+ \left( \frac{1-\delta^2}{2\delta_2} + 2\delta_2(1-\delta^2) + \frac{\mu C^2}{4\delta_2} + \frac{\mu C^2}{4\delta_4} \right)(\beta' \circ \nabla g u)(t)
+ \mu \delta_4 \int_\Omega u_1^2(x,t-\tau) dx - \frac{g(0) C^2}{4\delta_3} (\beta' \circ \nabla g u)(t),
\]

\[ (3.20) \]
where $\varepsilon_2$, $\delta_2$, $\delta_3$ and $\delta_4$ are arbitrary positive constants.

**Proof.** Differentiate (3.20) with respect to $t$, we have

$$
\chi'(t) = \int_{\Omega} \left( \nabla_g u, \int_0^t \beta(t-s) \nabla_g (u(t) - u(s)) ds \right)_g dx
- \int_{\Omega} \mu_1 \int_0^t (u(t) - u(s)) ds dx
- \int_{\Omega} \mu_2 \int_0^t (u(t) - u(s)) ds dx.
$$

(3.21)

In what follows we will estimate the right terms of (3.21) one by one. For the right-hand side first term, by Young’s inequality and (G1), we obtain for any $\delta_2 > 0$,

$$
\left| \int_{\Omega} \left( \nabla_g u, \int_0^t \beta(t-s) \nabla_g (u(t) - u(s)) ds \right)_g dx \right|
\leq \delta_2 \int_{\Omega} |\nabla_g u|^2 dx + \frac{1}{4\delta_2} \int_{\Omega} \left( \int_0^t \beta(t-s) |\nabla_g (u(s) - \nabla_g u(t))| ds \right)^2 dx
\leq \delta_2 \int_{\Omega} |\nabla_g u|^2 dx + \frac{1}{4\delta_2} (\beta \circ \nabla_g u)(t).
$$

(3.22)

For the right side second term, for any $\delta_3 > 0$, we have

$$
\int_{\Omega} \mu_1 \int_0^t (u(t) - u(s)) ds dx
\leq \delta_3 \int_{\Omega} u_1^2(x,t) dx - \frac{\beta(0)C^2}{4\delta_3} (\beta \circ \nabla_g u)(t),
$$

(3.23)

where we also use $\int_{\Omega} |v|^2 dx \leq c \int_{\Omega} |\nabla_g v|^2 dx$ for any $v \in H^1_{\Gamma_2}(\Omega)$. Similarly, the fifth term and the sixth can be estimated as follows

$$
\int_{\Omega} \mu_2 \int_0^t (u(t) - u(s)) ds dx
\leq \delta_3 \int_{\Omega} u_1^2(x,t) dx + \frac{C^2}{4\delta_3} (\beta \circ \nabla_g u)(t),
$$

(3.24)
and
\[
\int_{\Omega} u_t(x, t - \tau) \int_0^t \beta(t - s)(u(t) - u(s)) ds \, dx
\]
(3.25)
\[
\leq \delta_4 \int_{\Omega} u_t^2(x, t - \tau) \, dx + \frac{C^2}{4\delta_4} (\beta \circ \nabla g)(u)(t).
\]

For the third term, we have
\[
\int_{\Omega} \left( \int_0^t \beta(t - s) \nabla g u(s) ds, \int_0^t \beta(t - s) \nabla g (u(t) - u(s)) ds \right) \, dx
\]
\[
\leq \delta_2 \int_{\Omega} \left( \int_0^t \beta(t - s) \nabla g u(s) ds \right)^2 \, dx
\]
\[
+ \frac{1}{4\delta_2} \int_{\Omega} \left( \int_0^t \beta(t - s) \nabla g (u(t) - u(s)) ds \right)^2 \, dx
\]
\[
\leq \delta_2 \int_{\Omega} \left( \int_0^t \beta(t - s) \left( |\nabla g u(t) - \nabla g u(s)|_g + |\nabla g u(t)|_g \right) ds \right)^2 \, dx
\]
\[
+ \frac{1}{4\delta_2} \int_{\Omega} \left( \int_0^t \beta(t - s) \nabla g (u(t) - u(s)) ds \right)^2 \, dx
\]
\[
\leq (2\delta_2 + \frac{1}{4\delta_2}) \int_{\Omega} \left( \int_0^t \beta(t - s) \nabla g (u(t) - u(s)) ds \right)^2 \, dx
\]
\[
+ 2\delta_2 (1 - l)^2 \int_{\Omega} |\nabla g u(t)|_g^2 \, dx
\]
(3.26)
\[
\leq (2\delta_2 + \frac{1}{4\delta_2})(1 - l)(\beta \circ \nabla g)(u)(t) + 2\delta_2 (1 - l)^2 \int_{\Omega} |\nabla g u(t)|_g^2 \, dx.
\]

Inserting the above estimates (3.22)-(3.26) into (3.21), (3.20) is established. □

**Proof of Theorem 2.2 for \(|\mu_2| = \mu_1\)**. Define Lyapunov function \(\mathcal{L}\) as
(3.27)
\[
\mathcal{L}(t) := NE(t) + \epsilon_1 I(t) + \chi(t) + \epsilon_2 K(t),
\]
where \(N, \epsilon_1, \text{and} \epsilon_2\) are positive real numbers which will be chosen later. We claim that \(\mathcal{L}(t)\) and \(E(t)\) are equivalent for \(\epsilon_1\) and \(\epsilon_2\) small enough while \(N\) large enough, i.e., there exist two positive constants \(\alpha_3\) and \(\alpha_4\) depending on \(N, \epsilon_1\) and \(\epsilon_2\) such that
(3.28)
\[
\alpha_3 E(t) \leq \mathcal{L}(t) \leq \alpha_4 E(t), \quad \forall t \geq 0.
\]

Indeed, we consider the functional
\[
H(t) = \epsilon_1 I(t) + \chi(t) + \epsilon_2 K(t)
\]
and show that
(3.29)
\[
|H(t)| \leq CE(t), \quad C > 0.
\]
Using Young’s inequality, Poincaré’s inequality, we have

\[
|\chi(t)| = \left| \int_{\Omega} u_t \int_0^t \beta(t-s)(u(t) - u(s)) \, ds \, dx \right|
\]

\[
\leq \frac{1}{2} \int_{\Omega} u_t^2(x,t) \, dx + \frac{1}{2} \int_{\Omega} \left( \int_0^t \beta(t-s)(u(t) - u(s)) \, ds \right)^2 \, dx
\]

(3.30)

\[
\leq \frac{1}{2} \int_{\Omega} u_t^2(x,t) \, dx + \frac{1}{2}(1-l)C^2(\beta \circ \nabla g)(t).
\]

Similarly, we have

\[
|\epsilon_1 I(t) + \epsilon_2 K(t)| = \left| \epsilon_1 \int_{\Omega} u_t u \, dx \right| + \left| \epsilon_2 \int_{\Omega} \int_0^1 e^{-\tau \rho} z^2(x,\rho,t) \, d\rho \, dx \right|
\]

(3.31)

\[
\leq \frac{\epsilon_1}{2} \int_{\Omega} u_t^2(x,t) \, dx + \frac{\epsilon_1}{2} \int_{\Omega} |\nabla_g u|^2 \, dx + \epsilon_2 c \int_{\Omega} z^2(x,\rho,t) \, d\rho \, dx.
\]

By the definition of \( E(t) \), (3.30) and (3.31), we get (3.29) for some positive constant \( C \). Now, it is obvious that (3.28) holds by choosing \( \epsilon_1 \) and \( \epsilon_2 \) small enough while \( N \) large enough.

Since the function \( \beta \) is positive, continuous and \( \beta(0) > 0 \), then for any \( t \geq t_0 > 0 \), we have

\[
\int_0^t \beta(s) \, ds \geq \int_{t_0}^t \beta(s) \, ds = \beta_0.
\]

Now, using (3.5), (3.13), (3.18) and (3.20), we have

\[
\mathcal{L}''(t) \leq \left\{ \epsilon_1 (1 + \frac{\mu}{4\delta_1}) + \left( \delta_3(1 + \mu) - \beta_0 + \frac{\epsilon_2}{\tau} \right) \right\} \int_{\Omega} u_t^2(x,t) \, dx - \epsilon_2 K(t)
\]

\[
+ \left\{ \left( \delta_2 + 2\delta_2(1-l) \right)^2 - \epsilon_1 \left( \frac{1}{2} - 2\mu\delta_1 \right) \right\} \int_{\Omega} |\nabla_g u|^2 \, dx
\]

\[
+ \left( \frac{N}{2} - \frac{\beta(0)C^2}{4\delta_3} \right)(\beta \circ \nabla g)(t)
\]

\[
+ \left( \frac{\epsilon_1}{4\delta_1} + \mu\delta_4 - \frac{\epsilon_2}{\tau} \right) \int_{\Omega} u_t^2(x,t-\tau) \, dx
\]

(3.32)

\[
+ \left\{ \frac{\epsilon_1(1-l)}{2} + \left( 1 - \frac{l}{2\delta_2} + 2\delta_2(1-l) + \frac{\mu C^2}{4\delta_3} + \frac{\mu C^2}{4\delta_4} \right) \right\}(\beta \circ \nabla g)(t).
\]

Now we will choose the constants in (3.32) carefully. Firstly, let us take \( \delta_1 \) small enough such that

\[
2\mu\delta_1 C \leq \frac{l}{4}.
\]

Then, we select \( \delta_3 \) small enough such that

\[
\delta_3(1 + \mu) \leq \frac{\beta_0}{2}.
\]
After that, we pick $\epsilon_2$ so small that

$$
\frac{\epsilon_2}{\tau} < \frac{\beta_0}{8}.
$$

Once $\epsilon_2$ is fixed, then we choose $\delta_4$ small so that

$$
\mu \delta_4 \leq \frac{\epsilon_2 c}{2 \tau}.
$$

Further, we take $\epsilon_1$ small that

$$
\epsilon_1 < \min \left\{ \frac{32 \delta_1}{(4 \delta_1 + \mu) \beta_0} \frac{\delta_1 \epsilon_2 c}{\tau} \right\}.
$$

Also, let us take $\delta$ small so that

$$
\delta_2 (1 + 2(1-l)^2) \leq \frac{\epsilon_1 l}{8}.
$$

Finally, we choose $N$ large enough such that

$$
N > \frac{\beta(0) C^2}{\delta_3}.
$$

Consequently, there exist two positive constant $\gamma_1$ such that

$$
\mathcal{L}'(t) \leq -\gamma_1 E(t) + \epsilon \gamma_2 (\beta \circ \nabla g u)(t), \quad \forall t \geq t_0.
$$

The remaining part of the proof of Theorem 2.2 for $|\mu_2| = \mu_1$ can be obtained following the same steps as in the proof Theorem 2.2 for $|\mu_2| < \mu_1$, so we omit the details.

\[ \square \]

References


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