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SUPERCYCLICITY OF ℓ^p-SPHERICAL AND TORAL ISOMETRIES ON BANACH SPACES

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Abstract. Let $p \geq 1$ be a real number. A tuple $T = (T_1, \ldots, T_n)$ of commuting bounded linear operators on a Banach space X is called an ℓ^p-spherical isometry if $\sum_{i=1}^n \|T_i x\|^p = \|x\|^p$ for all $x \in X$. The tuple T is called a toral isometry if each T_i is an isometry. By a result of Ansari, Hedayatian, Khani-Robati and Moradi, for every $n \geq 1$, there is a supercyclic ℓ^2-spherical isometric n-tuple on $\mathbb{C}^n$ but there is no supercyclic ℓ^2-spherical isometry on an infinite-dimensional Hilbert space. In this article, we investigate the supercyclicity of ℓ^p-spherical isometries and toral isometries on Banach spaces. Also, we introduce the notion of semi-commutative tuples and we show that the Banach spaces ℓ^p $(1 \leq p < \infty)$ support supercyclic ℓ^p-spherical isometric semi-commutative tuples. As a result, all separable infinite-dimensional complex Hilbert spaces support supercyclic spherical isometric semi-commutative tuples.

1. Introduction

An n-tuple of operators is a finite sequence of length n of commuting bounded linear operators $T_1, T_2, \ldots, T_n$ acting on a Banach space X. For an n-tuple $T = (T_1, T_2, \ldots, T_n)$, let $\mathcal{F}_T$ be the multiplicative semigroup generated by T_i's, i.e., $\mathcal{F}_T = \{T_1^{k_1} \cdots T_n^{k_n} : k_i \geq 0,\ i = 1, 2, \ldots, n\}$. If there exists an element $x \in X$ such that the set $\operatorname{orb}(T, x) = \{Sx : S \in \mathcal{F}_T\}$ is dense in X, then T is said to be a *hypercyclic* tuple and x is called a *hypercyclic* vector for T. The n-tuple $T = (T_1, T_2, \ldots, T_n)$ is said to be *supercyclic* if there exists an element $x \in X$ such that $\mathbb{C}.\operatorname{orb}(T, x) = \{\lambda S x : \lambda \in \mathbb{C}, S \in \mathcal{F}_T\}$ is dense in X. In that case, the vector x is called a *supercyclic* vector for T. These definitions generalize the notions of hypercyclicity and supercyclicity of a single operator to a tuple of operators. The hypercyclicity of tuples of operators was first investigated by Feldman [4]. Also, the supercyclicity of tuples of operators was first investigated by Soltani, Hedayatian and Khani-Robati [9].

Recall that a tuple $(T_1, \ldots, T_n)$ on a Hilbert space H is called a spherical isometry if $\sum_{i=1}^n T_i^* T_i = I$.

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Theorem 1.1 (Theorem 2 of [2])**.** *For every $n \geq 1$, there is a supercyclic spherical isometric n-tuple on $\mathbb{C}^n$.*

Theorem 1.2 (Proposition 1 of [2])**.** *There is no supercyclic spherical isometry on an infinite-dimensional Hilbert space.*

In Section 2, we investigate ℓ^p-spherical isometries on Banach spaces and ask if there is an infinite-dimensional Banach space that supports a supercyclic ℓ^p-spherical isometry. We show that if $(T_1, \ldots, T_n)$ is a supercyclic ℓ^p-spherical isometry on ℓ^q $(1 \leq q < \infty)$, then none of T_i's can be a weighted backward shift. Also, we define semi-commutative tuples and we show that every separable infinite-dimensional Hilbert space supports a supercyclic spherical isometric semi-commutative tuple. We prove that there is no supercyclic toral isometry on any Banach space with dimension more than one.

2. ℓ^p-Spherical and toral isometries on Banach spaces

Recall from [6] that for a real number $p \geq 1$, a tuple $(T_1, T_2, \ldots, T_n)$ on a Banach space X is called ℓ^p-spherical isometry if $\sum_{i=1}^n \|T_i x\|^p = \|x\|^p$ for every $x \in X$. For complex Hilbert spaces, ℓ^2-spherical isometries are spherical isometries. Indeed, a tuple $(T_1, \ldots, T_n)$ on a complex Hilbert space H, is a spherical isometry if and only if $\sum_{i=1}^n \|T_i x\|^2 = \|x\|^2$ for every $x \in H$.

Question 2.1. Is there any infinite-dimensional Banach space which supports a supercyclic ℓ^p-spherical isometry?

If we think about the negative answer to the question, we may naturally try to show that a spherical isometric tuple may not include a supercyclic operator (a tuple which includes a supercyclic operator is clearly supercyclic). The following proposition shows that the famous supercyclic operator B_W may not be a member of an ℓ^p-spherical isometry on $X = C_0$ or ℓ^q $(1 \leq q < \infty)$. If $(e_n)_{n=0}^\infty$ is the canonical basis of X and $W = (w_n)_{n=1}^\infty$ is a bounded sequence of positive numbers, recall that the weighted backward shift B_W on X is defined by $B_W e_0 = 0$ and $B_W e_n = w_n e_{n-1}$ $(n \geq 1)$. It is known that B_W is always supercyclic [5].

Proposition 2.2. *Let $X = C_0$ or ℓ^q $(1 \leq q < \infty)$ and B_W be a weighted backward shift on X. Then there are no operators $T_1, \ldots, T_n \in L(X)$ such that $(B_W, T_1, \ldots, T_n)$ is an ℓ^p-spherical isometry.*

Proof. To get a contradiction, suppose that $(B_W, T_1, \ldots, T_n)$ is an ℓ^p-spherical isometry. Let $W = (w_n)_{n=1}^\infty$ be the weight sequence for B_W and $(e_n)_{n=0}^\infty$ be the canonical basis of X. If $x = \sum_{j=0}^{N-1} a_j e_j \in C_{00}$, then $B_W^N x = 0$ and so for $i = 1, \ldots, n$ we have $B_W^N T_i x = T_i B_W^N x = 0$ which shows that $T_i x = \sum_{j=0}^{N-1} b_{ij} e_j$. In particular, $T_i e_0 = c_i e_0$ and $T_i e_1 = a_i e_0 + b_i e_1$ $(1 \leq i \leq n)$. Then $\sum_{i=1}^n |c_i|^p = \sum_{i=1}^n \|T_i e_0\|^p = \sum_{i=1}^n \|T_i e_0\|^p + \|B_W e_0\|^p = \|e_0\|^p = 1$. On the other hand, for every $i = 1, \ldots, n$ we have $b_i w_1 e_0 = B_W T_i e_1 = T_i B_W e_1 =$

$T_i(w_1e_0) = w_1c_ie_0$ and so $b_i = c_i$. Now, the ℓ^p-spherical isometry condition for $x = e_1$ gives $\sum_{i=1}^n \|a_ie_0 + c_ie_1\|^p = 1 - w_1^p$. But, regarding the norm on X, we have $\|a_ie_0 + c_ie_1\| \geq |c_i|$ for all $i = 1, \dots, n$. This gives $1 - w_1^p \geq 1$ which is not true. □

Definition 2.3. We say that $(T_1, \dots, T_n)$ is a semi-commutative tuple on a Banach space X if for all $1 \leq i, j \leq n$, $\mathrm{Ker}(T_iT_j - T_jT_i)$ is either X or a hyperplane in X. The semi-commutative tuple $(T_1, \dots, T_n)$ is said to be supercyclic if there is a vector $x \in X$ such that the set $\{\lambda T_1^{k_1} \cdots T_n^{k_n} x : \lambda \in \mathbb{C},\ k_i \geq 0,\ i = 1, \dots, n\}$ is dense in X.

Proposition 2.4. *The Banach spaces ℓ^p $(1 \leq p < \infty)$ support supercyclic ℓ^p-spherical isometric semi-commutative tuples.*

Proof. Fix a real number $p \in [1, \infty)$ and choose $n \in \mathbb{N}$. For $1 \leq i \leq n$, we define T_i on ℓ^p by $T_i\big(a_0, a_1, a_2, a_3, \dots\big) = \big(r_ia_0, (\frac{3}{4n})^{\frac{1}{p}}a_1, (\frac{3}{4n})^{\frac{1}{p}}a_2, (\frac{3}{4n})^{\frac{1}{p}}a_3, \dots\big)$ where $\sum_{i=1}^n |r_i|^p = 1$. Also, define

$$S\big(a_0, a_1, a_2, a_3, \dots\big) = \big((\tfrac{1}{4})^{\frac{1}{p}}a_1, (\tfrac{1}{4})^{\frac{1}{p}}a_2, (\tfrac{1}{4})^{\frac{1}{p}}a_3, \dots\big).$$

Then it is easy to see that for all $1 \leq i, j \leq n$, $T_iT_j = T_jT_i$ and $\mathrm{Ker}(ST_i - T_iS)$ is either ℓ^p or M, where M is the hyperplane in ℓ^p consisting of all vectors x for which $a_1 = 0$. Thus, $(S, T_1, \dots, T_n)$ is a semi-commutative tuple on ℓ^p. On the other hand, if we put $x = (a_0, a_1, a_2, a_3, \dots)$, then we have $\|Sx\|^p + \sum_{i=1}^n \|T_ix\|^p = \|x\|^p$. This shows that the semi-commutative tuple $(S, T_1, \dots, T_n)$ is an ℓ^p-spherical isometry. Finally, the supercyclicity of this semi-commutative tuple follows from the supercyclicity of the weighted backward shift S. □

We saw in Theorem 1.2 that no infinite-dimensional Hilbert space can support a supercyclic spherical isometric tuple. We use Proposition 2.4 to get the following result.

Proposition 2.5. *Every separable infinite-dimensional complex Hilbert space supports a supercyclic spherical isometric semi-commutative tuple.*

Proof. By Proposition 2.4, there is a semi-commutative tuple $(T_1, \dots, T_n)$ on ℓ^2 which is supercyclic spherical isometry. If H is any separable infinite-dimensional complex Hilbert space and $U : H \to \ell^2$ is an isometric isomorphism, then it can be easily verified that $(U^{-1}T_1U, \dots, U^{-1}T_nU)$ is a supercyclic spherical isometric semi-commutative tuple on H. □

From [8], we recall that a bounded complex sequence $\xi \in \ell^\infty(\mathbb{N}, \mathbb{C})$ almost converges to a complex number c if $\lim\limits_{k\to\infty} \sup\limits_{n\in\mathbb{N}} |c - k^{-1} \sum\limits_{j=n}^{n+k-1} \xi(j)| = 0$. We say that the sequence ξ almost converges to c in the strong sense if $|\xi - c1|$ almost converges to zero, where 1 stands for the constant 1 sequence. A gauge function is a mapping $p : \mathbb{N} \to (0, \infty)$ with the property that $\{\frac{p(n+1)}{p(n)}\}_{n\in\mathbb{N}}$

almost converges in the strong sense to a positive number c. The set of all gauge functions is denoted by $\mathcal{P}$. Now suppose that X is a complex Banach space and let $\mathcal{L}(X)$ denote the set of bounded, linear operators acting on X. We say that the norm-sequence of an operator $T \in \mathcal{L}(X)$ is compatible with the gauge function $p \in \mathcal{P}$, if $\|T^n\| \leq p(n)$ holds for every $n \in \mathbb{N}$ and the sequence $\{\frac{\|T^n\|}{p(n)}\}_{n\in\mathbb{N}}$ does not almost converge to zero. The set of all such operators is denoted by $\mathcal{L}(p, X)$. It is shown in [7] that $\{\frac{p(n+1)}{p(n)}\}_{n\in\mathbb{N}}$ almost converges to the spectral radius $r(T)$ for every $T \in \mathcal{L}(p, X)$. The operator $T \in \mathcal{L}(p, X)$ belongs to the class $\mathcal{C}_1 \cdot (p, X)$ if $\{\frac{\|T^n x\|}{p(n)}\}_{n\in\mathbb{N}}$ does not almost converge to zero for all non-zero vectors $x \in X$. We remind the reader that a (closed) subspace $\mathcal{M}$ is hyperinvariant for T, if $C\mathcal{M} \subset \mathcal{M}$ holds for every operator C commuting with T.

Theorem 2.6 (Main Theorem of [8])**.** *Let $T \in \mathcal{L}(X)$ be an operator belonging to the class $\mathcal{C}_1 \cdot (p, X)$, $p \in \mathcal{P}$. Let us assume that there exists a sequence $\{x_n\}_{n\in\mathbb{Z}}$ in X such that the vectors $\{x_n\}_{n\in\mathbb{N}}$ span an infinite dimensional subspace, $Tx_n = x_{n+1}$ for every $n \in \mathbb{Z}$, and*

$$\sum_{n\in\mathbb{Z}} \frac{\log^*(r(T)^{-n}\|x_n\|)}{1+n^2} < \infty.$$

Then there exists a sequence $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ of non-zero hyperinvariant subspaces of T such that

$$\mathcal{X}_n \cap (\bigvee_{j\neq n} \mathcal{X}_j) = \{0\}$$

for every $n \in \mathbb{N}$. Furthermore, if $\sigma_p(T) \cap r(T)\mathbb{T} = \emptyset$, then

$$\bigcap_{n\in\mathbb{N}} (\bigvee_{j\geq n} \mathcal{X}_j) = \{0\}.$$

The authors in [1] proved that isometries on Banach spaces with dimension more than one are not supercyclic. In the following theorem, we generalize this result to toral isometries. Here $\sigma_p(T)$ stands for the point spectrum of T and $\mathbb{T}$ denotes the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the complex plane $\mathbb{C}$. Furthermore, $\log^* t := 0$ if $0 \leq t \leq 1$ and $\log^* t := \log t$ if $t \geq 1$.

Theorem 2.7. *Suppose that X is an infinite-dimensional Banach space. Then there does not exist a supercyclic toral isometry on X.*

Proof. We prove the theorem for 2-tuples; for other n-tuples ($n \geq 3$) the proof is similar. We argue by contradiction. Assume that x_0 is a supercyclic vector for the pair $T = (T_1, T_2)$. Let x be a nonzero vector in X. Therefore, there are two sequences of non-negative integers $\{k_i\}_i$ and $\{s_i\}_i$ and a sequence of scalars $\{\alpha_i\}_i$ such that

$$\alpha_i T_1^{k_i} T_2^{s_i} x_0 \longrightarrow x \tag{1}$$

which implies that for large i we have

$$\|x\| - |\alpha_i|\|x_0\| \leq \|\alpha_i T_1^{k_i} T_2^{s_i} x_0 - x\| < \frac{\|x\|}{2}.$$

Thus

$$|\alpha_i| > \frac{\|x\|}{2\|x_0\|} \qquad \forall i \geq i_0 \tag{2}$$

for some i_0. On the other hand, if z is an arbitrary element in X, then there are two sequences of non-negative integers $\{m_j\}_j$ and $\{n_j\}_j$ and a sequence of scalars $\{\beta_j\}_j$ such that

$$\beta_j T_1^{n_j} T_2^{m_j} x_0 \longrightarrow z.$$

Let ε be a positive number. Since T_1 and T_2 are isometries there is a positive integer j_0 such that

$$|\beta_j| < \frac{\|z\| + 1}{\|x_0\|} \tag{3}$$

and

$$\|\beta_j T_1^{n_j} T_2^{m_j} x_0 - z\| < \frac{\varepsilon}{2} \tag{4}$$

for all $j \geq j_0$. Now if i and j are sufficiently large, then (1), (2) and (3) imply that

$$\frac{|\beta_j|}{|\alpha_i|}\|x - \alpha_i T_1^{k_i} T_2^{s_i} x_0\| < \frac{\varepsilon}{2}. \tag{5}$$

Hence there are positive integers i and j such that $n_j > k_i$ and $m_j > s_i$ so that

$$\begin{aligned}\left\|\frac{\beta_j}{\alpha_i} T_1^{n_j - k_i} T_2^{m_j - s_i} x - z\right\| &\leq \left|\frac{\beta_j}{\alpha_i}\right| \left\|T_1^{n_j-k_i} T_2^{m_j-s_i} x - \alpha_i T_1^{n_j-k_i+k_i} T_2^{m_j-s_i+s_i} x_0\right\| \\ &\quad + \left\|\beta_j T_1^{n_j} T_2^{m_j} x_0 - z\right\| \\ &= \left|\frac{\beta_j}{\alpha_i}\right| \left\|x - \alpha_i T_1^{k_i} T_2^{s_i} x_0\right\| + \left\|\beta_j T_1^{n_j} T_2^{m_j} x_0 - z\right\| < \varepsilon.\end{aligned}$$

This implies that every nonzero vector x is a supercyclic vector for the pair (T_1, T_2). Thus, T_1 and T_2 do not admit common non-trivial (closed) invariant subspaces. Indeed, if N is such a subspace and x is a nonzero vector in N, then $\{\lambda T_1^k T_2^m x : \lambda \in \mathbb{C}, k, m \geq 0\} \subset N$ and so $N = \overline{N} = X$. This shows in particular that both T_1 and T_2 are surjective and hence invertible.

If $p(n) = 1$ for all $n \in \mathbb{N}$, then it is easily seen that the operators T_1 and T_2 are in the class $\mathcal{C}_1 \cdot (p, X)$. Put $x_n = T_1^n x_0$ and $y_n = T_2^n x_0$ for $n \in \mathbb{Z}$ and assume that $\bigvee_{n\in\mathbb{N}} x_n$ and $\bigvee_{n\in\mathbb{N}} y_n$ are finite-dimensional; therefore, $\dim X = \dim \mathbb{C} \cdot \overline{orb(T, x_0)} \leq (\dim \bigvee_{n\in\mathbb{N}} x_n)(\dim \bigvee_{n\in\mathbb{N}} y_n) < \infty$ which is absurd. So without loss of generality we can assume that $\{x_n\}_{n\in\mathbb{N}}$ spans an infinite-dimensional subspace. Since $r(T_1) = 1$, all conditions of Theorem 2.6 hold for the operator

T_1. It follows that T_1 and T_2 have a common nontrivial invariant subspace which is a contradiction. □

Remark 2.8. The assertion of Theorem 2.7 is also true for all Banach spaces X with $1 < \dim X < \infty$. Since two commuting complex matrices have a common eigenvector, we conclude that there is a non-trivial subspace N of X that is invariant under the operators T_1 and T_2. On the other hand, according to the proof of the above theorem, every nonzero vector x is a supercyclic vector for the pair (T_1, T_2). Hence for every nonzero element $x \in N$ the set $\mathbb{C}.\mathrm{orb}((T_1, T_2), x) \subset N$ is dense in X, which is a contradiction.

Denote by $\mathrm{Iso}(X)$ the set of all isometries on X.

Proposition 2.9. *Suppose that* $p \in [1, \infty)$ *and* $\sum_{i=1}^{n} \|T_i x\|^p = \|x\|^p$ *for every* $x \in X$. *If* $(n-1)$ *operators among* $T_1, \ldots, T_n$ *belong to* $\mathbb{C} \cdot \mathrm{Iso}(X)$*, then the last one also belongs to* $\mathbb{C} \cdot \mathrm{Iso}(X)$.

Proof. Without loss of generality, suppose that for $i = 1, \ldots, n-1$, $T_i = a_i A_i$ where $A_i \in \mathrm{Iso}(X)$ and $a_i \in \mathbb{C}$. If we put $a = \sum_{i=1}^{n-1} |a_i|^p$, then we have $a\|x\|^p + \|T_n x\|^p = \|x\|^p$ or $\|T_n x\|^p = (1-a)\|x\|^p$ for all $x \in X$. If $a = 1$, then $T_n = 0 = 0 \cdot I$ and we are done. Otherwise, if we put $S = (1-a)^{\frac{-1}{p}} T_n$, then S is clearly an isometry. Consequently, $T_n = (1-a)^{\frac{1}{p}} S$ and the proof is complete. □

It is clear that the tuple $(T_1, \ldots, T_n)$ is supercyclic if and only if $(a_1 T_1, \ldots, a_n T_n)$ is supercyclic where $a_1, \ldots, a_n$ are arbitrary non-zero scalars. Regarding this fact, together with Theorem 2.7 and Proposition 2.9, we have the following result.

Corollary 2.10. *Let* X *be a Banach space with dim*$X > 1$. *If* $(T_1, \ldots, T_n)$ *is a supercyclic* ℓ^p*-spherical isometry, then at most* $(n-2)$ *operators among* $T_1, \ldots, T_n$ *may belong to* $\mathbb{C}.Iso(X)$.

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