Properties of Nowhere Dense Sets in GTSs

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Abstract. In this paper, we analyze some new properties of nowhere dense and strongly nowhere dense sets. Also, we discuss the images of nowhere dense and strongly nowhere dense sets. Further, we study the properties of weak Baire space in generalized topological spaces.

1. Introduction

The notion of a generalized topological space was introduced by Császár in [1]. Let \( X \) be any nonempty set. A family \( \mu \subset \exp(X) \) is a generalized topology [3] in \( X \) if \( \emptyset \in \mu \) and \( \bigcup_{t \in T} G_t \in \mu \) whenever \( \{G_t \mid t \in T\} \subset \mu \) where \( \exp(X) \) is the power set of \( X \). We call the pair \((X, \mu)\) as a generalized topological space (GTS) [3]. If \( X \in \mu \), then the pair \((X, \mu)\) is called a strong generalized topological space (sGTS). If \( Y \subset X \), then the subspace generalized topology on \( Y \) is defined by, \( \mu_Y = \{Y \cap U \mid U \in \mu\} \) and \( (Y, \mu_Y) \) is called the subspace GTS. We use \( \tilde{\mu} \) [2] to denote \( \{U \in \mu \mid U \neq \emptyset\} \) and \( \mu(x) \) [2] to denote \( \{U \in \mu \mid x \in U\} \).

Let \( A \subset X \). The interior of \( A \) [3] denoted by \( i_{\mu}A \), is the union of all \( \mu \)-open sets contained in \( A \) and the closure of \( A \) [3] denoted by \( c_{\mu}A \), is the intersection of all \( \mu \)-closed sets containing \( A \). The elements in \( \mu \) are called the \( \mu \)-open sets, the complement of a \( \mu \)-open set is called the \( \mu \)-closed sets where the complement of \( \mu \) is denoted by \( \mu' \). Let \( \{(X_\alpha, \mu_\alpha) \mid \alpha \in \Gamma\} \) be a family of pairwise disjoint strong GTSs. Put \( X = \bigcup_{\alpha \in \Gamma} X_\alpha \) and \( \mu = \{A \subset X \mid A \cap X_\alpha \in \mu_\alpha \text{ for each } \alpha \in \Gamma\} \). Then \((X, \mu)\) is a GTS, which is denoted by \( \bigoplus_{\alpha \in \Gamma} X_\alpha \), and called the generalized topological sum (GT-sum) of \( \{(X_\alpha, \mu_\alpha) \mid \alpha \in \Gamma\} \) [3].

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2. Preliminaries

A subset $A$ of a GTS $(X, \mu)$ is said to be a $\mu$-nowhere dense [2] (resp. $\mu$-dense [3] and $\mu$-codense [3]) set if $i_\mu c_\mu A = \emptyset$ (resp. $c_\mu A = X$ and $c_\mu (X - A) = X$). A subset $A$ of a GTS $(X, \mu)$ is said to be $\mu$-strongly nowhere dense [2] if for any $V \in \tilde{\mu}$, there exists $U \in \tilde{\mu}$ such that $U \subset V$ and $U \cap A = \emptyset$. Every $\mu$-strongly nowhere dense set is $\mu$-nowhere dense set [2]. A subset $A$ of a GTS $(X, \mu)$ is said to be a $\mu$-meager [2] (resp. $\mu$-s-meager [2]) set if $A = \bigcup_{n \in \mathbb{N}} A_n$ and $A_n$ is $\mu$-nowhere dense (resp. $\mu$-strongly nowhere dense) for all $n \in \mathbb{N}$ where $\mathbb{N}$ denote the set of all natural numbers. Also, every subset of a $\mu$-meager set is a $\mu$-meager set. A subset $A$ of a GTS $(X, \mu)$ is said to be a $\mu$-second category (\(\mu\)-II category) [2] (resp. $\mu$-second category (\(\mu\)-s-II category) [2]) if $A$ is not a $\mu$-meager (resp. $\mu$-s-meager) set. A subset $A$ of a GTS $(X, \mu)$ is of $\mu$-residual [2] (resp. $\mu$-s-residual [2]) if $X - A$ is a $\mu$-meager (resp. $\mu$-s-meager) set. A GTS $(X, \mu)$ is called a Baire space (BS) [2] if every $V \in \tilde{\mu}$ is of $\mu$-II category set. A GTS $(X, \mu)$ is called a weak Baire space (wBS) [2] if each set $V \in \tilde{\mu}$ is of $\mu$-s-II category in $X$. A space $(X, \mu)$ is a strong Baire space (sBS) [2] if $V_1 \cap V_2 \cap \cdots \cap V_n$ is a $\mu$-II category set for all $V_1, V_2, \ldots, V_n \in \mu$ such that $V_1 \cap V_2 \cap \cdots \cap V_n \neq \emptyset$. Also, every sBS is BS [2]. In [2], it is defined that $\mu^* = \{\bigcup_i(U_{1i} \cup U_{2i} \cup U_{3i} \cdots \cup U_{ni}) \mid U_{1i}, U_{2i}, \ldots, U_{ni} \in \mu\}$ and $\mu^{**} = \{A \mid A$ is a $\mu$-II category set}. If $(X, \mu)$ is not sGTS, then $\mu^*$ is closed under finite intersection and $\mu \subset \mu^*, \mu \subset \mu^{**}$ if $(X, \mu)$ is BS [2].

Throughout this paper, $\mathcal{D}(\mu)$ (resp. $\mathcal{N}(\mu)$) denotes the set of all dense (resp. nowhere dense) sets in $(X, \mu)$. The notations $X_3, X_4, X_5$ are mean the sets $\{a, b, c\}, \{a, b, c, d\}, \{a, b, c, d, e\}$, respectively. The following lemmas will be useful in the sequel.

**Lemma 2.1.**([2, Property 2.2]) Let $(X, \mu)$ satisfy the condition: if $V_1, V_2 \in \mu$ and $V_1 \cap V_2 \neq \emptyset$, then $i_\mu (V_1 \cap V_2) \neq \emptyset$. Then the set $A \subset X$ is $\mu$-strongly nowhere dense set if and only if $A$ is a $\mu$-nowhere dense set.

**Lemma 2.2.**([2, Property 2.3]) Let $(X, \mu)$ be a GTS and $A \subset X$ be a $\mu$-strongly nowhere dense set. Then any subset of $A$ is $\mu$-strongly nowhere dense set.

**Lemma 2.3.**([2, Property 2.4]) If $A$ is a finite family of $\mu$-strongly nowhere dense sets, then $\bigcup A$ is a $\mu$-strongly nowhere dense set.

**Lemma 2.4.**([3, Proposition 3.4]) Let $(X, \mu)$ be a GTS and let $A \subset X$. Then $A \in \mathcal{N}(\mu)$ if and only if $c_\mu A$ is $\mu$-codense in $X$.

**Lemma 2.5.**([3, Lemma 6.12]) Let $(X, \mu)$ be the GT-sum of $\{(X_\alpha, \mu_\alpha) : \alpha \in \Gamma\}$. If $A \in \mathcal{D}(\mu)$, then $A \cap X_\alpha \in \mathcal{D}(\mu_\alpha)$ for any $\alpha \in \Gamma$.

3. Properties of Nowhere Dense Sets

In this section, we discuss the properties of two types of nowhere dense sets in generalized topological spaces. Also, we study the properties of $\mu^*, \mu^{**}$—nowhere dense and $\mu^*, \mu^{**}$—strongly nowhere dense sets in generalized topological spaces.
Theorem 3.1. Let \((X, \mu)\) be a GTS and let \(A \subseteq Y \subseteq X\). If \(Y\) is a \(\mu\)-s-meager set in \(X\), then \(A\) is a \(\mu\)-s-meager set in \(X\).

Proof. Suppose \(Y\) is a \(\mu\)-s-meager set in \(X\). Then \(Y = \bigcup_{n \in \mathbb{N}} B_n\) where \(B_n\) is a \(\mu\)-strongly nowhere dense set. Now, \(A = A \cap Y = \bigcup_{n \in \mathbb{N}} (A \cap B_n)\). Put \(A_n = A \cap B_n\) for \(n \in \mathbb{N}\). Then \(A_n\) is a \(\mu\)-strongly nowhere dense set for \(n \in \mathbb{N}\), by Lemma 2.2. Therefore, \(A\) is a \(\mu\)-s-meager set in \(X\).

The following Corollary 3.2 follows from Theorem 3.1 and the easy proof of the following Theorem 3.3 is omitted.

Corollary 3.2. Let \((X, \mu)\) be a GTS and let \(A \subseteq Y \subseteq X\). Then the following hold.

(a) If \(A\) is of \(\mu\)-s-II category set in \(X\), then \(Y\) is of \(\mu\)-s-II category set in \(X\).

(b) If \(A\) is a \(\mu\)-s-residual set in \(X\), then \(Y\) is a \(\mu\)-s-residual set in \(X\).

(c) If \(A\) is \(\mu\)-s-residual, then the closure of \(A\) is \(\mu\)-s-residual.

Theorem 3.3. Let \((X, \mu)\) be a GTS. Then the following hold.

(a) If \(G_n\) is \(\mu\)-s-meager for each \(n \in \mathbb{N}\), then \(\bigcup_{n \in \mathbb{N}} G_n\) is \(\mu\)-s-meager.

(b) If \(F_n\) is \(\mu\)-s-residual for each \(n \in \mathbb{N}\), then \(\bigcap_{n \in \mathbb{N}} F_n\) is \(\mu\)-s-residual.

Lemma 3.4. Let \((X, \mu)\) be a GTS and \(Y\) be a \(\mu\)-dense subset of \(X\). If \(A \subseteq Y \subseteq X\) is \(\mu\)-strongly nowhere dense in \(X\), then \(A\) is \(\mu_Y\)-strongly nowhere dense in \(Y\).

Proof. Suppose \(Y\) is a \(\mu\)-dense subset of \(X\). Let \(U \in \bar{\mu}_Y\). Then \(U = G \cap Y\) where \(G \in \bar{\mu}\). By hypothesis, there exists \(W \in \bar{\mu}\) such that \(W \subseteq G\) and \(W \cap A = \emptyset\). This implies \(W \cap Y \in \bar{\mu}_Y\) such that \(W \cap Y \subseteq G \cap Y\) and \(W \cap Y \cap A = \emptyset\). If \(V = W \cap Y\), then there exists \(V \in \bar{\mu}_Y\) such that \(V \subseteq U\) and \(V \cap A = \emptyset\). Therefore, \(A\) is \(\mu_Y\)-strongly nowhere dense in \(Y\).

The following Example 3.5 shows that the condition dense on \(Y\) cannot be dropped in Lemma 3.4.

Example 3.5. Consider the GTS \((X_4, \mu)\) where \(\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}\). Let \(Y = \{a, c, d\}\). Then \(\mu_Y = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}\). If \(A = \{c, d\}\), then \(A\) is a \(\mu\)-s-strongly nowhere dense set. For \(V = \{c\}\), there is no \(U \in \bar{\mu}_Y\) such that \(U \subseteq V\) and \(U \cap A = \emptyset\). Thus, \(A\) is not a \(\mu_Y\)-strongly nowhere dense set in \(Y\).

Theorem 3.6. Let \((X, \mu)\) be a GTS and \(Y\) be a \(\mu\)-dense subset of \(X\). If \(A \subseteq Y \subseteq X\) is a \(\mu\)-s-meager set in \(X\), then \(A\) is a \(\mu_Y\)-s-meager set in \(Y\).

Proof. Suppose \(A\) is a \(\mu\)-s-meager set in \(X\). Then \(A = \bigcup_{n \in \mathbb{N}} A_n\) where each \(A_n\) is a \(\mu\)-s-meager set in \(X\). By Lemma 3.4, each \(A_n\) is a \(\mu_Y\)-s-meager set in \(Y\).

Corollary 3.7. Let \((X, \mu)\) be a GTS and \(Y\) be a \(\mu\)-dense subset of \(X\). Then the following hold.
(a) If \( A \subset Y \subset X \) is a \( \mu_Y\)-s-II category set in \( Y \), then \( A \) is a \( \mu\)-s-II category set in \( X \).

(b) If \( A \subset Y \subset X \) is \( \mu\)-s-residual in \( X \), then \( A \) is \( \mu_Y\)-s-residual in \( Y \).

**Proof.** (a) Follows from Theorem 3.6.
(b) Suppose \( A \) is a \( \mu\)-s-residual set in \( X \). Then \( X - A \) is a \( \mu\)-s-meager set in \( X \). By hypothesis and Theorem 3.6, \( X - A \) is a \( \mu_Y\)-s-meager set in \( Y \). This implies \( Y - A \) is a \( \mu_Y\)-s-meager set in \( Y \), by Theorem 3.1. Therefore, \( A \) is a \( \mu_Y\)-s-residual set in \( Y \).

**Theorem 3.8.** Let \((X, \mu)\) be a GTS and \( A \subset X \). If \( A \) is a \( \mu\)-strongly nowhere dense set in \( X \), then \( A \) is \( \mu\)-codense (that is, \( X - A \in D(\mu) \)) in \( X \).

The following Example 3.9 shows that the converse of the above Theorem 3.8 is not be true and the Corollary 3.10 below follows from Lemma 2.3 and Theorem 3.8.

**Example 3.9.** Consider the GTS \((X_4, \mu)\) where \( \mu = \{\emptyset, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}\) . Let \( A = \{a,d\} \). Then \( A \) is a \( \mu\)-codense set. If \( V = \{a,c\} \), then there is no \( U \in \tilde{\mu} \) such that \( U \subset V \) and \( U \cap A = \emptyset \). Thus, \( A \) is not a \( \mu\)-strongly nowhere dense set in \( X \).

**Corollary 3.10.** Let \((X, \mu)\) be a GTS. If \( A \) is a finite family of \( \mu\)-strongly nowhere dense set, then \( \bigcup A \) is \( \mu\)-codense.

The following Theorem 3.11 follows from the definition of \( \mu^* \) and Lemma 2.1. Observe that for a sBS \((X, \mu)\), \( \mu^{**} \) is closed under finite intersections. Hence we have the Theorem 3.13 and Theorem 3.14 below follows from Theorem 3.13.

**Theorem 3.11.** Let \((X, \mu)\) be a GTS. Then the following hold.

(a) \( A \subset X \) is a \( \mu^*\)-nowhere dense set if and only if \( A \) is a \( \mu^*\)-strongly nowhere dense set.

(b) \( A \subset X \) is \( \mu^*\)-meager if and only if it is \( \mu^*\)-s-meager.

(c) \( A \subset X \) is \( \mu^*\)-residual if and only if it is \( \mu^*\)-s-residual.

(d) \( A \subset X \) is of \( \mu^*\)-II category if and only if it is of \( \mu^*\)-s-II category.

**Theorem 3.12.** Let \((X, \mu)\) be a GTS. Then the following are equivalent.

(a) \((X, \mu^*)\) is a sBS.

(b) \((X, \mu^*)\) is a BS.

(c) \((X, \mu^*)\) is a wBS.
Theorem 3.13. Let \((X, \mu)\) be a sBS. Then the following hold.

(a) \(A \subset X\) is a \(\mu^\star\)-nowhere dense set if and only if \(A\) is a \(\mu^\star\)-strongly nowhere dense set.

(b) \(A \subset X\) is \(\mu^\star\)-meager if and only if it is \(\mu^\star\)-s-meager.

(c) \(A \subset X\) is \(\mu^\star\)-residual if and only if it is \(\mu^\star\)-s-residual.

(d) \(A \subset X\) is of \(\mu^\star\)-II category if and only if it is of \(\mu^\star\)-s-II category.

Theorem 3.14. Let \((X, \mu)\) be a sBS. Then the following are equivalent.

(a) \((X, \mu^\star)\) is a sBS.

(b) \((X, \mu^\star)\) is a BS.

(c) \((X, \mu^\star)\) is a wBS.

4. Images of Nowhere Dense Sets in GTSs

In this section, we discuss the behavior of nowhere dense sets in GTS under feebly \((\mu, \lambda)\)-continuous, feebly \((\mu, \lambda)\)-open and \((\mu, \lambda)\)-open mappings. Let \((X, \mu)\) and \((Y, \lambda)\) be two GTSs. A function \(f : (X, \mu) \to (Y, \lambda)\) is called feebly \((\mu, \lambda)\)-continuous [4] if \(i_\mu(f^{-1}(B)) \neq \emptyset\) for every \(B \subset Y\) with \(i_\lambda(B) \neq \emptyset\). A function \(f : (X, \mu) \to (Y, \lambda)\) is called \((\mu, \lambda)\)-open (resp. feebly \((\mu, \lambda)\)-open) if \(f(U) \in \lambda\) for each \(U \in \mu\) (resp. \(i_\lambda(f(U)) \neq \emptyset\) for each \(U \in \hat{\mu}\)) [3].

Theorem 4.1. Let \((X, \mu)\) and \((Y, \lambda)\) be two GTSs and \(f : (X, \mu) \to (Y, \lambda)\) be a injective, \((\mu, \lambda)\)-open and feebly \((\mu, \lambda)\)-continuous mapping. If \(A \in \mathcal{N}(\mu)\), then \(f(A) \in \mathcal{N}(\lambda)\).

Proof. Suppose \(A \in \mathcal{N}(\mu)\). Then \(X - c_\mu(A)\) is \(\mu\)-dense and \(\mu\)-open in \(X\), by Lemma 2.4. Suppose \(Y - c_\lambda(f(A))\) is not \(\lambda\)-dense in \(Y\). Then there is a set \(U \in \hat{\lambda}\) such that \(U \cap (Y - c_\lambda(f(A))) = \emptyset\). This implies that \(U \subset Y - (Y - c_\lambda(f(A)))\) which implies that \(U \subset c_\lambda(f(A))\). Suppose \(x \notin f(c_\mu(A))\). Then \(f^{-1}(x) \notin c_\mu(A)\). This implies that there exists a set \(V \in \mu\) containing \(f^{-1}(x)\) such that \(V \cap A = \emptyset\) and so \(f(V \cap A) = \emptyset\). Since \(f\) is one-one, \(f(A) \cap f(V) = \emptyset\). Therefore, \(x \notin c_\lambda(f(A))\), since \(f\) is \((\mu, \lambda)\)-open and so \(c_\lambda(f(A)) \subset f(c_\mu(A))\). Hence \(U \subset f(c_\mu(A))\). By hypothesis, \(f^{-1}(U) \subset f^{-1}(c_\mu(A))\). Take \(G = i_\mu(f^{-1}(U))\). Then \(G \neq \emptyset\) and \(G \subset c_\mu(A)\), since \(f\) is feebly \((\mu, \lambda)\)-continuous. This implies that \(G \cap (X - c_\mu(A)) = \emptyset\), which is a contradiction to \(X - c_\mu(A)\) is \(\mu\)-dense. Therefore, \(Y - c_\lambda(f(A))\) is \(\lambda\)-dense in \(Y\). Thus, \(i_\lambda c_\lambda(f(A)) = \emptyset\). Hence \(f(A) \in \mathcal{N}(\lambda)\).  \(\Box\)
Example 4.2.

(a) Consider the two GTSs $(X_5, \mu)$ and $(X_4, \lambda)$ where $\mu = \{\emptyset, \{e\}, \{a, b\}, \{a, b, c\}, \{a, b, c, e\}\}$ and $\lambda = \{\emptyset, \{b\}, \{c, d\}, \{b, c, d\}\}$. Define a function $f : (X_5, \mu) \rightarrow (X_4, \lambda)$ by $f(a) = \emptyset, f(b) = c, f(c) = d, f(d) = a, f(e) = b$. Then $f$ is $(\mu, \lambda)$-open but not injective. Clearly, $f$ is a feebly $(\mu, \lambda)$-continuous mapping. Let $A = \{a\} \subset X_5$. Then $i_\mu c_\mu(A) = i_\mu(\{a, d\}) = \emptyset$ and so $A \in N(\mu)$. But $i_\lambda c_\lambda(f(A)) = i_\lambda(\{a, c, d\}) = \{c, d\} \neq \emptyset$. Therefore, $f(A) \notin N(\lambda)$.

(b) Consider the set $X_4$ with two generalized topologies $\mu, \lambda$ where $\mu = \{\emptyset, \{a\}, \{a, c\}, \{a, b, c\}\}$ and $\lambda = \{\emptyset, \{a\}, \{b, c\}, \{a, b, d\}, \{b, c, d\}\}$. Define a function $f : (X_4, \mu) \rightarrow (X_4, \lambda)$ by $f(a) = b, f(b) = c, f(c) = d, f(d) = a$. Then $f$ is an injective and $(\mu, \lambda)$-open mapping. Let $A = \{a\} \subset X_4$. Then $i_\mu f^{-1}(A) = i_\mu(\{d\}) = \emptyset$. Therefore, $f$ is not a feebly $(\mu, \lambda)$-continuous mapping. Let $B = \{d\} \subset X_4$. Then $i_\mu c_\mu(B) = i_\mu(\{d\}) = \emptyset$ and so $B \in N(\mu)$. But $i_\lambda c_\lambda(f(B)) = i_\lambda(\{a\}) = \{a\} \neq \emptyset$. Therefore, $f(B) \notin N(\lambda)$.

(c) Consider the set $X_4$ with two generalized topologies $\mu, \lambda$ where $\mu = \{\emptyset, \{a\}, \{a, c\}, \{a, b, c\}\}$ and $\lambda = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}\}$. Define a function $f : (X_4, \mu) \rightarrow (X_4, \lambda)$ by $f(a) = d, f(b) = c, f(c) = b, f(d) = a$. Clearly, $f$ is injective. Since each $A \subset X_4$ with $i_\lambda A \neq \emptyset, i_\mu f^{-1}(A) = \emptyset$, $f$ is a feebly $(\mu, \lambda)$-continuous mapping. But $f$ is not a $(\mu, \lambda)$-open. For, let $A = \{c\} \in \mu$. Then $f(A) = \{b\} \notin \lambda$. Let $B = \{b, d\} \subset X_4$. Then $i_\mu c_\mu(B) = i_\mu(\{b, d\}) = \emptyset$ and so $B \in N(\mu)$. But $i_\lambda c_\lambda(f(B)) = i_\lambda(\{X_4\}) = \{a, c, d\} \neq \emptyset$. Therefore, $f(B) \notin N(\lambda)$.

Corollary 4.3. Let $(X, \mu)$ and $(Y, \lambda)$ be two GTSs and $f : (X, \mu) \rightarrow (Y, \lambda)$ be a $(\mu, \lambda)$-open, feebly $(\mu, \lambda)$-continuous and injective mapping. Then the following hold.

(a) If $A \subset X$ is of $\mu$-meager, then $f(A)$ is of $\lambda$-meager.

(b) If $f$ is surjective and $B \subset X$ is $\mu$-residual in $X$, then $f(B)$ is $\lambda$-residual in $Y$.

(c) If $f$ is surjective and $C \subset Y$ is of $\lambda$-second category, then $f^{-1}(C)$ is of $\mu$-second category.

(d) If $A$ is a $\mu$-strongly nowhere dense set, then $f(A) \in N(\lambda)$ where $A \subset X$.

Proof. (a) Let $A$ be a $\mu$-meager set in $X$. Then $A = \bigcup_{n \in \mathbb{N}} A_n$ where each $A_n$ is $\mu$-nowhere dense set in $X$. Then by hypothesis and Theorem 4.1, each $f(A_n)$ is
λ-nowhere dense set in Y. Thus, \( f(A) = f(\bigcup_{n \in \mathbb{N}} A_n) = \bigcup_{n \in \mathbb{N}} f(A_n) \). Therefore, \( f(A) \) is \( \lambda \)-meager in Y.

(b) Let \( B \) be a \( \mu \)-residual set in \( X \). Then \( X - B \) is of \( \mu \)-meager set in \( X \). By (a), \( f(X - B) \) is of \( \lambda \)-meager in Y. Now \( f(X - B) = f(X) - f(B) = Y - f(B) \), since \( f \) is bijective map. Thus, \( Y - f(B) \) is of \( \lambda \)-meager set in Y. Therefore, \( f(B) \) is \( \lambda \)-residual in Y.

(c) Let \( C \subseteq Y \) be a \( \lambda \)-second category set in Y. Suppose \( f^{-1}(C) \) is of \( \mu \)-meager set in \( X \). By (a), \( f(f^{-1}(C)) \) is of \( \lambda \)-meager in Y. Since \( f \) is surjective, \( f^{-1}(C) = C \) and so \( C \) is of \( \lambda \)-meager in Y, which is not possible. Therefore, \( f^{-1}(C) \) is of \( \mu \)-second category in \( X \).

(d) The proof follows from Theorem 4.1 and the fact that every \( \mu \)-strongly nowhere dense set is \( \mu \)-nowhere dense set.

\[ \square \]

**Theorem 4.4.** Let \((X, \mu)\) and \((Y, \lambda)\) be two GTSs. Let \( f : (X, \mu) \to (Y, \lambda) \) be a feebly \((\mu, \lambda)\)-continuous, injective mapping. If \( A \) is \( \mu \)-codense, then \( f(A) \) is \( \lambda \)-codense.

**Proof.** Suppose \( f(A) \) is not a \( \lambda \)-codense set in \( Y \). Then there is \( G \subseteq X \) such that \( G \subseteq f(A) \) which implies \( f^{-1}(G) \subseteq f^{-1}(f(A)) = A \), since \( f \) is injective. Thus, \( i_{\mu}(f^{-1}(G)) \subseteq i_{\mu}(A) \). Since \( f \) is a feebly \((\mu, \lambda)\)-continuous, \( i_{\mu}(f^{-1}(G)) \neq \emptyset \). This implies that \( i_{\mu}A \neq \emptyset \) and hence \( A \) is not \( \mu \)-codense. This completes the proof. \( \square \)

The following Example 4.5 shows that the conditions injective, feebly \((\mu, \lambda)\)-continuous on \( f \) cannot be dropped in Theorem 4.4.

**Example 4.5.**

(a) Consider the two GTSs \((X_4, \mu)\) and \((X_3, \lambda)\) where \( \mu = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}\} \) and \( \lambda = \{\emptyset, \{a, b\}, \{a, c\}, \{b, c\}, X_3\} \). Define a function \( f : (X_4, \mu) \to (X_3, \lambda) \) by \( f(a) = c, f(b) = a, f(c) = b, f(d) = c \). Clearly, \( f \) is a feebly \((\mu, \lambda)\)-continuous but not injective mapping. Let \( A = \{c, d\} \). Then \( i_{\mu}A = \emptyset \) and hence \( A \) is \( \mu \)-codense. But \( c_{\lambda}(X_3 - f(A)) = c_{\lambda}(\{a\}) = \{a\} \neq X_3 \). Hence \( f(A) \) is not \( \lambda \)-codense.

(b) Consider the set \( X_4 \) with two generalized topologies \( \mu, \lambda \) where \( \mu = \{\emptyset, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}\} \) and \( \lambda = \{\emptyset, \{a, c\}, \{b, c\}, \{a, b, c\}\} \). Define a function \( f : (X_4, \mu) \to (X_4, \lambda) \) by \( f(a) = d, f(b) = c, f(c) = b, f(d) = a \). Then \( f \) is injective. Let \( B = \{a, b\} \) with \( i_{\lambda}B \neq \emptyset \). But \( i_{\mu}(f^{-1}(B)) = i_{\mu}(\{c, d\}) = \emptyset \). Thus, \( f \) is not a feebly \((\mu, \lambda)\)-continuous mapping on \( X_4 \). Let \( A = \{c, d\} \). Then \( A \) is a \( \mu \)-codense set in \( X_4 \). But \( c_{\lambda}(X_4 - f(A)) = c_{\lambda}(\{c, d\}) = \{b, c, d\} \neq X_4 \). Hence \( f(A) \) is not \( \lambda \)-codense.

**Theorem 4.6.** Let \((X, \mu)\) and \((Y, \lambda)\) be two GTSs. Let \( f : (X, \mu) \to (Y, \lambda) \) be a feebly \((\mu, \lambda)\)-open mapping. If \( A \) is \( \lambda \)-codense, then \( f^{-1}(A) \) is \( \mu \)-codense.

**Proof.** Suppose \( f^{-1}(A) \) is not a \( \mu \)-codense set in \( X \). Then there is \( G \subseteq \tilde{\mu} \) such that \( G \subseteq f^{-1}(A) \) which implies \( f(G) \subseteq f(f^{-1}(A)) \subseteq A \). Thus, \( i_{\lambda}(f(G)) \subseteq i_{\lambda}(A) \). Since
$f$ is a feebly $(\mu, \lambda)$-open, $i_\lambda(f(G)) \neq \emptyset$. This implies that $i_\lambda(A) \neq \emptyset$ and hence $A$ is not $\lambda$-codense. This completes the proof. \hfill \Box

The following Example 4.7 shows that the condition feebly $(\mu, \lambda)$-open on $f$ cannot be dropped in Theorem 4.6.

**Example 4.7.** Consider the two GTSs $(X_4, \mu)$ and $(X_3, \lambda)$ where $\mu = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $\lambda = \{\emptyset, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define a function $f : (X_4, \mu) \to (X_3, \lambda)$ by $f(a) = c, f(b) = a, f(c) = b, f(d) = c$. Clearly, $f$ is not a feebly $(\mu, \lambda)$-open mapping. Let $A = \{c\} \subset X_3$. Then $i_\lambda(A) = \emptyset$ and hence $A$ is $\lambda$-codense. But $c_\mu(X_4 - f^{-1}(A)) = c_\mu(\{b, c\}) = \{b, c, d\} \neq X_4$. Hence $f^{-1}(A)$ is not $\mu$-codense.

**Theorem 4.8.** Let $(X, \mu)$ and $(Y, \lambda)$ be two GTSs and $f : X \to Y$ be a feebly $(\mu, \lambda)$-continuous, feebly $(\mu, \lambda)$-open, injective mapping. Then $A$ is a $\mu$-strongly nowhere dense set in $X$ if and only if $f(A)$ is a $\lambda$-strongly nowhere dense set in $Y$.

*Proof.* Suppose $A$ is a $\mu$-strongly nowhere dense set in $X$. Let $U \in \lambda$. Then $i_\mu(f^{-1}(U)) \in \mu$. By hypothesis, there exists $G \in \hat{\mu}$ such that $G \subset i_\mu(f^{-1}(U))$ and $G \cap A = \emptyset$. This implies that $i_\lambda(f(G)) \in \lambda$ and $i_\lambda(f(G)) \subset f(i_\mu(f^{-1}(U))) \subset U$ and $i_\lambda(f(G)) \cap f(A) = \emptyset$. Therefore, $f(A)$ is a $\lambda$-strongly nowhere dense set in $Y$. Conversely, let $G \in \hat{\mu}$. Then $i_\lambda(f(G)) \in \lambda$. By hypothesis, there exists $W \in \hat{\lambda}$ such that $W \in i_\lambda(f(G))$ and $W \cap f(A) = \emptyset$. This implies that $i_\mu(f^{-1}(W)) \in \mu$ and $i_\mu(f^{-1}(W)) \subset f^{-1}(i_\lambda(f(G))) = G$ and $i_\mu(f^{-1}(W)) \cap A = \emptyset$. Therefore, $A$ is a $\mu$-strongly nowhere dense set in $X$. \hfill \Box

The following Example 4.9 shows that the conditions feebly $(\mu, \lambda)$-continuous, feebly $(\mu, \lambda)$-open, injective on $f$ cannot be dropped in Theorem 4.8. The easy proof of the Corollary 4.10 below is omitted.

**Example 4.9.**

(a) Consider the two GTSs $(X_4, \mu)$ and $(X_3, \lambda)$ where $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\lambda = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Define a function $f : (X_4, \mu) \to (X_3, \lambda)$ by $f(a) = a, f(b) = c, f(c) = b, f(d) = c$. Clearly, $f$ is a feebly $(\mu, \lambda)$-continuous, feebly $(\mu, \lambda)$-open mapping. But $f$ is not injective. Let $B = \{c, d\} \subset X_4$. Then $B$ is a $\mu$-strongly nowhere dense set in $X_4$. If $V = \{c\} \in \lambda$, then there is no $U \in \hat{\lambda}$ such that $U \subset V$ and $U \cap f(B) = \emptyset$. Therefore, $f(B)$ is not a $\lambda$-strongly nowhere dense set in $X_3$.

(b) Consider the two GTSs $(X_4, \mu)$ and $(X_3, \lambda)$ where $\mu = \{\emptyset, \{c\}, \{b, d\}, \{b, c, d\}\}$ and $\lambda = \{\emptyset, \{a\}, \{a, b\}\}$. Define a function $f : (X_4, \mu) \to (X_3, \lambda)$ by $f(a) = c, f(b) = b, f(c) = a, f(d) = a$. Then $f$ is a feebly $(\mu, \lambda)$-continuous, feebly $(\mu, \lambda)$-open mapping. But $f$ is not injective. Let $B = \{b\} \subset X_3$. Then $B$ is a $\lambda$-strongly nowhere dense set in $X_3$. If $V = \{b, d\} \in \hat{\mu}$, then there is no $U \in \hat{\mu}$ such that $U \subset V$ and $U \cap f^{-1}(B) = \emptyset$. Therefore, $f^{-1}(B)$ is not a $\mu$-strongly nowhere dense set in $X_4$. 

...
(c) Consider the set $X_4$ with two generalized topologies $\mu, \lambda$ where $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\lambda = \{\emptyset, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$. Define a function $f : (X_4, \mu) \to (X_4, \lambda)$ by $f(a) = c$, $f(b) = a$, $f(c) = d$, $f(d) = b$. Then $f$ is an injective and feebly $(\mu, \lambda)$-open mapping. If $A = \{d\} \in \lambda$, then $i_\mu(f^{-1}(A)) = \emptyset$. Therefore, $f$ is not feebly $(\mu, \lambda)$-continuous. Let $B = \{c, d\} \subset X_4$. Then $B$ is a $\mu$-strongly nowhere dense set in $X_4$. If $V = \{d\} \in \lambda$, then there is no $U \subset \lambda$ such that $U \subset V$ and $U \cap f(B) = \emptyset$. Therefore, $f(B)$ is not a $\lambda$-strongly nowhere dense set in $X_4$.

(d) Consider the set $X_4$ with two generalized topologies $\mu, \lambda$ where $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\lambda = \{\emptyset, \{a\}, \{b\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$. Define a function $f : (X_4, \mu) \to (X_4, \lambda)$ by $f(a) = c$, $f(b) = a$, $f(c) = d$, $f(d) = b$. Then $f$ is an injective and feebly $(\mu, \lambda)$-open mapping. If $A = \{d\} \in \lambda$, then $i_\mu(f^{-1}(A)) = \emptyset$. Therefore, $f$ is not feebly $(\mu, \lambda)$-continuous. Let $B = \{a\} \subset X_4$. Then $B$ is a $\lambda$-strongly nowhere dense set in $X_4$. If $V = \{b, c\} \in \tilde{\mu}$, then there is no $U \subset \lambda$ such that $U \subset V$ and $U \cap f^{-1}(B) = \emptyset$. Therefore, $f^{-1}(B)$ is not a $\mu$-strongly nowhere dense set in $X_4$.

(e) Consider the set $X_4$ with two generalized topologies $\mu, \lambda$ where $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\lambda = \{\emptyset, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}\}$. Define a function $f : (X_4, \mu) \to (X_4, \lambda)$ by $f(a) = c$, $f(b) = a$, $f(c) = d$, $f(d) = b$. Then $f$ is an injective and feebly $(\mu, \lambda)$-continuous mapping. If $A = \{a\} \in \lambda$, then $i_\lambda(f(A)) = \emptyset$. Therefore, $f$ is not feebly $(\mu, \lambda)$-open. Let $B = \{c, d\} \subset X_4$. Then $B$ is a $\mu$-strongly nowhere dense set in $X_4$. If $V = \{b, c\} \in \lambda$, then there is no $U \subset \lambda$ such that $U \subset V$ and $U \cap f(B) = \emptyset$. Therefore, $f(B)$ is not a $\lambda$-strongly nowhere dense set in $X_4$.

(f) Consider the set $X_4$ with two generalized topologies $\mu, \lambda$ where $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\lambda = \{\emptyset, \{a\}, \{c\}, \{b\}, \{b, c\}\}$. Define a function $f : (X_4, \mu) \to (X_4, \lambda)$ by $f(a) = c$, $f(b) = d$, $f(c) = b$, $f(d) = a$. Then $f$ is an injective and feebly $(\mu, \lambda)$-continuous mapping. If $A = \{b\} \in \tilde{\mu}$, then $i_\lambda(f(A)) = \emptyset$. Therefore, $f$ is not feebly $(\mu, \lambda)$-open. Let $B = \{d\} \subset X_4$. Then $B$ is a $\lambda$-strongly nowhere dense set in $X_4$. If $V = \{b\} \in \tilde{\mu}$, then there is no $U \subset \lambda$ such that $U \subset V$ and $U \cap f^{-1}(B) = \emptyset$. Therefore, $f^{-1}(B)$ is not a $\mu$-strongly nowhere dense set in $X_4$.

**Corollary 4.10.** Let $(X, \mu)$ and $(Y, \lambda)$ be two GTSs and $f : (X, \mu) \to (Y, \lambda)$ be a bijective, $(\mu, \lambda)$-continuous and $(\mu, \lambda)$-open. Then the following hold.

(a) $A \subset X$ is of $\mu$-s-meager in $X$ if and only if $f(A)$ is of $\lambda$-s-meager in $Y$.

(b) $B \subset X$ is $\mu$-s-residual in $X$ if and only if $f(B)$ is $\lambda$-s-residual in $Y$.

(c) $C \subset Y$ is of $\lambda$-s-II category in $Y$ if and only if $f^{-1}(C)$ is of $\mu$-s-II category in $X$. 


5. Weak Baire Spaces

In this section, we discuss the properties of weak Baire spaces in generalized topological spaces. Also, we characterize weak Baire spaces in terms of dense and codense subsets. The following Theorem 5.1 characterizes weak Baire spaces in GTSs.

**Theorem 5.1.** Let \((X, \mu)\) be a GTS. Then the following are equivalent.

(a) \(X\) is wBS.

(b) If \(A\) is \(\mu\)-s-residual in \(X\), then \(A \in \mathcal{D}(\mu)\).

(c) If \(B\) is \(\mu\)-s-meager in \(X\), then \(B\) is \(\mu\)-codense.

**Proof.** (a) \(\Rightarrow\) (b). Let \(A\) be a \(\mu\)-s-residual set in \(X\). Then \(X - A = \bigcup_{n \in \mathbb{N}} A_n\) where each \(A_n\) is a \(\mu\)-strongly nowhere dense set. Let \(G \in \hat{\mu}\). Suppose \(A \cap G = \emptyset\). Then \(G \subset X - A\) and so \(G\) is a \(\mu\)-s-meager set in \(X\), by Theorem 3.1 which is a contradiction to \(X\) is wBS. Therefore, \(A \cap G \neq \emptyset\).

Hence \(A \in \mathcal{D}(\mu)\).

(b) \(\Rightarrow\) (c). Suppose \(B\) is \(\mu\)-s-meager in \(X\). Then \(X - B\) is a \(\mu\)-s-residual set in \(X\) and so \(X - B \in \mathcal{D}(\mu)\), by (b). Therefore, \(B\) is \(\mu\)-codense.

(c) \(\Rightarrow\) (a). Suppose that there exists a nonempty \(\mu\)-open subset \(U\) which is a \(\mu\)-s-meager set in \(X\). By (c), \(U\) is a \(\mu\)-codense set in \(X\). Then \(X - U\) is \(\mu\)-dense which implies that \((X - U) \cap U \neq \emptyset\) which is not possible. Therefore, \(U\) is of \(\mu\)-s-II category. Hence \(X\) is a wBS. \(\square\)

**Theorem 5.2.** Let \((X, \mu)\) be a wBS. Then the following hold and also they are equivalent.

(a) If \(\bigcap_{n \in \mathbb{N}} G_n \neq \emptyset\) is \(\mu\)-s-residual set where \(\{G_n : n \in \mathbb{N}\}\) is a family of subsets of \(X\), then \(G_n \in \mathcal{D}(\mu)\) for each \(n \in \mathbb{N}\).

(b) If \(\bigcup_{n \in \mathbb{N}} F_n \neq X\) is \(\mu\)-s-meager set where \(\{F_n : n \in \mathbb{N}\}\) is a family of subsets of \(X\), then \(F_n\) is \(\mu\)-codense for each \(n \in \mathbb{N}\).

**Proof.** (a). Suppose \(\bigcap_{n \in \mathbb{N}} G_n \neq \emptyset\) is \(\mu\)-s-residual set where \(\{G_n : n \in \mathbb{N}\}\) is a family of subsets of \(X\). Then by Corollary 3.2 (b), each \(G_n\) is a \(\mu\)-s-residual set. By Theorem 5.1, \(G_n \in \mathcal{D}(\mu)\) for each \(n \in \mathbb{N}\).

(b). Suppose \(\bigcup_{n \in \mathbb{N}} F_n \neq X\) is \(\mu\)-s-meager set where \(\{F_n : n \in \mathbb{N}\}\) is a family of subsets of \(X\). By Theorem 3.1, each \(F_n\) is a \(\mu\)-s-meager set. Then \(X - F_n\) is a \(\mu\)-s-residual set for each \(n \in \mathbb{N}\). By Theorem 5.1, \(X - F_n \in \mathcal{D}(\mu)\) for each \(n \in \mathbb{N}\). Therefore, \(F_n\) is \(\mu\)-codense for each \(n \in \mathbb{N}\).

(a) \(\Rightarrow\) (b). Suppose \(\bigcup_{n \in \mathbb{N}} F_n \neq X\) is \(\mu\)-s-meager set where \(\{F_n : n \in \mathbb{N}\}\) is a family of subsets of \(X\). Then \(X - \bigcup_{n \in \mathbb{N}} F_n \neq \emptyset\) and so \(\bigcap_{n \in \mathbb{N}} (X - F_n) \neq \emptyset\). Since each \(F_n\)
is a $\mu$-s-meager set, $X - F_n$ is a $\mu$-s-residual set for each $n \in \mathbb{N}$. By Theorem 3.3, $\bigcap_{n \in \mathbb{N}} (X - F_n)$ is a $\mu$-s-residual set. By (a), $X - F_n \in \mathcal{D}(\mu)$ for each $n \in \mathbb{N}$. Therefore, $F_n$ is $\mu$-codense for each $n \in \mathbb{N}$.

(b) $\Rightarrow$ (a). Suppose $\bigcap_{n \in \mathbb{N}} G_n \neq \emptyset$ is $\mu$-s-residual set where $\{G_n : n \in \mathbb{N}\}$ is a family of subsets of $X$. Then $X - \bigcap_{n \in \mathbb{N}} G_n \neq X$ and so $\bigcup_{n \in \mathbb{N}} (X - G_n) \neq X$. Since each $G_n$ is a $\mu$-s-residual set, $X - G_n$ is a $\mu$-s-meager set for each $n \in \mathbb{N}$. By Theorem 3.3, $\bigcup_{n \in \mathbb{N}} (X - G_n) \neq X$ is a $\mu$-s-meager set. By (b), each $X - G_n$ is $\mu$-codense and so $G_n \in \mathcal{D}(\mu)$ for each $n \in \mathbb{N}$.

\begin{theorem}
Let $(X, \mu)$ be a wBS and $(Y, \lambda)$ be a GTS. Let $f : (X, \mu) \to (Y, \lambda)$ be a feebly $(\mu, \lambda)$-continuous, injective mapping. Then the following hold.

(a) If $A$ is a $\mu$-s-meager set in $X$, then $f(A)$ is a $\lambda$-dense set in $Y$.

(b) If $f$ is a surjective mapping and $B$ is a $\mu$-s-residual set in $X$, then $f(B)$ is a $\lambda$-dense set in $Y$.
\end{theorem}

\begin{proof}
(a) Follows from Theorem 4.4 and Theorem 5.1.
(b) Suppose $B$ is a $\mu$-s-residual set in $X$. Then by (a), $f(X - B)$ is a $\lambda$-dense set in $Y$. This implies $Y - f(X - B)$ is a $\lambda$-dense set in $Y$ which implies $f(B)$ is a $\lambda$-dense set in $Y$.
\end{proof}

\begin{theorem}
Let $(X, \mu)$ and $(Y, \lambda)$ be two GTSs and $f : X \to Y$ be an feebly $(\mu, \lambda)$-open mapping. If $Y$ is a wBS, then the following hold.

(a) If $A$ is $\lambda$-s-meager in $Y$, then $f^{-1}(A)$ is $\mu$-dense in $X$.

(b) If $A$ is $\lambda$-s-residual in $Y$, then $f^{-1}(A) \in \mathcal{D}(\mu)$.
\end{theorem}

\begin{proof}
(a) Suppose $A$ is $\lambda$-s-meager in $Y$. By Theorem 5.1, $A$ is $\lambda$-codense. By Theorem 4.6, $f^{-1}(A)$ is $\mu$-codense in $X$.
(b) Suppose $A$ is $\lambda$-s-residual in $Y$. Then $Y - A$ is $\lambda$-s-meager in $Y$. By (a), $f^{-1}(Y - A)$ is $\mu$-codense in $X$. Now $f^{-1}(Y - A) = f^{-1}(Y) - f^{-1}(A) = X - f^{-1}(A)$. Therefore, $f^{-1}(A)$ is dense in $X$.
\end{proof}

\begin{theorem}
Let $(X, \mu)$ be a GTS. If every $\mu$-dense subset of $X$ is a wBS, then $X$ is a wBS.
\end{theorem}

\begin{proof}
Let $A$ be a $\mu$-dense subset of $X$. Suppose that there exists a nonempty $\mu$-open subset $U$ is a $\mu$-s-meager set in $X$. This implies $U \cap A$ is a $\mu$-s-meager set in $X$. Then by Theorem 3.6, $U \cap A$ is a $\mu_A$-s-meager set in $A$. Since $A$ is $\mu$-dense and $U \in \mu$, $U \cap A \in \mu_A$. Thus, $U \cap A$ is a nonempty $\mu_A$-open, $\mu_A$-s-meager set in $A$ and so $A$ is not a wBS, which is a contradiction to our hypothesis. Therefore, $U$ is of $\mu$-s-II category set. Hence $X$ is a wBS.
\end{proof}

\begin{theorem}
Let $(X, \mu)$ be the GT-sum of $\{(X_\alpha, \mu_\alpha) : \alpha \in \Gamma\}$ and $X$ be a wBS. If $A$ is $\mu$-s-residual, then $A \cap X_\alpha \in \mathcal{D}(\mu_\alpha)$ for any $\alpha \in \Gamma$.
\end{theorem}
Proof. Suppose \( A \) is a \( \mu \)-s-residual set in \( X \). By Theorem 5.1, \( A \) is a \( \mu \)-dense set in \( X \). By hypothesis and Lemma 2.5, \( A \cap X_\alpha \in \mathcal{D}(\mu_\alpha) \) for any \( \alpha \in \Gamma \). □

Theorem 5.7. Let \((X, \mu)\) and \((Y, \lambda)\) be two GTSs and \( f : X \to Y \) be an feebly \((\mu, \lambda)\)-open, feebly \((\mu, \lambda)\)-continuous, injective mapping. Then \((X, \mu)\) is a wBS if and only if \((Y, \lambda)\) is a wBS.

Proof. Suppose \( X \) is a wBS. Let \( V \in \mathcal{\lambda} \). Suppose \( V \) is a \( \lambda \)-s-meager set. Then \( V = \bigcup V_n \) where each \( V_n \) is a \( \lambda \)-strongly nowhere dense set. By hypothesis and Theorem 4.8, \( f^{-1}(V_n) \) is a \( \mu \)-strongly nowhere dense set in \( X \) for each \( n \in \mathbb{N} \). Therefore, \( f^{-1}(V) \) is a \( \mu \)-s-meager set which implies \( i_\mu f^{-1}(V) \) is a nonempty \( \mu \)-open, \( \mu \)-s-meager set in \( X \), since \( f \) is feebly \((\mu, \lambda)\)-continuous. Therefore, \( X \) is not a wBS, which is not possible. Hence \( Y \) is a wBS. Conversely, assume that \( Y \) is a wBS. Let \( U \in \mathcal{\mu} \). Suppose \( U \) is a \( \mu \)-s-meager set. Then \( U = \bigcup U_n \) where each \( U_n \) is a \( \mu \)-strongly nowhere dense set in \( X \). By hypothesis and Theorem 4.8, \( f(U_n) \) is a \( \lambda \)-strongly nowhere dense set in \( Y \) for each \( n \in \mathbb{N} \). Therefore, \( f(V) \) is a \( \lambda \)-s-meager set which implies \( i_\lambda f(V) \) is a nonempty \( \lambda \)-open, \( \lambda \)-s-meager set in \( Y \), since \( f \) is feebly \((\mu, \lambda)\)-open. Therefore, \( Y \) is not a wBS, which is not possible. Hence \( X \) is a wBS. □

The following Corollary 5.8 follows from Theorem 5.5 and Theorem 5.7.

Corollary 5.8. Let \((X, \mu)\) and \((Y, \lambda)\) be two GTSs and \( f : X \to Y \) be an feebly \((\mu, \lambda)\)-open, feebly \((\mu, \lambda)\)-continuous, injective mapping. Then the following hold.

(a) If every \( \mu \)-dense subset of \( X \) is a wBS, then \( Y \) is a wBS.

(b) If every \( \lambda \)-dense subset of \( Y \) is a wBS, then \( X \) is a wBS.

References


