Almost Ricci Soliton and Gradient Almost Ricci Soliton on 3-dimensional $f$-Kenmotsu Manifolds

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Abstract. The object of the present paper is to study almost Ricci solitons and gradient almost Ricci solitons in 3-dimensional $f$-Kenmotsu manifolds.

1. Introduction

The study of almost Ricci soliton was introduced by Pigola et al. [18], where essentially they modified the definition of Ricci soliton by adding the condition on the parameter $\lambda$ to be a variable function, more precisely, we say that a Riemannian manifold $(M^n, g)$ admits an almost Ricci soliton, if there exists a complete vector field $V$, called potential vector field and a smooth soliton function $\lambda : M^n \rightarrow \mathbb{R}$ satisfying

\[ \text{Ric} + \frac{1}{2} \mathcal{L}_V g = \lambda g, \]

where $\text{Ric}$ and $\mathcal{L}$ stand, respectively, for the Ricci tensor and Lie derivative. We shall refer to this equation as the fundamental equation of an almost Ricci soliton $(M^n, g, V, \lambda)$. It will be called expanding, steady or shrinking, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Otherwise it will be called indefinite. When the vector field $V$ is gradient of a smooth function $f : M^n \rightarrow \mathbb{R}$ the metric will be called gradient almost Ricci soliton. In this case the preceding equation becomes

\[ \text{Ric} + \nabla^2 f = \lambda g, \]
where $\nabla^2 f$ stands for the Hessian of $f$. Sometimes classical theory of tensorial calculus is more convenient to make computations. Then, we can write the fundamental equation in this language as follows:

\[(1.3) \quad R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}.\]

Moreover, if the vector field $X$ is trivial, or the potential $f$ is constant, the almost Ricci soliton will be called trivial, otherwise it will be a non-trivial almost Ricci soliton. We notice that when $n \geq 3$ and $X$ is a Killing vector field an almost Ricci soliton will be a Ricci soliton, since in this case we have an Einstein manifold, from which we can apply Schur’s lemma to deduce that $\lambda$ is constant. Taking into account that the soliton function $\lambda$ is not necessarily constant, certainly comparison with soliton theory will be modified. In particular the rigidity result contained in Theorem 1.3 of [18] indicates that almost Ricci solitons should reveal a reasonably broad generalization of the fruitful concept of classical soliton. In fact, we refer the reader to [18] to see some of this changes.

In the direction to understand the geometry of almost Ricci soliton, Barros and Ribeiro Jr. proved in [2] that a compact gradient almost Ricci soliton with non-trivial conformal vector field is isometric to a Euclidean sphere. In the same paper they proved an integral formula for compact case, which was used to prove several rigidity results, for more details see [2].

The existence of Ricci almost soliton has been confirmed by Pigola et. al. [18] on some certain class of warped product manifolds. Some characterization of Ricci almost soliton on a compact Riemannian manifold can be found in ([1], [2], [3]). It is interesting to note that if the potential vector field $V$ of the Ricci almost soliton $(M, g, V, \lambda)$ is Killing then the soliton becomes trivial, provided the dimension of $M > 2$. Moreover, if $V$ is conformal then $M^n$ is isometric to Euclidean sphere $S^n$. Thus the Ricci almost soliton can be considered as a generalization of Einstein metric as well as Ricci soliton.

In [6], authors studied Ricci solitons and gradient Ricci solitons on 3-dimensional normal almost contact metric manifolds. In [10] authors studied compact Ricci soliton. Beside these, A. Ghosh [12] studied $K$-contact and Sasakian manifolds whose metric is gradient almost Ricci solitons. Conditions of $K$-contact and Sasakian manifolds are more stronger than normal almost contact metric manifolds in the sense that the 1-form $\eta$ of normal almost contact metric manifolds are not contact form. The Ricci soliton and gradient Ricci soliton have been studied by several authors such as ([5], [7], [9]) and many others.

The present paper is organized as follows:

After preliminaries, in section 3 we study almost Ricci soliton in 3-dimensional $f$-Kenmotsu manifolds. Finally, we consider gradient almost Ricci solitons in 3-dimensional $f$-Kenmotsu manifolds.

2. Preliminaries

Let $M$ be an almost contact manifold, i.e., $M$ is a connected $(2n+1)$-dimensional...
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A differentiable manifold endowed with an almost contact metric structure \((\phi, \xi, \eta, g)\) [4]. As usually, denote by \(\Phi\) the fundamental 2-form of \(M\), \(\Phi(X, Y) = g(X, \phi Y)\), \(X, Y \in \chi(M)\), \(\chi(M)\) being the Lie algebra of differentiable vector fields on \(M\).

For further use, we recall the following definitions ([4], [11], [19]). The manifold \(M\) and its structure \((\phi, \xi, \eta, g)\) is said to be:

(i) normal if the almost complex structure defined on the product manifold \(M \times \mathbb{R}\) is integrable (equivalently \([\phi, \phi] + 2 d\eta \otimes \xi = 0\)),

(ii) almost cosymplectic if \(d\eta = 0\) and \(d\Phi = 0\),

(iii) cosymplectic if it is normal and almost cosymplectic (equivalently, \(\nabla \phi = 0\), \(\nabla\) being covariant differentiation with respect to the Levi-Civita connection).

The manifold \(M\) is called locally conformal cosymplectic (respectively, almost cosymplectic) if \(M\) has an open covering \(\{U_t\}\) endowed with differentiable functions \(\sigma_t: U_t \rightarrow \mathbb{R}\) such that over each \(U_t\) the almost contact metric structure \((\phi_t, \xi_t, \eta_t, g_t)\) defined by

\[
\phi_t = \phi, \quad \xi_t = e^{\sigma_t} \xi, \quad \eta_t = e^{-\sigma_t} \eta, \quad g_t = e^{-2\sigma_t} g
\]

is cosymplectic (respectively, almost cosymplectic).

Olszak and Rosca [16] studied normal locally conformal almost cosymplectic manifold. They gave a geometric interpretation of \(f\)-Kenmotsu manifolds and studied some curvature properties. Among others they proved that a Ricci symmetric \(f\)-Kenmotsu manifold is an Einstein manifold.

By an \(f\)-Kenmotsu manifold we mean an almost contact metric manifold which is normal and locally conformal almost cosymplectic.

Let \(M\) be a real \((2n + 1)\)-dimensional differentiable manifold endowed with an almost contact structure \((\phi, \xi, \eta, g)\) satisfying

\[
\begin{align*}
\phi^2 &= -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \\
\phi \xi &= 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi), \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X) \eta(Y),
\end{align*}
\]

for any vector fields \(X, Y \in \chi(M)\), where \(I\) is the identity of the tangent bundle \(TM\), \(\phi\) is a tensor field of \((1,1)\)-type, \(\eta\) is a 1-form, \(\xi\) is a vector field and \(g\) is a metric tensor field. We say that \((M, \phi, \xi, \eta, g)\) is an \(f\)-Kenmotsu manifold if the covariant differentiation of \(\phi\) satisfies [15]:

\[
(\nabla_X \phi)(Y) = f(g(\phi X, Y) \xi - \eta(Y) \phi X),
\]

where \(f \in C^\infty(M)\) such that \(df \wedge \eta = 0\). If \(f = \alpha = \text{constant} \neq 0\), then the manifold is a \(\alpha\)-Kenmotsu manifold [13]. 1-Kenmotsu manifold is a Kenmotsu manifold ([14], [17]). If \(f = 0\), then the manifold is cosymplectic [13]. An \(f\)-Kenmotsu manifold is said to be regular if \(f^2 + f' \neq 0\), where \(f' = \xi f\).
For an $f$-Kenmotsu manifold from (2.2) it follows that

\[(2.3) \nabla_X \xi = f\{X - \eta(X)\xi\}.\]

The condition $df \wedge \eta = 0$ holds if $\dim M \geq 5$. In general this does not hold if $\dim M = 3$ [16].

In a 3-dimensional Riemannian manifold, we always have

\[(2.4) R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}.\]

In a 3-dimensional $f$-Kenmotsu manifold we have [16]

\[(2.5) R(X,Y)Z = \left(\frac{r}{2} + 2f^2 + 2f'\right)(X \wedge Y)Z - \left(\frac{r}{2} + 3f^2 + 3f'\right)\{\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z\},\]

\[(2.6) S(X,Y) = \left(\frac{r}{2} + f^2 + f'\right)g(X,Y) - \left(\frac{r}{2} + 3f^2 + 3f'\right)\eta(X)\eta(Y),\]

where $r$ is the scalar curvature of $M$ and $f' = \xi(f)$.

From (2.5), we obtain

\[(2.7) R(X,Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y],\]

and (2.6) yields

\[(2.8) S(X,\xi) = -2(f^2 + f')\eta(X).\]

**Example.** ([8]) We consider the three-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where $(x,y,z)$ are the standard coordinates in $\mathbb{R}^3$. The vector fields

\[e_1 = z^2 \frac{\partial}{\partial x}, \quad e_2 = z^2 \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}\]

are linearly independent at each point of $M$. Let $g$ be the Riemannian metric defined by

\[g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0, \quad g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1.\]

Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z,e_3)$ for any $Z \in \chi(M)$. Let $\phi$ be the $(1,1)$ tensor field defined by $\phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3) = 0$. 

Then using linearity of $\phi$ and $g$ we have
\[
\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \\
g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),
\]
for any $Z, W \in \chi(M)$. Now, by direct computations we obtain
\[
[e_1, e_2] = 0, \quad [e_2, e_3] = -\frac{2}{z}e_2, \quad [e_1, e_3] = -\frac{2}{z}e_1.
\]

The Riemannian connection $\nabla$ of the metric tensor $g$ is given by the Koszul’s formula which is
\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\
- g([X, Y], Z) - g(Y, [X, Z]) + g(Z, [X, Y]).
\]

Using (2.9) we have
\[
2g(\nabla_{e_1} e_3, e_1) = 2g(-\frac{2}{z} e_1, e_1),
\]
\[
2g(\nabla_{e_1} e_3, e_2) = 0 \quad \text{and} \quad 2g(\nabla_{e_1} e_3, e_3) = 0.
\]
Hence $\nabla_{e_1} e_3 = -\frac{2}{z} e_1$. Similarly, $\nabla_{e_2} e_3 = -\frac{2}{z} e_2$ and $\nabla_{e_3} e_3 = 0$. (2.9) further yields
\[
\nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = \frac{2}{z} e_3, \\
\nabla_{e_2} e_2 = \frac{2}{z} e_3, \quad \nabla_{e_2} e_1 = 0, \\
\nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0.
\]

From the above it follows that the manifold satisfies $\nabla_X \xi = f\{X - \eta(X)\xi\}$ for $\xi = e_3$, where $f = -\frac{2}{z}$. Hence we conclude that $M$ is an $f$-Kenmotsu manifold. Also $f^2 + f' \neq 0$. Hence $M$ is a regular $f$-Kenmotsu manifold.

3. Almost Ricci Soliton

In this section we consider almost Ricci solitons on 3-dimensional $f$-Kenmotsu manifolds. In particular, let the potential vector field $V$ be pointwise collinear with $\xi$ i.e., $V = b\xi$, where $b$ is a function on $M$. Then from (1.1) we have
\[
g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X, Y) = 2\lambda g(X, Y).
\]
Using (2.3) in (3.1), we get
\[
2fb[g(X, Y) - \eta(X)\eta(Y)] + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X, Y) = 2\lambda g(X, Y).
\]
Putting $Y = \xi$ in (3.2) and using (2.8) yields
\[
(Xb) + (\xi b)\eta(X) - 4(f^2 + f')\eta(X) = 2\lambda \eta(X).
\]
Putting \( X = \xi \) in (3.3) we obtain
\[
(3.4) \quad \xi b = 2(f^2 + f') + \lambda.
\]

Putting the value of \( \xi b \) in (3.3) yields
\[
(3.5) \quad db = [\lambda + 2(f^2 + f')]\eta.
\]

Applying \( d \) on (3.5) and using \( d^2 = 0 \), we get
\[
(3.6) \quad 0 = d^2 b = [\lambda + 2(f^2 + f')]d\eta.
\]
Taking wedge product of (3.6) with \( \eta \), we have
\[
(3.7) \quad [\lambda + 2(f^2 + f')]\eta \wedge d\eta = 0.
\]
Since \( \eta \wedge d\eta \neq 0 \) in a 3-dimensional \( f \)-Kenmotsu manifold, therefore
\[
(3.8) \quad \lambda + 2(f^2 + f') = 0 \Rightarrow \lambda = -2(f^2 + f').
\]
Using (3.8) in (3.5) gives \( db = 0 \) i.e., \( b = \text{constant} \). Therefore from (3.2) we have
\[
(3.9) \quad S(X,Y) = (\lambda - fb)g(X,Y) + fb\eta(X)\eta(Y).
\]

In view of (3.9) we can state the following:

**Theorem 3.1.** If in a 3-dimensional \( f \)-Kenmotsu manifold the metric \( g \) admits almost Ricci soliton and \( V \) is pointwise collinear with \( \xi \), then \( V \) is constant multiple of \( \xi \) and the manifold is \( \eta \)-Einstein of the form (3.9).

The converse of the above theorem is not true, in general. However if we take \( f = \text{constant} \), i.e., if we consider a 3-dimensional \( \eta \)-Einstein \( f \)-Kenmotsu manifold, then it admits a Ricci soliton. This can be proved as follows:

Let \( M \) be a 3-dimensional \( \eta \)-Einstein \( f \)-Kenmotsu manifold and \( V = \xi \). Then
\[
(3.10) \quad S(X,Y) = \gamma g(X,Y) + \delta \eta(X)\eta(Y),
\]
where \( \gamma \) and \( \delta \) are certain scalars.

Now using (2.3)
\[
(\mathcal{L}_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)
= 2f\{g(X,Y) - \eta(X)\eta(Y)\}.
\]
Therefore
\[
(3.11) \quad (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) - 2\lambda g(X,Y) = 2(f + \gamma - \lambda)g(X,Y)
-2(f - \delta)\eta(X)\eta(Y).
\]
From equation (3.11) it follows that \( M \) admits a Ricci soliton \((g, \xi, \lambda)\) if \( f + \gamma - \lambda = 0 \) and \( \delta = f = \text{constant} \). From (3.10) we have using (2.8), \(-2f^2 = \gamma + \delta\). Hence \( \gamma = -2f^2 - f = \text{constant} \). Therefore \( \lambda = (\gamma + \delta) = \text{constant} \). So we have the following:

**Theorem 3.2.** If a 3-dimensional \( f \)-Kenmotsu manifold is \( \eta \)-Einstein of the form \( S = \gamma g + \delta \eta \otimes \eta \), then a Ricci almost soliton \((M, g, \xi, \lambda)\) reduces to a Ricci soliton \((g, \xi, (\gamma + \delta))\).

Now let \( V = \xi \). Then (3.1) reduces to

\[
(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) = 2\lambda g(X, Y). \tag{3.12}
\]

Now, in view of (2.6) we have

\[
(\mathcal{L}_\xi g)(X, Y) = -2 \left[ \left( \frac{r}{2} + f^2 + f' \right) g(X, Y) \right. \\
- \left. \left( \frac{r}{2} + 3f^2 + 3f' \right) \eta(X) \eta(Y) \right] + 2\lambda g(X, Y). \tag{3.13}
\]

\[
2f \{g(X, Y) - \eta(X) \eta(Y)\} = 2\lambda g(X, Y) - 2 \left[ \left( \frac{r}{2} + f^2 + f' \right) g(X, Y) \right. \\
- \left. \left( \frac{r}{2} + 3f^2 + 3f' \right) \eta(X) \eta(Y) \right]. \tag{3.14}
\]

Putting \( X = Y = \xi \) in (3.14) yields

\[
\lambda = 4(f^2 + f'). \tag{3.15}
\]

Assuming that \( f = \text{constant} \), we get \( f' = \xi f = 0 \). This implies \( \lambda = 4f^2 = \text{constant} \). Thus we can state the following:

**Theorem 3.3.** If a 3-dimensional \( f \)-Kenmotsu manifold with \( f = \text{constant} \) admits almost Ricci soliton then it reduces to a Ricci soliton.

4. **Gradient Almost Ricci Soliton**

This section is devoted to study 3-dimensional \( f \)-Kenmotsu manifolds admitting gradient almost Ricci soliton. For a gradient almost Ricci soliton, we have

\[
\nabla_Y Df = \lambda Y - QY, \tag{4.1}
\]

where \( D \) denotes the gradient operator of \( g \).

Differentiating (4.1) covariantly in the direction of \( X \) yields

\[
\nabla_X \nabla_Y Df = d\lambda(X)Y + \lambda \nabla_X Y - (\nabla_X Q)Y. \tag{4.2}
\]
Similarly, we get
\[ \nabla_Y \nabla_X Df = d\lambda(Y)X + \lambda \nabla_Y X - (\nabla_Y Q)X, \] (4.3)
and
\[ \nabla_{[X,Y]} Df = \lambda [X,Y] - Q[X,Y]. \] (4.4)

In view of (4.2), (4.3) and (4.4), we have
\[ R(X,Y) Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df \]
\[ = (\nabla_Y Q)X - (\nabla_X Q)Y - (Y\lambda)X + (X\lambda)Y. \] (4.5)

We get from (2.6)
\[ QY = \left( \frac{r}{2} + f^2 + f' \right) Y - \left( \frac{r}{2} + 3f^2 + 3f' \right) \eta(Y)\xi, \] (4.6)

Differentiating (4.6) covariantly in the direction of \( X \) and using (2.3), we get
\[ (\nabla_X Q)Y = \left\{ \left( \frac{Xr}{2} + 2f(Xf) + (Xf') \right) \right\} Y \]
\[ - \left\{ \left( \frac{Xr}{2} + 6f(Xf) + 3(Xf') \right) \right\} \{ f\eta(X)\xi + f\eta(Y)X - 2f\eta(X)\eta(Y)\xi \}. \] (4.7)

In view of (4.7), we get from (4.5)
\[ R(X,Y) Df \]
\[ = \left\{ \left( \frac{Yr}{2} + 2f(Yf) + (Yf') \right) \right\} X - \left\{ \left( \frac{Xr}{2} + 2f(Xf) + (Xf') \right) \right\} Y \]
\[ - \left\{ \left( \frac{Yr}{2} + 6f(Yf) + 3(Yf') \right) \right\} \{ f\eta(X)\xi + f\eta(Y)X - 2f\eta(X)\eta(Y)\xi \} \]
\[ + \left\{ \left( \frac{Xr}{2} + 6f(Xf) + 3(Xf') \right) \right\} \{ f\eta(X)\xi + f\eta(Y)X - 2f\eta(X)\eta(Y)\xi \} \]
\[ - (Y\lambda)X + (X\lambda)Y. \] (4.8)

This implies
\[ g(R(X,\xi) Df, \xi) = \left\{ \left( \frac{\xi r}{2} + 2f(\xi f) + (\xi f') \right) \right\} \eta(X) \]
\[ - \left\{ \left( \frac{Xr}{2} + 2f(Xf) + (Xf') \right) \right\} \]
\[ - (\xi \lambda)\eta(X) + (X\lambda). \] (4.9)

Also, we have from (2.5)
\[ g(R(X,\xi) Df, \xi) = (f^2 + f') \{ (Xf) - (\xi f)\eta(X) \}. \] (4.10)
In view of (4.9) and (4.10) we obtain

\[
(f^2 + f')\{(Xf) - (\xi f)\eta(X)\} = \left\{\frac{\xi r}{2} + 2f(\xi f) + (\xi f')\right\}\eta(X)
- \left\{\frac{(Xr)}{2} + 2f(Xf) + (Xf')\right\}
- (\xi \lambda)\eta(X) + (X\lambda).
\]

(4.11)

Assuming that the scalar curvature \(r\) and \(f\) are constants. Then it follows from (4.11) that

(12) \hspace{1cm} d\lambda - (\xi \lambda)\eta = 0.

Applying \(d\) both sides of (12), we get

(13) \hspace{1cm} \xi \lambda = 0.

Using (13) in (12), we have

(14) \hspace{1cm} d\lambda = 0.

This implies \(\lambda = \text{constant}\). Thus we can state the following:

**Theorem 4.1.** If a 3-dimensional \(f\)-Kenmotsu manifold admits gradient almost Ricci soliton then it reduces to a Ricci soliton provided the scalar curvature \(r\) and \(f\) are constants.

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**References**


