Canal Surfaces in Galilean 3-Spaces

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Abstract. In this paper, we defined the admissible canal surfaces with isotropic radial vector in Galilean 3-spaces and obtained their position vectors. Also we gave some important results by using their Gauss and mean curvatures.

1. Introduction

A canal surface is defined as envelope of a one-parameter set of spheres, centered at a spine curve $\gamma(s)$ with radius $r(s)$. When $r(s)$ is a constant function, the canal surface is the envelope of a moving sphere and is called a pipe surface. Canal surfaces have wide applications in CAGD, such as construction of blending surfaces, shape reconstruction, transition surfaces between pipes, robotic path planning, etc. An envelope of a 1-parameter family of surfaces is constructed in the same way that we constructed a 1-parameter family of curves. The family is described by a differentiable function $F(x, y, z, \lambda) = 0$, where $\lambda$ is a parameter. When $\lambda$ can be eliminated from the equations

$$F(x, y, z, \lambda) = 0$$

and

$$\frac{\partial F(x, y, z, \lambda)}{\partial \lambda} = 0,$$

we get the envelope, which is a surface described implicitly as $G(x, y, z) = 0$. For example, for a 1-parameter family of planes we get a develople surface([1], [2], [3], [5], [7] and [9]).

A general canal surface is an envelope of a 1-parameter family of surfaces. The envelope of a 1-parameter family $s \mapsto S^2(s)$ of spheres in $IR^3$ is called a general canal surface [3]. The curve formed by the centers of the spheres is called center curve of the canal surface. The radius of general canal surface is the function $r$ such that $r(s)$ is the radius of the sphere $S^2(s)$. Suppose that the center curve of...
a canal surface is a unit speed curve $\alpha : I \to \mathbb{R}^3$. Then the general canal surface can be parametrized by the formula

$$C(s, t) = \alpha(s) - R(s)T - Q(s)\cos(t)N + Q(s)\sin(t)B,$$

where

$$R(s) = r(s)r'(s),$$

$$Q(s) = \pm r(s)\sqrt{1 - r'(s)^2}.$$

All the tubes and the surfaces of revolution are subclass of the general canal surface.

**Theorem 1.1** ([3]). Let $M$ be a canal surface. The center curve of $M$ is a straight line if and only if $M$ is a surface of revolution for which no normal line to the surface is parallel to the axis of revolution. The following conditions are equivalent for a canal surface $M$:

i. $M$ is a tube parametrized by (1.1);

ii. the radius of $M$ is constant;

iii. the radius vector of each sphere in family that defines the canal surface $M$ meets the center curve orthogonally.

**2. Canal Surfaces in Galilean Space**

The Galilean space $G_3$ is a Cayley-Klein space defined from a 3-dimensional projective space $P(\mathbb{R}3)$ with the absolute figure that consists of an ordered triple $\{\omega, f, I\}$, where $\omega$ is the ideal (absolute) plane, $f$ the line (absolute line) in $\omega$ and $I$ the fixed elliptic involution of points off. We introduce homogeneous coordinates in $G_3$ in such a way that the absolute plane $\omega$ is given by $x_0 = 0$, the absolute line $f$ by $x_0 = x_1 = 0$ and the elliptic involution by $(0 : 0 : x_2 : x_3) \mapsto (0 : 0 : x_3 : -x_2)$. With respect to the absolute figure, there are two types of lines in the Galilean space, isotropic lines which intersect the absolute line $f$ and non-isotropic lines which do not. A plane is called Euclidean if it contains $f$, otherwise it is called isotropic. In the given affine coordinates, isotropic vectors are of the form $(0, y, z)$, whereas Euclidean planes are of the form $x = k, k \in \mathbb{R}$.

The scalar product in Galilean space $G_3$ is defined by

$$g(A, B) = \begin{cases} a_1b_1, & \text{if } a_1 \neq 0 \lor b_1 \neq 0, \\ a_2b_2 + a_3b_3, & \text{if } a_1 = 0 \land b_1 = 0, \end{cases}$$

where $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. The Galilean cross product is defined by

$$A \wedge_{G_3} B = \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \text{ if } a_1 \neq 0 \lor b_1 \neq 0.$$
The unit Galilean sphere is defined by

\[ S^2_\pm = \{ X \in G_3 \mid g(X, X) = \mp r^2 \}. \]

An admissible curve \( \alpha : I \subseteq \mathbb{R} \to G_3 \) in the Galilean space \( G_3 \) which parameterized by the arc length \( s \) defined by

\[ \alpha(s) = (s, y(s), z(s)), \]

where \( s \) is a Galilean invariant and the arc length on \( \alpha \). The curvature \( \kappa(s) \) and the torsion \( \tau(s) \) are defined by

\[ \kappa(s) = \sqrt{(y''(s))^2 + (z''(s))^2}, \quad \tau(x) = \frac{\det (\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa^2(s)}. \]

The orthonormal frame in the sense of Galilean space \( G_3 \) is defined by

\[ T(s) = \alpha'(s) = (1, y'(s), z'(s)), \]
\[ N(s) = \frac{1}{\kappa(s)} (0, y''(s), z''(s)), \]
\[ B(s) = \frac{1}{\kappa(s)} (0, -z''(s), y''(s)). \]

The vectors \( T, N \) and \( B \) in (2.3) are called the vectors of the tangent, principal normal and the binormal line of \( \alpha \), respectively. They satisfy the following Frenet equations

\[ T' = \kappa N, \quad N' = \tau B, \quad B' = -\tau N. \]

A \( C^r \)-surface \( M, r \geq 1 \), immersed in the Galilean space, \( x : U \to M, U \subset \mathbb{R}^2 \),

\[ x(u, v) = (x(u, v), y(u, v), z(u, v)), \]

has the following first fundamental form

\[ I = (g_1 du + g_2 dv)^2 + \epsilon(h_{11} du^2 + 2h_{12} du dv + h_{22} dv^2), \]

where the symbols \( g_i = x_i \) and \( h_{ij} = g(\tilde{x}_i, \tilde{x}_j) \) stand for derivatives of the first coordinate function \( x(u, v) \) with respect to \( u, v \) and for the Euclidean scalar product of the projections \( \tilde{x}_k \) of vectors \( x_k \) onto the \( yz \)-plane, respectively. Furthermore,

\[ \epsilon = \begin{cases} 0, & \text{if direction } du : dv \text{ is non-isotropic,} \\ 1, & \text{if direction } du : dv \text{ is isotropic.} \end{cases} \]

In every point of a surface there exists a unique isotropic direction defined by \( g_1 du + g_2 dv = 0 \). In that direction, the arc length is measured by

\[ ds^2 = \frac{1}{(g_1)^2} \left\{ h_{11} (g_2)^2 - 2h_{12} g_1 g_2 + h_{22} (g_1)^2 \right\} dv^2 = \frac{W^2}{(g_1)^2} dv^2, \quad g_1 \neq 0, \]
where
\[ h_{11} = \frac{x_1^2}{W^2}, \quad h_{12} = -\frac{x_1x_2}{W^2}, \quad h_{12} = \frac{x_2^2}{W^2}, \]
\[ x_1 = \frac{\partial x}{\partial u}, \quad x_2 = \frac{\partial x}{\partial v}, \quad W^2 = (x_2x_1 - x_1x_2)^2. \]

A surface is called *admissible* if it has no Euclidean tangent planes. Therefore, for an admissible surface \( g_1 \neq 0 \) or \( g_2 \neq 0 \) holds. An admissible surface can always locally be expressed as \( z = f(x, y) \).

The Gaussian \( K \) and mean curvature \( H \) are \( C^{r-2} \)-functions, \( r \geq 2 \), defined by
\[ K = \frac{L_{11}L_{22} - L_{12}^2}{W^2}, \quad H = \frac{(g_2)^2 L_{11} - 2g_1g_2L_{12} + (g_1)^2 L_{22}}{2W^2}, \]
where
\[ L_{ij} = g \left( \frac{x_1x_{ij} - x_{ij}x_1}{x_1}, \eta \right), \quad x_1 = g_1 \neq 0. \]

The unit normal vector \( \eta \) given by an isotropic vector is defined by
\[ \eta = \frac{x_1 \land G_3 x_2}{W} = \frac{1}{W}(0, -x_2z_1 + x_1z_2, x_2y_1 - x_1y_2) \]
([4], [6], [8]).

In Galilean geometry, there are two types sphere depending radious vector whether it is a isotropic or non-isotropic. Spheres with non-isotropic radious vector are Euclidean circles in yoz-plane and spheres with isotropic radious vector are parallel planes such as \( x = \pm r \). We denote the Euclidean circles by \( S^1_{\pm}(s) \).

**Definition 2.1.** The envelope of a 1-parameter family \( s \rightarrow S^1_{\pm}(s) \) of the circles in \( G_3 \) is called a *canal surface* in Galilean 3-space. The curve formed by the centers of the Euclidean circles is called *center curve* of the canal surface. The radius of the canal surface is the function \( r \) such that \( r(s) \) is the radius of the Euclidean circles \( S^1_{\pm}(s) \).

Let \( \gamma(s) \) be an admissible curve as centered curve and canal surface is a patch that parametrizes the envelope of Euclidean circles which can be defined as
\[ C(s, t) = \gamma(s) + \psi(s, t) T(s) + \varphi(s, t) N(s) + \omega(s, t) B(s) \]
with the regularity conditions \( C_s \neq 0, C_t \neq 0 \) and \( C_s \times C_t \neq 0 \), where \( \varphi(s, t) \) and \( \omega(s, t) \) are \( C^\infty \)-functions of \( s \) and \( t \). Since \( C(s, t) - \gamma(s) \) is the surface normal of \( S^1_{\pm}(s) \) and \( C(s, t) \) is non-isotropic then \( \psi(s, t) = 0 \) and
\[ g(C(s, t) - \gamma(s), C(s, t) - \gamma(s)) = \varphi(s, t)^2 + \omega(s, t)^2 = r(s)^2 \]
and by differentiating (2.6) with respect to \( s \) and \( t \) we get
\[ \varphi_t(s, t) \varphi(s, t) + \omega_t(s, t) \omega(s, t) = 0, \]
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\[
\varphi_s(s,t) \varphi(s,t) + \omega_s(s,t) \omega(s,t) = r'(s) r(s),
\]

\[
g(C(s,t) - \gamma(s), C_s(s,t)) = 0,
\]

\[
g(C(s,t) - \gamma(s), C_t(s,t)) = 0,
\]

and also we find the functions \( \varphi(s,t) \) and \( \omega(s,t) \) are

\[
\varphi(s,t) = r(s) \cos(t), \quad \omega(s,t) = r(s) \sin(t)
\]

by using (2.6), (2.7) and (2.8). Thus, we give the following corollary.

**Corollary 2.2.** Let \( \gamma(s) \) be an admissible curve. Then the position vector of canal surface with isotropic radius vector and centered curve \( \gamma(s) \) is

\[
C(s,t) = \gamma(s) + r(s) \cos(t) N(s) + r(s) \sin(t) B(s).
\]

The natural basis \( \{C_s, C_t\} \) are given by

\[
C_s = T + \{r' \cos(t) - r \tau \sin(t)\} N + \{r' \sin(t) + r \tau \cos(t)\} B,
\]

\[
C_t = -r \sin(t) N + r \cos(t) B.
\]

From (2.4) and (2.12), the components \( h_{ij} \) and \( g_i \) are

\[
h_{11} = (r'(s))^2 + r^2(s) \tau^2(s), \quad h_{12} = r^2(s) \tau(s), \quad h_{22} = r^2(s),
\]

\[
g_1 = 1, \quad g_2 = 0.
\]

Thus, the first fundamental form of canal surface is

\[
I_C = \left(1 + (r'(s))^2 + r^2(s) \tau^2(s)\right) du^2 + 2r^2(s) \tau(s) du dv + r^2(s) dv^2.
\]

By using (2.4), the second derivations of (2.12)

\[
C_{ss} = \left\{k + (r'' - r \tau^2) \cos(t) - (2r' \tau + r \tau') \sin(t)\right\} N
\]

\[
+ \left\{(2r' \tau + r \tau') \cos(t) + (r'' - r \tau^2) \sin(t)\right\} B
\]

\[
C_{tt} = -r \cos(t) N - r \sin(t) B
\]

\[
C_{st} = -(r' \sin(t) + r \tau \cos(t)) N + (r' \cos(t) - r \tau \sin(t)) B
\]

and the unit normal vector

\[
\eta(s,t) = -\cos(t) N(s) - \sin(t) B(s)
\]

coefficients \( L_{ij} \) are
\[
L_{11} = - \left\{ \kappa(s) \cos(t) + r''(s) - r(s) \tau^2(s) \right\}, \quad L_{12} = r(s) \tau(s), \quad L_{22} = r(s)
\]
so the second fundamental form is
\[
II_C = - \left\{ \kappa(s) \cos(t) + r''(s) - r(s) \tau^2(s) \right\} \, du^2 + 2r(s) \tau(s) \, dudv + r(s) \, dv^2.
\]
The Gauss curvature and mean curvature of a non-isotropic canal surface in the Galilean space are given by
\[
K(s,t) = r''(s) - \kappa(s) \cos(t) r(s), \quad H(s,t) = \frac{1}{2r(s)}.
\]
In the case \(K(s,t) = 0\), the centered curve has to be planar and there are two K-flat canal surfaces for \(r(s) = c_1 s + c_2\) and \(r(s) = c\). Thus, we conclude the following cases.

**Theorem 2.3.** Let \(M\) be a canal surface in Galilean 3-space. Then followings are true.

i. There is no minimal canal surface.

ii. The Gauss and mean curvatures of canal surface satisfy the relation
\[
K(s,t) + 2H(s,t) \{ \kappa(s) \cos(t) - r''(s) \} = 0,
\]

iii. \(M\) is a \(K\)-flat canal surface if and only if \(M\) is a elliptic cone and its position vector is
\[
C(s,t) = (s, (c_1 s + c_2)(c_3 \cos(t) \mp \sqrt{1 - (c_3)^2} \sin(t))
, (c_1 s + c_2)(\mp \sqrt{1 - (c_3)^2} \cos(t) - c_3 \sin(t))),
\]
where \(c_1 \neq 0, c_2 \in IR, c_3 \in [0,1]\), see Figure 1(a).

iv. \(M\) is a \(K\)-flat tubular surface if and only if \(M\) is a elliptic cylinder and its position vector is
\[
C(s,t) = (s, c_1 c_2 \cos(t) \mp c_1 \sqrt{1 - (c_2)^2} \sin(t), \mp c_1 \sqrt{1 - (c_2)^2} \cos(t) - c_1 c_2 \sin(t)),
\]
where \(c_1 \in IR^+, c_2 \in [0,1]\), see Figure 1(b).

v. All the tubes are surface with constant mean curvature.

On the other hand, a surface is said to be a Weingarten surface if its Gauss and mean curvatures satisfy the Jacobi condition \(\Phi(H,K) = K_t H_s - H_t K_s = 0\). Thus, we can give also following theorem.
Theorem 2.4. Let \( M \) be a canal surface in Galilean 3-space. Then \( M \) is a Weingarten surface if and only if \( M \) is either a tubular surface or a surface of revolution with
\[
r(s) = \pm c_3 e^{-\left(\frac{2 \pi^2}{c_1}\right)} \left(e^{2\left(\frac{2 \pi^2}{c_1}\right)} + 1\right)
\]
and \( r(s) = c_1 s + c_2 \), where \( c_1 \neq 0 \), \( c_2 \in \mathbb{R} \) and \( c_3 \in \mathbb{R}^+ \).

Proof. Let us assume that \( M \) be a Weingarten surface. Differentiating \( K(s,t) \) and \( H(s,t) \) with respect to \( s \) and \( t \) gives

\[
K_s(s,t) = \frac{r'''(s)r(s) - r''(s)r'(s) + (\kappa(s)r'(s) - \kappa'(s)r(s))\cos(t)}{r(s)^3}
\]

\[
K_t(s,t) = \frac{r''(s) + \kappa(s)\sin(t)}{r(s)}
\]

and

\[
H_s(s,t) = -\frac{r'(s)}{2r(s)}, \quad H_t(s,t) = 0.
\]

From the Jacobi equation \( \Phi(H,K) = K_tH_s - H_tK_s = 0 \), we get

\[
\frac{1}{r(s)^3} \left\{ \begin{array}{c}
\tau''(s)r'''(s) - \tau''(s)r'(s) - \tau''(s)r'(s)
+ \kappa(s)(\kappa(s)r'(s) - \kappa'(s)r(s))\cos(t)
\end{array} \right\} \sin(t) = 0.
\]

Since \( \{1, \sin(t), \cos(t)\} \) is linearly independent then

\[
\begin{align*}
\tau''(s)r'''(s) - \tau''(s)r'(s) - \tau''(s)r'(s) &= 0, \\
\kappa(s)(\kappa(s)r'(s) - \kappa'(s)r(s)) &= 0, \\
\kappa(s)(\kappa(s)r'(s) - \kappa'(s)r(s)) &= 0, \\
\tau''(s)(\kappa(s)r'(s) - \kappa'(s)r(s)) &= 0.
\end{align*}
\]

(14)

If \( \kappa(s) \) is non-zero constant then from (14)_3, \( r(s) \) is non-zero constant. If \( \kappa(s) = 0 \) then from (14)_1, either \( r(s) = c_1 s + c_2 \) or

\[
r(s) = \pm c_3 e^{-\left(\frac{2 \pi^2}{c_1}\right)} \left(e^{2\left(\frac{2 \pi^2}{c_1}\right)} + 1\right).
\]

It is easy to see that the Jacobi equation \( \Phi(H,K) = 0 \) satisfies in each case of \( \kappa(s) \) and \( r(s) \) are non-zero constants, \( \kappa(s) = 0 \), and \( r(s) = c_1 s + c_2 \) and \( \kappa(s) = 0 \), and \( r(s) = \pm c_3 e^{-\left(\frac{2 \pi^2}{c_1}\right)} \left(e^{2\left(\frac{2 \pi^2}{c_1}\right)} + 1\right) \), for the necessary part. Thus proof is completed.

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Figure 1: Some Galilean Canal Surfaces. For (a); $c_1 = c_2 = 1$, $c_3 = 1/2$, for (b); $c_1 = 1$, $c_2 = 1/2$.

References


