Euler Characteristics of Log Calabi–Yau Threefolds

Nam-Hoon Lee
Department of Mathematics Education, Hongik University 42-1, Sangsu-Dong, Mapo-Gu, Seoul 121-791, Korea
e-mail: nhlee@hongik.ac.kr

Abstract. For any even integer \( n \), we show that there exists a log Calabi–Yau threefold \((Y, D)\) such that the Euler characteristic of \( Y \) is \( n \). Furthermore \( Y \) is smooth and \( D \) is smooth anticanonical section of \( Y \) that is a \( K3 \) surface.

A log Calabi–Yau pair \((Y, D)\) consists of a proper variety \( Y \) and an effective \( \mathbb{Q} \)-divisor \( D \) such that \((Y, D)\) is log canonical and \( K_Y + D \) is \( \mathbb{Q} \)-linearly equivalent to zero. Recently much interest has been given to log Calabi–Yau pairs ([5, 7, 3, 2]). Some types of boundedness properties for klt log Calabi–Yau pairs were proved in [5, 3]. For a Fano variety \( Y \), \((Y, D)\) is a log Calabi–Yau pair, where \( D \) is an anticanonical section of \( Y \). For a Calabi–Yau variety \( Y \), \((Y, 0)\) is also a log Calabi–Yau pair. It is well-known that there are only finitely many deformation types for Fano threefolds. The Euler Characteristics of elliptic Calabi–Yau threefolds are bounded ([6, 4]) and many expect that it is true for the general Calabi–Yau threefolds. So it would be a natural question to ask whether the Euler Characteristics for log Calabi–Yau threefolds is also bounded. Suppose that \( Y \) is smooth and \( D \) is a smooth anticanonical section of \( Y \) that is a \( K3 \) surface. Then the pair \((Y, D)\) is a log Calabi–Yau threefold. The authors in [2] gave an unbounded family of such log Calabi–Yau threefolds with the Euler characteristics \( e(Y) = -48n - 46 \), where \( n \geq N_0 \) for some integer \( N_0 \). In this note, we show the following theorem, which is an improvement of the result in [2].

Theorem 1.1. For any even integer \( n \), there exists a log Calabi–Yau threefold \((Y, D)\) such that the Euler characteristic \( e(Y) \) of \( Y \) is \( n \). Moreover, \( Y \) is smooth and \( D \) is smooth anticanonical section of \( Y \) that is a \( K3 \) surface.

Its proof is short and quite elementary. Note that the Euler characteristic \( e(Y) \) of \( Y \) in Theorem 1.1 is given by

\[
2(1 + h^{1,1}(Y) - h^{1,2}(Y)),
\]

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which is even. So Theorem 1.1 basically asserts that the Euler characteristic of such a log Calabi–Yau threefold $Y$ can be any possible number.

**Proof of Theorem 1.1.** Let $(X, D)$ be a log Calabi–Yau threefold such that $X$ is smooth and that $D$ is a smooth anticanonical section of $X$ which is a $K3$ surface. Furthermore assume that there is a smooth curve $c$ on $D$. We let $X_0 = X$, $D_0 = D$ and $c_0 = c$. Let $X_1 \to X_0$ be the blow-up along $c_0$ and $D_1$ be the proper transform of $D_0$. Then $D_1$ is isomorphic to $D_0$ and it is an anticanonical section of $X_1$. So $(X_1, D_1)$ is also a log Calabi–Yau threefold. Let $c_1$ be the intersection of $D_1$ with the exceptional divisor. Then $c_1$ is a smooth curve on $D_1$ and it is isomorphic to $c_0$. Inductively we repeat this procedure to construct log Calabi–Yau threefolds: Let $X_{i+1} \to X_i$ be the blow-up along $c_i$, $D_{i+1}$ be the proper transform of $D_i$ and $c_{i+1}$ be the intersection of $D_{i+1}$ with the exceptional divisor. Then $(X_{i+1}, D_{i+1})$ is also a log Calabi–Yau threefold. Note that Euler characteristic $e(X_{i+1})$ of $X_{i+1}$ is

$$e(X_{i+1}) = e(X_i) + 2 - 2g(c_i)$$

and $g(c_{i+1}) = g(c_i)$, where $g(c_i)$ is the genus of the curve $c_i$. Hence we have

$$e(X_i) = e(X) + i(2 - 2g(c)).$$

Let $X = \mathbb{P}^3$ and $D$ be a smooth quartic that is a Kummer surface (e.g., the Fermat quartic). Then $D$ has two canonical elliptic fibrations with fibers $\Gamma_1, \Gamma_2$ respectively such that $\Gamma_1 \cdot \Gamma_2 = 1$ (see [1] for example). Note that the linear system $|\Gamma_1 + \Gamma_2|$ is base-point-free. So we can choose a smooth divisor $c$ from the linear system. Note that $g(c) = 2$. Apply the above procedure to get the log Calabi–Yau threefolds $(X_i, D_i)$’s. We have

$$(1.1) \quad e(X_i) = e(X) + i(2 - 2g(c)) = e(\mathbb{P}^3) - 2i = 4 - 2i,$$

where $i \geq 0$.

We note that $D$ also contains a smooth rational curve $c'$. Apply the above procedure again to $c'$ to get log Calabi–Yau threefolds $(X_i', D'_i)$’s. Then we have

$$(1.2) \quad e(X'_i) = e(X) + i(2 - 2g(c')) = 4 + 2i,$$

where $i \geq 0$. By (1.1) and (1.2), any even integer $n$ is the Euler characteristic of one of $X_i$’s or $X'_i$’s.

**References**


