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CLASSIFICATION OF FREE ACTIONS OF FINITE GROUPS ON 3-DIMENSIONAL NILMANIFOLDS

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Abstract. We study free actions of finite groups on 3-dimensional nilmanifolds with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_p$. By the works of Bieberbach and Waldhausen, this classification problem is reduced to classifying all normal nilpotent subgroups of almost Bieberbach groups of finite index, up to affine conjugacy.

1. Introduction

The classifying finite group actions on a 3-dimensional nilmanifold can be understood by the works of Bieberbach, L. Auslander and Waldhausen [5,6,16]. Free actions of cyclic, abelian and finite groups on the 3-torus were studied in [7], [10] and [4], respectively. In [4], the authors generalized the result of [10] by changing the finite abelian groups condition into the finite groups condition. Our motivation is analogous to this situation.

Let $\mathcal{H}$ be the 3-dimensional Heisenberg group; i.e., $\mathcal{H}$ consists of all 3×3 real upper triangular matrices with diagonal entries 1. Thus $\mathcal{H}$ is a simply connected, 2-step nilpotent Lie group, and it fits an exact sequence

$$1 \to \mathbb{R} \to \mathcal{H} \to \mathbb{R}^2 \to 1,$$

where $\mathbb{R} = \mathcal{Z}(\mathcal{H})$, the center of $\mathcal{H}$. Hence $\mathcal{H}$ has the structure of a line bundle over $\mathbb{R}^2$. We take a left invariant metric coming from the orthonormal basis

$$\left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

for the Lie algebra of $\mathcal{H}$. This is, what is called, the Nil-geometry and its isometry group is $\mathrm{Isom}(\mathcal{H}) = \mathcal{H} \rtimes O(2)$ [13, 14]. All isometries of $\mathcal{H}$ preserve orientation and the bundle structure.

We say that a closed 3-dimensional manifold M has a Nil-geometry if there is a subgroup π of $\mathrm{Isom}(\mathcal{H})$ so that π acts properly discontinuously and freely

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with quotient $M = \mathcal{H}/\pi$. The simplest such a manifold is the quotient of $\mathcal{H}$ by the lattice consisting of integral matrices. For each integer $p > 0$, let

$$\Gamma_p = \left\{ \begin{bmatrix} 1 & l & \frac{n}{p} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \middle| \; l, m, n \in \mathbb{Z} \right\}.$$

Then Γ_1 is the discrete subgroup of $\mathcal{H}$ consisting of all matrices with integer entries and Γ_p is a lattice of $\mathcal{H}$ containing Γ_1 with index p. Clearly

$$\mathrm{H}_1(\mathcal{H}/\Gamma_p; \mathbb{Z}) = \Gamma_p/[\Gamma_p, \Gamma_p] = \mathbb{Z}^2 \oplus \mathbb{Z}_p.$$

Note that these Γ_p's produce infinitely many distinct nilmanifolds

$$\mathcal{N}_p = \mathcal{H}/\Gamma_p$$

covered by $\mathcal{N}_1$.

It is interesting that if a finite group acts freely on the (standard) 3-dimensional nilmanifold $\mathcal{N}_1$ with the first homology $\mathbb{Z}^2$, then it is cyclic (see [2]). Free actions of finite abelian groups on the 3-dimensional nilmanifold with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_p$ were classified in [1]. In this paper we study free actions of finite groups deleting an abelian condition on 3-dimensional nilmanifolds with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_p$ by utilizing the method used in [1] and classify all such group actions, up to topological conjugacy. This classification problem is reduced to classifying all normal nilpotent subgroups of almost Bieberbach groups of finite index, up to affine conjugacy. This work contains the results of [1] as corollaries.

Let G be a finite group acting freely on the nilmanifold $\mathcal{N}_p$. Then clearly, $M = \mathcal{N}_p/G$ is a topological manifold, and $\pi = \pi_1(M) \subset \mathrm{TOP}(\mathcal{H})$ is isomorphic to an almost Bieberbach group. Let π' be an embedding of π into $\mathrm{Aff}(\mathcal{H})$. Such an embedding always exists. Since any isomorphism between lattices extends uniquely to an automorphism of $\mathcal{H}$, we may assume the subgroup Γ_p goes to itself by the embedding $\pi \to \pi' \subset \mathrm{Aff}(\mathcal{H})$. Then the quotient group $G' = \pi'/\Gamma_p$ acts freely on the nilmanifold $\mathcal{N}_p = \mathcal{H}/\Gamma_p$. Moreover, $M' = \mathcal{N}_p/G'$ is an infra-nilmanifold. Thus, a finite free topological action $(G, \mathcal{N}_p)$ gives rise to an isometric action $(G', \mathcal{N}_p)$ on the nilmanifold $\mathcal{N}_p$. Clearly, $\mathcal{N}_p/G$ and $\mathcal{N}_p/G'$ are sufficiently large, see [6, Proposition 2]. By works of Waldhausen and Heil [5, 16], M is homeomorphic to M'.

Definition 1.1. Let groups G_i act on manifolds M_i, for $i = 1, 2$. The action (G_1, M_1) is *topologically conjugate* to (G_2, M_2) if there exists an isomorphism $\theta: G_1 \to G_2$ and a homeomorphism $h: M_1 \to M_2$ such that

$$h(g \cdot x) = \theta(g) \cdot h(x)$$

for all $x \in M_1$ and all $g \in G_1$. When $G_1 = G_2$ and $M_1 = M_2$, topologically conjugate is the same as *weakly equivariant*.

For $\mathcal{N}_p/G$ and $\mathcal{N}_p/G'$ being homeomorphic implies that the two actions $(G, \mathcal{N}_p)$ and $(G', \mathcal{N}_p)$ are topologically conjugate. Consequently, a finite free action $(G, \mathcal{N}_p)$ is topologically conjugate to an isometric action $(G', \mathcal{N}_p)$. Such a pair $(G', \mathcal{N}_p)$ is not unique. However, by the result obtained by Lee and Raymond [9], all the others are topologically conjugate.

2. Criteria for affine conjugacy

In this section, we develop a technique for finding and classifying all possible finite group actions on 3-dimensional nilmanifolds with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_p$. The problem will be reduced to a purely group-theoretic one. We quote most of Introduction and Section 2 of [1] in this section for the reader's conveniences.

Let Γ be any lattice of $\mathcal{H}$ and $\mathcal{Z}(\mathcal{H})$ be the center of $\mathcal{H}$. Then $\mathbb{Z} = \Gamma \cap \mathcal{Z}(\mathcal{H})$ and $\Gamma/\Gamma \cap \mathcal{Z}(\mathcal{H})$ are lattices of $\mathcal{Z}(\mathcal{H})$ and $\mathcal{H}/\mathcal{Z}(\mathcal{H})$, respectively. Therefore, the lattice Γ is an extension of $\mathbb{Z}$ by $\mathbb{Z}^2$, that is, there is an exact sequence:

$$1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z}^2 \to 1.$$

Let a, b and c be elements of Γ such that the images of a and b in $\mathbb{Z}^2$ generate $\mathbb{Z}^2$ and c generates the center $\mathbb{Z}$. Then it is known that such Γ is isomorphic to one of the following groups, for some k:

$$\Gamma_k = \langle a, b, c \mid [b, a] = c^k,\ [c, a] = [c, b] = 1 \rangle, \quad k \neq 0,$$

where $[b, a] = b^{-1}a^{-1}ba$. This group is realized as a uniform lattice of $\mathcal{H}$ if one takes

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ c = \begin{bmatrix} 1 & 0 & \frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark that Γ_k is equal to Γ_{-k}.

The following proposition gives a characterization of an almost Bieberbach group (see [8]).

Proposition 2.1. *An abstract group π is the fundamental group of a 3-dimensional infra-nilmanifold if and only if π is torsion-free and contains Γ_k for some $k > 0$ as a maximal normal nilpotent subgroup of finite index.*

It is well known that all 3-dimensional infra-nilmanifolds are Seifert manifolds (see [9,12]). Assume that M is a 3-dimensional infra-nilmanifold. Then M has a Seifert bundle structure; namely, M is a circle bundle over a 2-dimensional orbifold with singularities. It is known that there are 15 classes of distinct closed 3-dimensional manifolds M with a Nil-geometry up to Seifert local invariant [3, Proposition 6.1].

Note that if $M = \mathcal{H}/\pi$ is a 3-dimensional infra-nilmanifold, then there is a diffeomorphism f between $\mathcal{H}$ and $\mathbb{R}^3$, and an isomorphism φ between π and π', where π' is a subgroup of

$$\mathrm{Aff}(\mathbb{R}^3) = \mathbb{R}^3 \rtimes \mathrm{GL}(3, \mathbb{R})$$

such that $(\pi, \mathcal{H})$ and $(\pi', \mathbb{R}^3)$ are weakly equivariant. Therefore, an infra-nilmanifold $M = \mathcal{H}/\pi$ is diffeomorphic to an affine manifold $M' = \mathbb{R}^3/\pi'$.

The following is the list for 15 kinds of the 3-dimensional almost Bieberbach groups imbedded in $\mathrm{Aff}(\mathcal{H}) = \mathcal{H} \rtimes (\mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R}))$ ([1, pp. 799–801]). We shall use

$$t_1 = \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I \right), \quad t_2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, I \right),$$
$$t_3 = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I \right),$$

respectively, where I is the identity in $\mathrm{Aut}(\mathcal{H}) = \mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R})$. In each presentation, n is any positive integer and t_3 is central except π_3 and π_4. Note that t_1 and t_2 are fixed, but k in t_3 varies for each $\pi_{i,j}$. For example, $k = n$ for π_1; $k = 2n$ for π_2, etc.

$$\begin{aligned}
\pi_1 &= \langle\, t_1, t_2, t_3 \mid [t_2, t_1] = t_3^n \,\rangle, \\
\pi_2 &= \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{2n},\ \alpha^2 = t_3, \alpha t_1 \alpha^{-1} = t_1^{-1}, \alpha t_2 \alpha^{-1} = t_2^{-1} \,\rangle, \\
\pi_3 &= \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{2n},\ [t_3, t_1] = [t_3, t_2] = 1,\ \alpha t_3 \alpha^{-1} = t_3^{-1}, \\
&\qquad \alpha t_1 \alpha^{-1} = t_1,\ \alpha t_2 = t_2^{-1} \alpha t_3^{-n},\ \ \alpha^2 = t_1 \,\rangle, \\
\pi_4 &= \langle\, t_1, t_2, t_3, \alpha, \beta \mid [t_2, t_1] = t_3^{4n},\ [t_3, t_1] = [t_3, t_2] = [\alpha, t_3] = 1, \\
&\qquad \beta t_3 \beta^{-1} = t_3^{-1},\ \alpha t_1 = t_1^{-1} \alpha t_3^{2n}, \alpha t_2 = t_2^{-1} \alpha t_3^{-2n}, \alpha^2 = t_3, \beta^2 = t_1, \\
&\qquad \beta t_1 \beta^{-1} = t_1,\ \beta t_2 = t_2^{-1} \beta t_3^{-2n},\ \ \alpha\beta = t_1^{-1} t_2^{-1} \beta \alpha t_3^{-(2n+1)} \,\rangle, \\
\pi_{5,1} &= \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{4n-2},\ \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^4 = t_3 \,\rangle, \\
\pi_{5,2} &= \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{4n}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^4 = t_3^3 \,\rangle, \\
\pi_{5,3} &= \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{4n}, \alpha t_1 \alpha^{-1} = t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1},\ \alpha^4 = t_3 \,\rangle, \\
\pi_{6,1} &= \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{3n}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1}, \alpha^3 = t_3 \,\rangle, \\
\pi_{6,2} &= \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{3n}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1}, \alpha^3 = t_3^2 \,\rangle, \\
\pi_{6,3} &= \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{3n-2}, \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1}, \alpha^3 = t_3^2 \,\rangle, \\
\pi_{6,4} &= \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{3n-1},\ \alpha t_1 \alpha^{-1} = t_2, \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1}, \alpha^3 = t_3 \,\rangle, \\
\pi_{7,1} &= \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{6n}, \alpha t_1 \alpha^{-1} = t_1 t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^6 = t_3 \,\rangle, \\
\pi_{7,2} &= \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{6n-2}, \alpha t_1 \alpha^{-1} = t_1 t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^6 = t_3 \,\rangle, \\
\pi_{7,3} &= \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{6n},\ \alpha t_1 \alpha^{-1} = t_1 t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^6 = t_3^5 \,\rangle, \\
\pi_{7,4} &= \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{6n-4},\ \alpha t_1 \alpha^{-1} = t_1 t_2, \alpha t_2 \alpha^{-1} = t_1^{-1}, \alpha^6 = t_3^5 \,\rangle.
\end{aligned}$$

Let $(G, \mathcal{N}_p)$ be a free affine action of a finite group G on the nilmanifold $\mathcal{N}_p$. Then $\mathcal{N}_p/G$ is an infra-nilmanifold. Let $\pi = \pi_1(\mathcal{N}_p/G)$ and $\Gamma_p = \pi_1(\mathcal{N}_p)$.

Then π is an almost Bieberbach group. In fact, since the covering projection $\mathcal{N}_p \to \mathcal{N}_p/G$ is regular, Γ_p is a normal subgroup of π.

Definition 2.2. Let $\pi \subset \mathrm{Aff}(\mathcal{H}) = \mathcal{H} \rtimes \mathrm{Aut}(\mathcal{H})$ be an almost Bieberbach group, and let N_1, N_2 be subgroups of π. We say that (N_1, π) is *affinely conjugate* to (N_2, π), denoted by $N_1 \sim N_2$, if there exists an element $(t, T) \in \mathrm{Aff}(\mathcal{H})$ such that $(t, T)\pi(t, T)^{-1} = \pi$ and $(t, T)N_1(t, T)^{-1} = N_2$.

Our classification problem of free finite group actions $(G, \mathcal{N}_p)$ with

$$\pi_1(\mathcal{N}_p/G) \cong \pi$$

can be solved by finding all normal nilpotent subgroups N of π each of which is isomorphic to Γ_p, and classify (N, π) up to affine conjugacy. This procedure is a purely group-theoretic problem and can be handled by affine conjugacy.

The following proposition [1, Proposition 3.1] is a working criterion for determining all normal nilpotent subgroups of π isomorphic to Γ_p.

Proposition 2.3. *Let N be a normal nilpotent subgroup of an almost Bieberbach group π and isomorphic to Γ_p. Then N can be represented by a set of generators*

$$N = \langle\, t_1^{d_1} t_2^{m} t_3^{n_1},\, t_2^{d_2} t_3^{n_2},\, t_3^{\frac{K d_1 d_2}{p}} \,\rangle,$$

where d_1, d_2 are divisors of p; K is determined by $t_3^K = [t_2, t_1]$; $0 \le m < d_2$, $0 \le n_i < \frac{K d_1 d_2}{p}$ $(i = 1, 2)$.

"Realization" for the action of G on $\mathcal{H}/N$ as an action of G on $\mathcal{H}/\Gamma_p = \mathcal{N}_p$ can be done by the following procedure. To describe the natural affine action of G on the nilmanifold $\mathcal{H}/N$ as an action of G on $\mathcal{N}_p$, we must make the nilmanifold $\mathcal{H}/N$ the nilmanifold $\mathcal{N}_p$ whose fundamental group is Γ_p and describe the action on the universal covering level. In other words, the action of G should be defined on $\mathcal{H}$ as affine maps (this is really explaining the liftings of a set of generators of G in π), and simply say that our action is the affine action modulo the lattice Γ_p. It is quite easy to achieve this.

Let $N = \langle\, {t_1}^{d_1} t_2^{m} t_3^{n_1},\, t_2^{d_2} t_3^{n_2},\, t_3^{\frac{K d_1 d_2}{p}} \,\rangle$ be a normal nilpotent subgroup of an almost Bieberbach group π which is isomorphic to Γ_p. Then we can find an automorphism

$$\mu = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} \frac{n_2}{d_1 d_2 K} \\ \frac{m}{2d_2} - \frac{n_1}{d_1 d_2 K} \end{bmatrix},\ \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{m}{d_1 d_2} & \frac{1}{d_2} \end{bmatrix} \right) \right) \in \mathrm{Aut}(\mathcal{H})$$

such that $\mu N \mu^{-1} = \Gamma_p$, using the following relations:

$$\mu(t_1^{d_1} t_2^{m} t_3^{n_1})\mu^{-1} = t_1,\quad \mu(t_2^{d_2} t_3^{n_2})\mu^{-1} = t_2,\quad \mu(t_3^{\frac{K d_1 d_2}{p}})\mu^{-1} = t_3^{\frac{K}{p}}.$$

Therefore, the conjugation by $\mu \in \mathrm{Aff}(\mathcal{H})$ maps π into another almost Bieberbach group in such a way that N maps onto Γ_p. Suppose $\{\alpha_1, \ldots, \alpha_k\}$

generates the quotient group G when projected down via $\pi \to G$, then

$$\{\mu\alpha_1\mu^{-1}, \dots, \mu\alpha_k\mu^{-1}\}$$

describes the action of G on the nilmanifold $\mathcal{N}_p$.

3. Free actions of finite groups on the nilmanifold

In this section, we shall find all possible finite groups acting freely (up to topological conjugacy) on the 3-dimensional nilmanifold $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_i$ $(1 \le i \le 7)$. This, as in other parts of calculations, was done by using the program Mathematica [17] and hand-checked.

Now we shall find all possible finite groups acting freely (up to topological conjugacy) on the 3-dimensional nilmanifold $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_2$.

Lemma 3.1. *Let N be a normal nilpotent subgroup of an almost Bieberbach group $\pi_2, \pi_{5,i}$ $(i = 1,2,3)$ or $\pi_{7,j}$ $(j = 1,2,3,4)$ which is isomorphic to Γ_p. Then N can be represented by one of the following sets of generators*

$$N_1 = \langle t_1^{d_1} t_2^{m},\ t_2^{d_2},\ t_3^{\frac{Kd_1d_2}{p}} \rangle, \qquad N_2 = \langle t_1^{d_1} t_2^{m},\ t_2^{d_2} t_3^{\frac{Kd_1d_2}{2p}},\ t_3^{\frac{Kd_1d_2}{p}} \rangle,$$

$$N_3 = \langle t_1^{d_1} t_2^{m} t_3^{\frac{Kd_1d_2}{2p}},\ t_2^{d_2},\ t_3^{\frac{Kd_1d_2}{p}} \rangle, \qquad N_4 = \langle t_1^{d_1} t_2^{m} t_3^{\frac{Kd_1d_2}{2p}},\ t_2^{d_2} t_3^{\frac{Kd_1d_2}{2p}},\ t_3^{\frac{Kd_1d_2}{p}} \rangle,$$

where d_1, d_2 are divisors of p; $0 \le m < \bar{d} = gcd(d_1, d_2)$, $\frac{pm}{d_1d_2} \in \mathbb{Z}$ in the case of π_2, $\frac{d_1}{d_2} + \frac{m^2}{d_1d_2} \in \mathbb{Z}$ and d_1 is a common divisor of m and d_2 in the case of $\pi_{5,i}$, $\frac{d_1}{d_2} + \frac{m(m-d_1)}{d_1d_2} \in \mathbb{Z}$ and d_1 is a common divisor of m and d_2 in the case of $\pi_{7,j}$.

Proof. Let N be a normal nilpotent subgroup of π_2 isomorphic to Γ_p. Then by Proposition 2.3, we have

$$N = \langle\, t_1^{d_1} t_2^{m} t_3^{\ell},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{2nd_1d_2}{p}} \rangle, \quad \left(0 \le m < d_2,\ 0 \le \ell, r < \frac{2nd_1d_2}{p}\right).$$

Note that we obtained the normalizer $N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$ of π_2 in [1]: for $r, s \in \mathbb{Z}$,

$$N_{\mathrm{Aff}(\mathcal{H})}(\pi_2) = \left\{ \left(\begin{bmatrix} 1 & \frac{r}{2} & * \\ 0 & 1 & \frac{s}{2} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z}) \right\}.$$

Let $\bar{d} = \gcd(d_1, d_2)$. Then there exist $s, t \in \mathbb{Z}$ such that $\bar{d} = sd_1 + td_2$. Also there exist $q, w \in \mathbb{Z}$ such that $m = \bar{d}q + w$ $(0 \le w < \bar{d})$. It is not hard to see

$$N \sim \langle\, t_1^{d_1} t_2^{w} t_3^{\ell'},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{2nd_1d_2}{p}} \,\rangle$$

by using

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{sq}{2} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -sq & 1 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2).$$

Note that the relation

$$t_1(t_1^{d_1}t_2^m t_3^\ell)t_1^{-1} = (t_1^{d_1}t_2^m t_3^\ell)t_3^{\frac{2nd_1d_2}{p}(-\frac{pm}{d_1d_2})} \in N$$

shows $\frac{pm}{d_1d_2} \in \mathbb{Z}$.

Let

$$\mu = \left(\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \right) \in \mathrm{Aff}(\mathcal{H}).$$

Then we have

$$\mu(t_1^{d_1}t_2^m t_3^\ell)\mu^{-1} = \left(\begin{bmatrix} 1 & -d_1 & d_1m - \frac{\ell}{K} + d_1v - mu \\ 0 & 1 & -m \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right),$$

$$\mu(t_2^{d_2}t_3^r)\mu^{-1} = \left(\begin{bmatrix} 1 & 0 & -\frac{r}{K} - d_2u \\ 0 & 1 & -d_2 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right).$$

Let $\mu = \alpha, \alpha^2, \alpha^3$ for the case of $\pi_2, \pi_{5,i}$ or $\pi_{7,j}$, respectively. Since N is a normal nilpotent subgroup of $\pi_2, \pi_{5,i}$ or $\pi_{7,j}$, the following two relations

$$\mu(t_1^{d_1}t_2^m t_3^\ell)\mu^{-1} = (t_1^{d_1}t_2^m t_3^\ell)^{-1}(t_3^{\frac{Kd_1d_2}{p}})^x \in N,$$

$$\mu(t_2^{d_2}t_3^r)\mu^{-1} = (t_2^{d_2}t_3^r)^{-1}(t_3^{\frac{Kd_1d_2}{p}})^y \in N$$

show that

$$x = \frac{2p\ell}{Kd_1d_2} - \frac{pm}{d_2} \in \mathbb{Z}, \quad y = \frac{2pr}{Kd_1d_2} \in \mathbb{Z}, \quad (\text{for } \pi_2 \text{ and } \pi_{5,i})$$

$$x = \frac{2p\ell}{Kd_1d_2} - \frac{pm}{d_2} + \frac{p(m-d_1)}{d_1d_2} \in \mathbb{Z}, \quad y = \frac{2pr}{Kd_1d_2} + \frac{p}{d_1} \in \mathbb{Z}, \quad (\text{for } \pi_{7,j}).$$

Similarly, the following two relations

$$\alpha(t_1^{d_1}t_2^m t_3^\ell)\alpha^{-1} \in N, \quad \alpha(t_2^{d_2}t_3^r)\alpha^{-1} \in N,$$

show that

$$\frac{d_1}{d_2} + \frac{m^2}{d_1d_2} \in \mathbb{Z}, \quad \frac{m}{d_1} \in \mathbb{Z}, \quad \frac{d_2}{d_1} \in \mathbb{Z}, \quad (\text{for } \pi_{5,i})$$

$$\frac{d_1}{d_2} + \frac{m(m-d_1)}{d_1d_2} \in \mathbb{Z}, \quad \frac{m}{d_1} \in \mathbb{Z}, \quad \frac{d_2}{d_1} \in \mathbb{Z}, \quad (\text{for } \pi_{7,j}).$$

Therefore we can get $\frac{2p\ell}{Kd_1d_2} \in \mathbb{Z}$ and $\frac{2pr}{Kd_1d_2} \in \mathbb{Z}$. Since $0 \leq \ell, r < \frac{Kd_1d_2}{p}$ by Proposition 2.3, we have $\ell = 0$ or $\frac{Kd_1d_2}{2p}$ and $r = 0$ or $\frac{Kd_1d_2}{2p}$. □

Remark. In the case of π_2, the condition $\frac{pm}{d_1d_2} \in \mathbb{Z}$ in Lemma 3.1 is crucial to determine the number of affinely non-conjugacy classes when d_1, d_2 and p are given. In fact, for $\bar{d} = (d_1, d_2)$ and $p = kD$ where $k \in \mathbb{N}$, D is the least common multiple of d_1 and d_2, we have $\frac{pm}{d_1 d_2} \in \mathbb{Z}$ if and only if $\frac{km}{d} \in \mathbb{Z}$. Let $q = (\bar{d}, k)$. Then $\frac{km}{d} \in \mathbb{Z}$ if and only if $\frac{k'm}{d'} \in \mathbb{Z}$, where

$$k = qk',\ \bar{d} = q\bar{d}',\ (k',\bar{d}') = 1.$$

Thus $\bar{d}'$ is a divisor of m. Since $0 \leq m < \bar{d} = q\bar{d}'$, we can get $m = 0, \bar{d}', \ldots, (q-1)\bar{d}'$.

Let $N^m = \langle {t_1}^{d_1} t_2^m t_3^\ell,\, t_2^{d_2} t_3^r,\, t_3^{\frac{Kd_1d_2}{p}} \rangle$ and $N^{m'} = \langle {t_1}^{d_1} t_2^{m'} t_3^{\ell'},\, t_2^{d_2} t_3^{r'},\, t_3^{\frac{Kd_1d_2}{p}} \rangle$ be normal nilpotent subgroups of an almost Bieberbach group π which are isomorphic to Γ_p. If N^m is affinely conjugate to $N^{m'}$, then there exists

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi)$$

satisfying either

$$\mu({t_1}^{d_1} {t_2}^m t_3^\ell)\mu^{-1} = t_1^{d_1} {t_2}^{m'} t_3^{\ell'},\quad \mu(t_2^{d_2} t_3^r)\mu^{-1} = t_2^{d_2} t_3^{r'}, \tag{3.1}$$

or

$$\mu({t_1}^{d_1} {t_2}^m t_3^\ell)\mu^{-1} = t_2^{d_2} {t_3}^{r'},\quad \mu(t_2^{d_2} t_3^r)\mu^{-1} = t_1^{d_1} {t_2}^{m'} t_3^{\ell'}. \tag{3.2}$$

From (3.1) or (3.2), we obtain the following relations respectively:

$$bd_2 = 0,\quad dd_2 = d_2,\quad ad_1 + bm = d_1,\quad cd_1 + dm = m', \tag{3.3}$$

or

$$bd_2 = d_1,\quad dd_2 = m',\quad ad_1 + bm = 0,\quad cd_1 + dm = d_2. \tag{3.4}$$

From these two relations and the normalizer $N_{\mathrm{Aff}(\mathcal{H})}(\pi)$ of each almost Bieberbach group π, we can get either

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ m = m' \tag{3.5}$$

or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix},\ m = m' = 0,\ bc = 1. \tag{3.6}$$

So, we obtain the following theorem.

Theorem 3.2. *Let N^m and $N^{m'}$ be normal nilpotent subgroups of an almost Bieberbach group π which are isomorphic to Γ_p and whose sets of generators are*

$$N^m = \langle {t_1}^{d_1} t_2^m t_3^\ell,\, t_2^{d_2} t_3^r,\, t_3^{\frac{Kd_1d_2}{p}} \rangle,\quad N^{m'} = \langle {t_1}^{d_1} t_2^{m'} t_3^{\ell'},\, t_2^{d_2} t_3^{r'},\, t_3^{\frac{Kd_1d_2}{p}} \rangle.$$

If $m \neq m'$, then N^m is not affinely conjugate to $N^{m'}$.

Proposition 3.3 (π_2). *Let N_i $(i = 1, 2, 3, 4)$ be a normal nilpotent subgroup of π_2 in Lemma 3.1 and isomorphic to Γ_p. Then we have the following*:

(1) $N_1 \sim N_2$ *if and only if* $m = 0, d_1 = p$.
(2) $N_1 \sim N_3$ *if and only if* $m = 0, d_2 = p$.
(3) $N_1 \sim N_4$ *if and only if* $m = 0, d_1 = d_2 = p$.
(4) $N_2 \sim N_3$ *if and only if* $m = 0, d_1 = d_2$.
(5) $N_2 \sim N_4$ *if and only if either* $m = 0, d_2 = p$, *or* $2m = d_2, 2d_1 = p$.
(6) $N_3 \sim N_4$ *if and only if* $m = 0, d_1 = p$.

Proof. (1) Suppose that N_1 is affinely conjugate to N_2. Then there exists

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$$

satisfying either (3.1) or (3.2). From (3.5), we can get $x = -\frac{d_1}{2p}$ and $y = -\frac{m}{2p}$. Since $\mu \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$, we have $2x = -\frac{d_1}{p} \in \mathbb{Z}$ and $2y = -\frac{m}{p} \in \mathbb{Z}$. Note that d_1, d_2 are divisors of p and $0 \le m < \bar{d}$ by Lemma 3.1. Thus we have $d_1 = p$ and $m = 0$. Similarly, from (3.6), we can get $d_1 = d_2 = p$ and $m = 0$. Conversely, suppose that $d_1 = p$ and $m = 0$. Then $N_1 \sim N_2$ by using

$$\mu = \left(\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2).$$

(4) Suppose that N_2 is affinely conjugate to N_3. Note that

$$N_3 \sim \langle\, t_1^{d_1} t_2^{m} t_3^{-\frac{nd_1d_2}{p}},\ t_2^{d_2},\ t_3^{-\frac{2nd_1d_2}{p}} \rangle.$$

Then there exists $\mu \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$ satisfying either (3.1) or (3.2). From (3.5), we can get $x = \frac{d_1}{2p}$ and $y = \frac{m+d_2}{2p}$. Since $\mu \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$, we have $2x = \frac{d_1}{p} \in \mathbb{Z}$ and $2y = \frac{m+d_2}{p} \in \mathbb{Z}$. Note that d_1, d_2 are divisors of p and $0 \le m < d_2$. Thus we have $d_1 = d_2 = p$ and $m = 0$. Next, from (3.6), a similar calculation shows that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ d_1 = d_2,\ m = 0.$$

The converse is easy by using

$$\mu = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2).$$

The other cases can be done similarly. □

The following theorem can be obtained easily by using Proposition 3.3. From now on, we shall denote affine conjugacy classes by AC classes.

Notation. ${}^{\xi}\langle \alpha_1, \dots, \alpha_k \rangle$ means the subgroup generated by conjugations of $\alpha_1, \dots, \alpha_k$ by ξ.

Theorem 3.4 (π_2). *Table 2 gives a complete list of all free actions (up to topological conjugacy) of finite groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_2$.*

TABLE 2

Generators	AC classes of normal nilpotent subgroups	Conditions ($m \geq 0$)
${}^{\mu_i}\langle t_1, t_2, \alpha\rangle$	$N_1 = \langle t_1^{d_1} t_2^{m},\ t_2^{d_2},\ t_3^{\frac{2nd_1d_2}{p}}\rangle$	
	$N_2 = \langle t_1^{d_1} t_2^{m},\ t_2^{d_2} t_3^{\frac{nd_1d_2}{p}},\ t_3^{\frac{2nd_1d_2}{p}}\rangle$	if $m = 0$, then $d_1 \neq p$
	$N_3 = \langle t_1^{d_1} t_2^{m} t_3^{\frac{nd_1d_2}{p}},\ t_2^{d_2},\ t_3^{\frac{2nd_1d_2}{p}}\rangle$	if $m = 0$, then $d_2 \neq p, d_1 \neq d_2$
	$N_4 = \langle t_1^{d_1} t_2^{m} t_3^{\frac{nd_1d_2}{p}},\ t_2^{d_2} t_3^{\frac{nd_1d_2}{p}},\ t_3^{\frac{2nd_1d_2}{p}}\rangle$	if $m \neq 0$, then either $2m \neq d_2$ or $2d_1 \neq p$, if $m = 0$, then $d_1 \neq p, d_2 \neq p$

where

$$\mu_1 = \left(I,\ \left(\begin{bmatrix} 0 \\ \frac{m}{2d_2} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{m}{d_1d_2} & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_2 = \left(I,\ \left(\begin{bmatrix} \frac{1}{2p} \\ \frac{m}{2d_2} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{m}{d_1d_2} & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_3 = \left(I,\ \left(\begin{bmatrix} 0 \\ \frac{m}{2d_2} - \frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{m}{d_1d_2} & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_4 = \left(I,\ \left(\begin{bmatrix} \frac{1}{2p} \\ \frac{m}{2d_2} - \frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{m}{d_1d_2} & \frac{1}{d_2} \end{bmatrix}\right)\right).$$

Here I is the identity in $\mathcal{H}$.

The realization for the action of $G = \pi_2/N_i$ on the nilmanifold $\mathcal{H}/N_i$, as an affine action on the nilmanifold $\mathcal{N}_p$, is easy provided that we follow the "Realization" procedure. The generators of the group $G = \pi_2/N_i$ can be obtained from $t_1, t_2, \alpha \in \pi_2$. For example, we observe that N_2 in π_2 is not equal to Γ_p, but isomorphic to Γ_p. To obtain an action of $G = \pi_2/N_2$ on $\mathcal{N}_p$, one has to conjugate the representation of π_2 so that N_2 becomes Γ_p by means of an automorphism $\mu_2 \in \mathrm{Aut}(\mathcal{H})$, where

$$\mu_2 = \left(I,\ \left(\begin{bmatrix} \frac{1}{2p} \\ \frac{m}{2d_2} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{m}{d_1d_2} & \frac{1}{d_2} \end{bmatrix}\right)\right) \in \mathrm{Aff}(\mathcal{H}).$$

Thus we can see that $\mu_2\, N_2\, \mu_2^{-1} = \Gamma_p$, and the following three elements of $\mathrm{Aff}(\mathcal{H})$

$$\mu_2\, t_1\, \mu_2^{-1} = \left(\begin{bmatrix} 1 & \frac{1}{d_1} & -\frac{m(p+(1+p)d_1)}{2pd_1^2d_2} \\ 0 & 1 & -\frac{m}{d_1d_2} \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right),$$

$$
\begin{aligned}
\mu_2\, t_2\, \mu_2^{-1} &= \left(\begin{bmatrix}1 & 0 & \frac{1}{2pd_2}\\ 0 & 1 & \frac{1}{d_2}\\ 0 & 0 & 1\end{bmatrix},\ \left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\right),\\
\mu_2\, \alpha\, \mu_2^{-1} &= \left(\begin{bmatrix}1 & 0 & -\frac{1}{4nd_1d_2}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\ \left(\begin{bmatrix}\frac{1}{p}\\ \frac{m}{d_2}\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right)\right)
\end{aligned}
$$

describe actions of π_2/N_2 on $\mathcal{N}_p$. That is, these act on $\mathcal{H}$ by

$$
\begin{aligned}
\mu_2\, t_1\, \mu_2^{-1}\cdot \begin{bmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix} &= \begin{bmatrix}1 & x+\frac{1}{d_1} & z+\frac{y}{d_1}-\frac{m(p+(1+p)d_1)}{2pd_1{}^2d_2}\\ 0 & 1 & y-\frac{m}{d_1d_2}\\ 0 & 0 & 1\end{bmatrix},\\
\mu_2\, t_2\, \mu_2^{-1}\cdot \begin{bmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix} &= \begin{bmatrix}1 & x & z+\frac{1}{2pd_2}\\ 0 & 1 & y+\frac{1}{d_2}\\ 0 & 0 & 1\end{bmatrix},\\
\mu_2\, \alpha\, \mu_2^{-1}\cdot \begin{bmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix} &= \begin{bmatrix}1 & -x & z-\frac{y}{p}+\frac{mx}{d_2}-\frac{1}{4nd_1d_2}\\ 0 & 1 & -y\\ 0 & 0 & 1\end{bmatrix}.
\end{aligned}
$$

The other cases can be done similarly.

Note that π_2/N is abelian if and only if $N \supset [\pi_2, \pi_2] = \langle t_1^2,\ t_2^2,\ t_3^{2n}\rangle$. Thus we obtain the following result, which is the same as Theorem 3.3 of [1].

Corollary 3.5. *The following table gives a complete list of all free actions* (*up to topological conjugacy*) *of finite abelian groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_2$.*

Group G	Generators	AC classes of normal nilpotent subgroups
$\mathbb{Z}_{\frac{4n}{p}}$	$\langle\alpha\rangle$	$\frac{2n}{p}\in\mathbb{N}$, $\quad N_1=\langle t_1,\ t_2,\ t_3^{\frac{2n}{p}}\rangle$
	${}^{\xi_2}\langle\alpha\rangle$	$\frac{n}{p}\in\mathbb{N},\ p\neq 1$, $\quad N_2=\langle t_1,\ t_2t_3^{\frac{n}{p}},\ t_3^{\frac{2n}{p}}\rangle$
	${}^{\xi_3}\langle\alpha\rangle$	$N_3=\langle t_1t_3^{\frac{n}{p}},\ t_2t_3^{\frac{n}{p}},\ t_3^{\frac{2n}{p}}\rangle$
$\mathbb{Z}_2\times\mathbb{Z}_{\frac{8n}{p}}$	${}^{\eta_1}\langle t_1,\ \alpha\rangle$	$\frac{4n}{p}\in\mathbb{N},\ p\in 2\mathbb{N}$, $\quad L_1=\langle t_1^2,\ t_2,\ t_3^{\frac{4n}{p}}\rangle$
	${}^{\eta_2}\langle t_1,\ \alpha\rangle$	$\frac{2n}{p}\in\mathbb{N},\ p\in 2\mathbb{N}+2$, $\ L_2=\langle t_1^2,\ t_2t_3^{\frac{2n}{p}},\ t_3^{\frac{4n}{p}}\rangle$
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{\frac{16n}{p}}$	${}^{\zeta}\langle t_1,\ t_2,\ \alpha\rangle$	$\frac{8n}{p}\in\mathbb{N},\ p\in 4\mathbb{N}$, $\quad N=\langle t_1^2,\ t_2^2,\ t_3^{\frac{8n}{p}}\rangle$

where

$$
\begin{aligned}
\xi_2 &= \left(I, \left(\begin{bmatrix}\frac{1}{2p}\\ 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\right), & \xi_3 &= \left(I, \left(\begin{bmatrix}\frac{1}{2p}\\ -\frac{1}{2p}\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\right),\\
\eta_1 &= \left(I, \left(\begin{bmatrix}0\\ 0\end{bmatrix}, \begin{bmatrix}\frac{1}{2} & 0\\ 0 & 1\end{bmatrix}\right)\right), & \eta_2 &= \left(I, \left(\begin{bmatrix}\frac{1}{2p}\\ 0\end{bmatrix}, \begin{bmatrix}\frac{1}{2} & 0\\ 0 & 1\end{bmatrix}\right)\right),
\end{aligned}
$$

$$\zeta = \left(I, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \right).$$

Here I is the identity in $\mathcal{H}$.

From now on, for the remaining cases we will omit to state a corollary which is the same as the theorem in [1] respectively. The following example shows that a free action of a finite group on the nilmanifold $\mathcal{N}_2$ has more affinely non-conjugate classes than a free abelian group action on $\mathcal{N}_2$ which yields an orbit manifold homeomorphic to $\mathcal{H}/\pi_2$.

Example. Let N be a normal nilpotent subgroup of π_2 isomorphic to Γ_2. Then $p = 2$ and d_1, d_2 are divisors of p, $0 \leq m < \bar{d} = \gcd(d_1, d_2)$, $\frac{pm}{d_1 d_2} \in \mathbb{Z}$ by Lemma 3.1. Thus the possible pairs of (d_1, d_2) are (1,1), (1,2), (2,1), or (2,2). By Proposition 3.3, we have the following results:

(i) When $d_1 = d_2 = 1$: Since $0 \leq m < d_2$, we have $m = 0$. Then, the possible normal nilpotent subgroups are

$$N_1 = \langle t_1,\ t_2,\ t_3^n \rangle, \quad N_2 = \langle t_1,\ t_2 t_3^{\frac{n}{2}},\ t_3^n \rangle, \quad N_4 = \langle t_1 t_3^{\frac{n}{2}},\ t_2 t_3^{\frac{n}{2}},\ t_3^n \rangle.$$

Since $N \supset [\pi_2, \pi_2] = \langle t_1^2, t_2^2, t_3^{2n} \rangle$, we can conclude that π_2/N_i $(i = 1, 2, 4)$ is abelian.

(ii) When $d_1 = 1, d_2 = 2$, the possible normal nilpotent subgroups are

$$N_1 = \langle t_1,\ t_2^2,\ t_3^{2n} \rangle, \quad N_2 = \langle t_1,\ t_2^2 t_3^n,\ t_3^{2n} \rangle.$$

Note that π_2/N_1 is abelian and π_2/N_2 is nonabelian.

(iii) When $d_1 = 2, d_2 = 1$, there exist 2 affinely non-conjuate normal subgroups

$$N_1' = \langle t_1^2,\ t_2,\ t_3^{2n} \rangle, \quad N_3' = \langle t_1^2 t_3^n,\ t_2,\ t_3^{2n} \rangle.$$

It is easy to see that $N_1' \sim N_1$ and $N_3' \sim N_2$ in the case (ii).

(iv) When $d_1 = 2, d_2 = 2$, there exists only one normal subgroup N_1 of π_2,

$$N_1 = \langle t_1^2,\ t_2^2,\ t_3^{4n} \rangle.$$

Note that π_2/N_1 is nonabelian.

Lemma 3.6. *Let N be a normal nilpotent subgroup of an almost Bieberbach group π_3 and isomorphic to Γ_p. Then N can be represented by one of the following sets of generators*

$$N_1 = \langle t_1^{d_1},\ t_2^{d_2} t_3^r,\ t_3^{\frac{2nd_1d_2}{p}} \rangle, \qquad N_2 = \langle t_1^{d_1} t_3^{\frac{2nd_1d_2}{2p}},\ t_2^{d_2} t_3^{r'},\ t_3^{\frac{2nd_1d_2}{p}} \rangle,$$
$$N_3 = \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\ell},\ t_2^{d_2} t_3^{s},\ t_3^{\frac{2nd_1d_2}{p}} \rangle,$$

where $2d_1$ is a divisor of p, $s = 2\ell$ if $p = 4kd_1$, or $s = 2\ell + \frac{2nd_1d_2}{2p}$ if $p = 2(2k-1)d_1$ for $k \in \mathbb{N}$.

Proof. Let N be a normal nilpotent subgroup of π_3 and isomorphic to Γ_p. Then by Proposition 2.3, we have

$$N = \langle\, t_1^{d_1} t_2^{m} t_3^{\ell},\, t_2^{d_2} t_3^{r},\, t_3^{\frac{2nd_1d_2}{p}} \rangle, \quad \left(0 \le m < d_2,\ 0 \le \ell, r < \frac{2nd_1d_2}{p}\right).$$

Since N is a normal nilpotent subgroup of π_3, the following two relations

$$\alpha(t_1^{d_1} t_2^{m} t_3^{\ell})\alpha^{-1} = (t_1^{d_1} t_2^{d_2} t_3^{\ell})(t_2^{d_2} t_3^{r})^{-\frac{2m}{d_2}} (t_3^{\frac{2nd_1d_2}{p}})^{x} \in N,$$
$$\alpha(t_2^{d_2} t_3^{r})\alpha^{-1} = (t_2^{d_2} t_3^{r})^{-1} (t_3^{\frac{2nd_1d_2}{p}})^{\frac{p}{2d_1}} \in N$$

show that

$$x = \frac{2prm}{2nd_1{d_2}^2} - \frac{2p\ell}{2nd_1d_2} + \frac{pm}{2d_1d_2} \in \mathbb{Z}, \quad \frac{2m}{d_2} \in \mathbb{Z}, \quad \frac{p}{2d_1} \in \mathbb{Z}.$$

Thus d_2 is a divisor of $2m$ and $2d_1$ is a divisor of p. Since $0 \le m < d_2$, we have $m = 0$ or $d_2 = 2m$.

(i) When $m = 0$:
Since $\frac{2p\ell}{2nd_1d_2} \in \mathbb{Z}$ and $0 \le \ell < \frac{2nd_1d_2}{p}$, we have $\ell = 0$ or $\frac{2nd_1d_2}{2p}$. Therefore N can be represented by one of the following sets of generators

$$N_1 = \langle t_1^{d_1},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{2nd_1d_2}{p}} \rangle, \quad N_2 = \langle t_1^{d_1} t_3^{\frac{2nd_1d_2}{2p}},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{2nd_1d_2}{p}} \rangle.$$

(ii) When $m = \frac{d_2}{2}$:
In this case, we have

$$\frac{2prm}{2nd_1{d_2}^2} - \frac{2p\ell}{2nd_1d_2} + \frac{pm}{2d_1d_2} = \frac{pr}{2nd_1d_2} - \frac{2p\ell}{2nd_1d_2} + \frac{p}{4d_1} \in \mathbb{Z}. \tag{3.7}$$

Since $0 \le \frac{pr}{2nd_1d_2} < 1$ and $0 \le \frac{2p\ell}{2nd_1d_2} < 2$, we have

$$-2 < \frac{pr}{2nd_1d_2} - \frac{2p\ell}{2nd_1d_2} < 1. \tag{3.8}$$

From (3.7) and (3.8), it is easy to show that N can be represented by the following set of generators

$$N = \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\ell},\ t_2^{d_2} t_3^{2\ell},\ t_3^{\frac{2nd_1d_2}{p}} \rangle \qquad (p = 4kd_1,\quad k \in \mathbb{N}),$$
$$N = \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\ell},\ t_2^{d_2} t_3^{2\ell + \frac{2nd_1d_2}{2p}},\ t_3^{\frac{2nd_1d_2}{p}} \rangle \qquad (p = (2k-1)2d_1,\quad k \in \mathbb{N}).$$

Therefore we have proved the lemma. $\square$

Let

$$N_1^r = \langle t_1^{d_1},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{2nd_1d_2}{p}} \rangle, \quad N_2^r = \langle t_1^{d_1} t_3^{\frac{2nd_1d_2}{2p}},\ t_2^{d_2} t_3^{r'},\ t_3^{\frac{2nd_1d_2}{p}} \rangle,$$
$$N_3^{\ell} = \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\ell},\ t_2^{d_2} t_3^{s},\ t_3^{\frac{2nd_1d_2}{p}} \rangle.$$

Then we can obtain the following proposition which is easily proved by using the methods in Proposition 3.3 and Lemma 3.6.

Proposition 3.7 (π_3)**.** *Let N_i $(i=1,2,3)$ be a normal nilpotent subgroup of π_3 in Lemma 3.6 and isomorphic to Γ_p. Then we have the following*:

(1) $N_1 \sim N_2$ *if and only if* $d_2 = p$, $r \equiv r' \pmod{d_2}$.
(2) $N_1 \nsim N_3$, $N_2 \nsim N_3$.
(3) $N_1^r \sim N_1^{r'}$ *if and only if* $r \equiv r' \pmod{d_2}$.
(4) $N_2^r \sim N_2^{r'}$ *if and only if* $r \equiv r' \pmod{d_2}$.
(5) $N_3^\ell \sim N_3^{\ell'}$ *if and only if* $2\ell \equiv 2\ell' \pmod{d_2}$.

Theorem 3.8 (π_3)**.** *Table 3 gives a complete list of all free actions (up to topological conjugacy) of finite groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_3$.*

Table 3

Generators	AC classes of normal nilpotent subgroups	Conditions
${}^{\mu_i}\langle t_2, t_3, \alpha\rangle$	$N_1 = \langle t_1^{d_1},\ t_2^{d_2}t_3^{r_1},\ t_3^{\frac{2nd_1d_2}{p}}\rangle$	
	$N_2 = \langle t_1^{d_1},\ t_2^{d_2}t_3^{r_2},\ t_3^{\frac{2nd_1d_2}{p}}\rangle$	$r_1 \not\equiv r_2 \pmod{d_2}$
	$N_3 = \langle t_1^{d_1}t_3^{\frac{nd_1d_2}{p}},\ t_2^{d_2}t_3^{s_1},\ t_3^{\frac{2nd_1d_2}{p}}\rangle$	$d_2 \neq p$ or $r_i \not\equiv s_1 \pmod{d_2}$
	$N_4 = \langle t_1^{d_1}t_3^{\frac{nd_1d_2}{p}},\ t_2^{d_2}t_3^{s_2},\ t_3^{\frac{2nd_1d_2}{p}}\rangle$	($d_2 \neq p$ or $r_i \not\equiv s_2 \pmod{d_2}$) and $s_1 \not\equiv s_2 \pmod{d_2}$
	$N_5 = \langle t_1^{d_1}t_2^{\frac{d_2}{2}}t_3^{\ell_1},\ t_2^{d_2}t_3^{2\ell_1},\ t_3^{\frac{2nd_1d_2}{p}}\rangle$	$p = 4kd_1$
	$N_6 = \langle t_1^{d_1}t_2^{\frac{d_2}{2}}t_3^{\ell_2},\ t_2^{d_2}t_3^{2\ell_2},\ t_3^{\frac{2nd_1d_2}{p}}\rangle$	$p = 4kd_1, 2\ell_1 \not\equiv 2\ell_2 \pmod{d_2}$
	$N_7 = \langle t_1^{d_1}t_2^{\frac{d_2}{2}}t_3^{\ell_1},\ t_2^{d_2}t_3^{2\ell_1+\frac{nd_1d_2}{p}},\ t_3^{\frac{2nd_1d_2}{p}}\rangle$	$p = (2k-1)2d_1$
	$N_8 = \langle t_1^{d_1}t_2^{\frac{d_2}{2}}t_3^{\ell_2},\ t_2^{d_2}t_3^{2\ell_2+\frac{nd_1d_2}{p}},\ t_3^{\frac{2nd_1d_2}{p}}\rangle$	$p = (2k-1)2d_1, 2\ell_1 \not\equiv 2\ell_2 \pmod{d_2}$

where $r = r_1$ for $i = 1$, $r = r_2$ for $i = 2$, $s = s_1$ for $i = 3$, $s = s_2$ for $i = 4$, $\ell = \ell_1$ for $i = 5, 7$, $\ell = \ell_2$ for $i = 6, 8$, and

$$\mu_1 = \left(I,\ \left(\begin{bmatrix} \frac{r}{2nd_1d_2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_2 = \left(I,\ \left(\begin{bmatrix} \frac{s}{2nd_1d_2} \\ -\frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_3 = \left(I,\ \left(\begin{bmatrix} \frac{\ell}{nd_1d_2} \\ \frac{1}{4} - \frac{\ell}{2nd_1d_2} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{1}{2d_1} & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_4 = \left(I,\ \left(\begin{bmatrix} \frac{\ell}{nd_1d_2} + \frac{1}{2p} \\ \frac{1}{4} - \frac{\ell}{2nd_1d_2} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{1}{2d_1} & \frac{1}{d_2} \end{bmatrix}\right)\right).$$

Note that π_3/N is abelian if and only if $N \supset [\pi_3, \pi_3] = \langle t_2^2 t_3^n,\ t_3^2\rangle$. Therefore it is easy to get Theorem 3.4 of [1] as a corollary of the above theorem.

The following lemma shows all possible finite groups acting freely (up to topological conjugacy) on the 3-dimensional nilmanifold $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_4$.

Lemma 3.9. *Let N be a normal nilpotent subgroup of an almost Bieberbach group π_4 and isomorphic to Γ_p. Then N can be represented by one of the following sets of generators: for $s, w \in \mathbb{N}$,*

(A) $p = 4sd_1$, $p = 2wd_2$:

$$N_{(1,1)} = \langle t_1^{d_1},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle, \qquad N_{(1,2)} = \langle t_1^{d_1},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,$$

$$N_{(2,1)} = \langle t_1^{d_1} t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle, \quad N_{(2,2)} = \langle t_1^{d_1} t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,$$

$$N_{(3,1)} = \langle t_1^{d_1} t_2^{\frac{d_2}{2}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle, \qquad N_{(3,3)} = \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle.$$

(B) $p = 2(2s-1)d_1$, $p = 2wd_2$:

$$\begin{aligned}
N_{(1,1)} &= \langle t_1^{d_1},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,\\
N_{(1,2)} &= \langle t_1^{d_1},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,\\
N_{(2,1)} &= \langle t_1^{d_1} t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,\\
N_{(2,2)} &= \langle t_1^{d_1} t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,\\
N_{(4,1)} &= \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\frac{12nd_1d_2}{4p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,\\
N_{(4,3)} &= \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\frac{4nd_1d_2}{4p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle.
\end{aligned}$$

(C) $p = 4sd_1$, $p = (2w-1)d_2$:

$$\begin{aligned}
N_{(3,2)} &= \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\frac{4nd_1d_2}{4p}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,\\
N_{(3,4)} &= \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\frac{12nd_1d_2}{4p}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle.
\end{aligned}$$

(D) $p = 2(2s-1)d_1$, $p = (2w-1)d_2$:

$$\begin{aligned}
N_{(4,2)} &= \langle t_1^{d_1} t_2^{\frac{d_2}{2}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,\\
N_{(4,4)} &= \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle.
\end{aligned}$$

Proof. Let N be a normal nilpotent subgroup of π_4 and isomorphic to Γ_p. Then by Proposition 2.3,

$$N = \langle\, t_1^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{4nd_1d_2}{p}} \rangle, \quad \left(0 \le m < d_2,\ 0 \le \ell, r < \frac{4nd_1d_2}{p}\right).$$

Since N is a normal nilpotent subgroup of π_4, the following two relations

$$\begin{aligned}
\beta(t_1^{d_1} t_2^m t_3^\ell)\beta^{-1} &= (t_1^{d_1} t_2^m t_3^\ell)(t_2^{d_2} t_3^r)^{-\frac{2m}{d_2}} (t_3^{\frac{4nd_1d_2}{p}})^x \in N,\\
\beta(t_2^{d_2} t_3^r)\beta^{-1} &= (t_2^{d_2} t_3^r)^{-1} (t_3^{\frac{4nd_1d_2}{p}})^{\frac{p}{2d_1}} \in N
\end{aligned}$$

show that $-\frac{2m}{d_2} \in \mathbb{Z}$ $\frac{p}{2d_1} \in \mathbb{Z}$ and

$$x = \frac{2prm}{4nd_1d_2^2} - \frac{2p\ell}{4nd_1d_2} + \frac{pm}{2d_1d_2} \in \mathbb{Z}. \tag{3.9}$$

Since $0 \le m < d_2$ and $\frac{2m}{d_2} \in \mathbb{Z}$, we have $m = 0$ or $\frac{d_2}{2}$. Moreover the following two relations

$$\alpha(t_1^{d_1}t_2^m t_3^\ell)\alpha^{-1} = (t_1^{d_1}t_2^m t_3^\ell)^{-1}(t_3^{\frac{4nd_1d_2}{p}})^y \in N,$$
$$\alpha(t_2^{d_2}t_3^r)\alpha^{-1} = (t_2^{d_2}t_3^r)^{-1}(t_3^{\frac{4nd_1d_2}{p}})^{\frac{2pr}{4nd_1d_2}-\frac{p}{2d_1}} \in N$$

show that

$$y = \frac{2p\ell}{4nd_1d_2} + \frac{p}{2d_2} - \frac{pm}{2d_1d_2} \in \mathbb{Z}. \tag{3.10}$$

Since $\frac{2pr}{4nd_1d_2} - \frac{p}{2d_1} \in \mathbb{Z}$, $\frac{p}{2d_1} \in \mathbb{Z}$ and $0 \le r < \frac{4nd_1d_2}{p}$, we have $r = 0$ or $\frac{4nd_1d_2}{2p}$. Using (3.9) and (3.10), we can classify the normal nilpotent subgroups representing N. Now we can consider the following two cases.

(I) When $m = 0$:
From (3.9), we have $\frac{2p\ell}{4nd_1d_2} \in \mathbb{Z}$. Since $0 \le \ell < \frac{4nd_1d_2}{p}$, we obtain $\ell = 0$ or $\frac{4nd_1d_2}{2p}$. Thus N can be represented by the following two groups:

$$N_1 = \langle t_1^{d_1},\ t_2^{d_2}t_3^r,\ t_3^{\frac{4nd_1d_2}{p}}\rangle, \quad N_2 = \langle t_1^{d_1}t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2}t_3^r,\ t_3^{\frac{4nd_1d_2}{p}}\rangle.$$

Since $r = 0$ or $\frac{4nd_1d_2}{2p}$, we have the following four types of normal nilpotent subgroups representing N:

$$N_{(1,1)} = \langle t_1^{d_1},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}}\rangle, \qquad N_{(1,2)} = \langle t_1^{d_1},\ t_2^{d_2}t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}}\rangle,$$
$$N_{(2,1)} = \langle t_1^{d_1}t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}}\rangle, \quad N_{(2,2)} = \langle t_1^{d_1}t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2}t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}}\rangle.$$

From (3.10), we have $\frac{p}{2d_2} \in \mathbb{Z}$.

(II) When $2m = d_2$:
From (3.9), we have

$$\frac{pr}{4nd_1d_2} - \frac{2p\ell}{4nd_1d_2} + \frac{p}{4d_1} \in \mathbb{Z}. \tag{3.11}$$

Since $0 \le \frac{pr}{4nd_1d_2} < 1$ and $0 \le \frac{2p\ell}{4nd_1d_2} < 2$, we have

$$-2 < \frac{pr}{4nd_1d_2} - \frac{2p\ell}{4nd_1d_2} < 1. \tag{3.12}$$

Since $\frac{p}{2d_1} \in \mathbb{Z}$, there are two cases. But we only deal with the case of $p = 4sd_1$ ($s \in \mathbb{N}$). From (3.11) and (3.12), we have $\frac{pr}{4nd_1d_2} - \frac{2p\ell}{4nd_1d_2} = 0,\ -1$. Hence $r = 2\ell,\ 2\ell - \frac{4nd_1d_2}{p}$, respectively.
(a) When $r = 2\ell$: Since $r = 0$ or $\frac{4nd_1d_2}{2p}$, we have $\ell = 0$ or $\frac{4nd_1d_2}{4p}$, respectively.

(i) $r=0$, $\ell=0$:

$$N_{(3,1)}=\langle t_1^{d_1}t_2^{\frac{d_2}{2}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}}\rangle.$$

From (3.10), we have $\frac{p}{2d_2}\in\mathbb{Z}$.

(ii) $r=\frac{4nd_1d_2}{2p}$ or $\ell=\frac{4nd_1d_2}{4p}$:

$$N_{(3,2)}=\langle t_1^{d_1}t_2^{\frac{d_2}{2}}t_3^{\frac{4nd_1d_2}{4p}},\ t_2^{d_2}t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}}\rangle.$$

From (3.10), we have $\frac{1}{2}+\frac{p}{2d_2}\in\mathbb{Z}$.
Similarly we can obtain the following results.
(b) When $r=2\ell-\frac{4nd_1d_2}{p}$:

$$N_{(3,3)}=\langle t_1^{d_1}t_2^{\frac{d_2}{2}}t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}}\rangle,\quad \frac{p}{2d_2}\in\mathbb{Z}.$$

$$N_{(3,4)}=\langle t_1^{d_1}t_2^{\frac{d_2}{2}}t_3^{\frac{12nd_1d_2}{4p}},\ t_2^{d_2}t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}}\rangle,\quad \frac{1}{2}+\frac{p}{2d_2}\in\mathbb{Z}.$$

Note that $\frac{1}{2}+\frac{p}{2d_2}\in\mathbb{Z}$ if and only if $p=(2w-1)d_2, w\in\mathbb{N}$. By (I) and (II), we have proved (A) and (C). The other cases can be done similarly. $\square$

Proposition 3.10 (π_4)**.** *Let $N_{(i,j)}$ $(i=1,2, j=1,2,3,4)$ be a normal nilpotent subgroup of π_4 in Lemma 3.9 and isomorphic to Γ_p. Then we have the following*:

(1) $N_{(1,2)}\sim N_{(2,1)}$ *if and only if* $d_1=d_2$.
$N_{(1,1)}\nsim N_{(1,2)},\ N_{(1,1)}\nsim N_{(2,1)},\ N_{(1,1)}\nsim N_{(2,2)}$,
$N_{(1,2)}\nsim N_{(2,2)},\ N_{(2,1)}\nsim N_{(2,2)}$.
(2) $N_{(3,1)}\nsim N_{(3,3)},\ N_{(4,1)}\nsim N_{(4,3)}$.
(3) $N_{(3,2)}\sim N_{(3,4)}$ *if and only if* $d_2=p$.
$N_{(4,2)}\sim N_{(4,4)}$ *if and only if* $d_2=p$ *or* $2d_1=p$.
(4) $N_{(1,k)}\nsim N_{(3,j)},\ N_{(1,k)}\nsim N_{(4,j)},\ N_{(2,k)}\nsim N_{(3,j)},\ N_{(2,k)}\nsim N_{(4,j)}$ $(k=1,2)$.

Proof. (1) First we need to find the normalizer $N_{\mathrm{Aff}(\mathcal{H})}(\pi_4)$ by applying the method used in Theorem 3.3 of [1]:

$$\mu(x,y,z,u,v)=\left(\begin{bmatrix}1&x&z\\0&1&y\\0&0&1\end{bmatrix},\ \left(\begin{bmatrix}u\\v\end{bmatrix},\begin{bmatrix}a&b\\c&d\end{bmatrix}\right)\right),$$

where $2x\in\mathbb{Z}$, $2y\in\mathbb{Z}$, $z\in\mathbb{R}$ and

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathbb{Z}_4\rtimes\mathbb{Z}_2=\left\langle\begin{bmatrix}0&-1\\1&0\end{bmatrix},\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right\rangle.$$

Note that $\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]\in\mathrm{Aut}(\mathcal{H})$ can be evaluated respectively by the elements of $\mathbb{Z}_4\rtimes\mathbb{Z}_2$. More precisely, the values of $\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]\in\mathrm{Aut}(\mathcal{H})$ are $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}-\frac{1}{2}\\-\frac{1}{2}\end{smallmatrix}\right],\left[\begin{smallmatrix}0\\-\frac{1}{2}\end{smallmatrix}\right],\left[\begin{smallmatrix}-\frac{1}{2}\\0\end{smallmatrix}\right]$.

Note that the four types of normal nilpotent subgroups $N_{(1,1)},\ N_{(1,2)},\ N_{(2,1)}$, $N_{(2,2)}$ of π_4 are of the same forms as the four types of normal nilpotent subgroups N_1,N_2,N_3,N_4 of π_2, respectively. Thus all calculations to determine affine conjugacy among $N_{(i,j)}$ $(1 \le i, j \le 2)$ are similar to those used in the proof of Proposition 3.3. Therefore we can obtain the following results:

(a) $N_{(1,1)} \sim N_{(1,2)}$ if and only if $m = 0, d_1 = p$.
(b) $N_{(1,1)} \sim N_{(2,1)}$ if and only if $m = 0, d_2 = p$.
(c) $N_{(1,1)} \sim N_{(2,2)}$ if and only if $m = 0, d_1 = d_2 = p$.
(d) $N_{(1,2)} \sim N_{(2,1)}$ if and only if $m = 0, d_1 = d_2$.
(e) $N_{(1,2)} \sim N_{(2,2)}$ if and only if either $m = 0, d_2 = p$ or $2m = d_2, d_1 = \frac{p}{2}$.
(f) $N_{(2,1)} \sim N_{(2,2)}$ if and only if $m = 0, d_1 = p$.

Note that if $2d_1$ and $2d_2$ are divisors of p, then there exist $N_{(i,j)}$ $(1 \le i, j \le 2)$ by (A) and (B) of Lemma 3.9. Therefore we can conclude that $N_{(1,2)} \sim N_{(2,1)}$ if and only if $d_1 = d_2$,

$$\begin{aligned}&N_{(1,1)} \nsim N_{(1,2)},\ N_{(1,1)} \nsim N_{(2,1)},\ N_{(1,1)} \nsim N_{(2,2)},\\ &N_{(1,2)} \nsim N_{(2,2)},\ N_{(2,1)} \nsim N_{(2,2)}.\end{aligned}$$

(3) Suppose that $N_{(4,2)}$ is affinely conjugate to $N_{(4,4)}$, where

$$\begin{aligned}N_{(4,2)} &= \langle t_1^{d_1} t_2^{\frac{d_2}{2}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,\\ N_{(4,4)} &= \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle.\end{aligned}$$

Then there exists $\mu \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_4)$ satisfying either (3.1) or (3.2). From (3.5), we can get $x = 0$ and $y = \frac{d_2}{2p}$. Since $2y = \frac{d_2}{p} \in \mathbb{Z}$, we have $d_2 = p$. The converse is easy by using

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_4).$$

In (3.6), since $\frac{d_2}{2} \neq 0$, we have a contradiction. Another possibility is as follows:

$$\mu(t_1^{d_1} t_2^{\frac{d_2}{2}})\mu^{-1} = t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{-\frac{4nd_1d_2}{2p}}, \quad \mu(t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}})\mu^{-1} = t_2^{d_2} t_3^{-\frac{4nd_1d_2}{2p}}.$$

From these two relations, we can get $x = \frac{d_1}{p}$ and $y = 0$.

Note that in this case we have $p = 2(2s-1)d_1,\ s \in \mathbb{N}$. Therefore $2x = \frac{2d_1}{p} = \frac{2d_1}{2(2s-1)d_1} = \frac{1}{2s-1} \in \mathbb{Z} \Leftrightarrow s = 1 \Leftrightarrow p = 2d_1$. The converse is easy by using

$$\left(\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_4).$$

The other cases can be done similarly. □

Theorem 3.11 (π_4)**.** *Table 4 gives a complete list of all free actions (up to topological conjugacy) of finite groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_4$.*

TABLE 4

Generators	AC classes of normal nilpotent subgroups	Conditions
$^{\mu_i}\langle t_2, \alpha, \beta\rangle$	$N_1 = \langle t_1^{d_1},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}}\rangle$	$p = 2sd_1,\ p = 2wd_2$
	$N_2 = \langle t_1^{d_1},\ t_2^{d_2}t_3^{\frac{2nd_1d_2}{p}},\ t_3^{\frac{4nd_1d_2}{p}}\rangle$	$p = 2sd_1,\ p = 2wd_2$
	$N_3 = \langle t_1^{d_1}t_3^{\frac{2nd_1d_2}{p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}}\rangle$	$p = 2sd_1,\ p = 2wd_2,\ d_1 \neq d_2$
	$N_4 = \langle t_1^{d_1}t_3^{\frac{2nd_1d_2}{p}},\ t_2^{d_2}t_3^{\frac{2nd_1d_2}{p}},\ t_3^{\frac{4nd_1d_2}{p}}\rangle$	$p = 2sd_1,\ p = 2wd_2$
	$N_5 = \langle t_1^{d_1}t_2^{\frac{d_2}{2}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}}\rangle$	$p = 4sd_1,\ p = 2wd_2$
	$N_6 = \langle t_1^{d_1}t_2^{\frac{d_2}{2}}t_3^{\frac{nd_1d_2}{p}},\ t_2^{d_2}t_3^{\frac{2nd_1d_2}{p}},\ t_3^{\frac{4nd_1d_2}{p}}\rangle$	$p = 4sd_1,\ p = (2w-1)d_2$
	$N_7 = \langle t_1^{d_1}t_2^{\frac{d_2}{2}}t_3^{\frac{2nd_1d_2}{p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}}\rangle$	$p = 4sd_1,\ p = 2wd_2$
	$N_8 = \langle t_1^{d_1}t_2^{\frac{d_2}{2}}t_3^{\frac{3nd_1d_2}{p}},\ t_2^{d_2}t_3^{\frac{2nd_1d_2}{p}},\ t_3^{\frac{4nd_1d_2}{p}}\rangle$	$p = 4sd_1,\ p = (2w-1)d_2,\ d_2 \neq p$
	$N_9 = \langle t_1^{d_1}t_2^{\frac{d_2}{2}}t_3^{\frac{3nd_1d_2}{p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}}\rangle$	$p = (2s-1)2d_1,\ p = 2wd_2$
	$N_{10} = \langle t_1^{d_1}t_2^{\frac{d_2}{2}},\ t_2^{d_2}t_3^{\frac{2nd_1d_2}{p}},\ t_3^{\frac{4nd_1d_2}{p}}\rangle$	$p = (2s-1)2d_1,\ p = (2w-1)d_2$
	$N_{11} = \langle t_1^{d_1}t_2^{\frac{d_2}{2}}t_3^{\frac{nd_1d_2}{p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}}\rangle$	$p = (2s-1)2d_1,\ p = 2wd_2$
	$N_{12} = \langle t_1^{d_1}t_2^{\frac{d_2}{2}}t_3^{\frac{2nd_1d_2}{p}},\ t_2^{d_2}t_3^{\frac{2nd_1d_2}{p}},\ t_3^{\frac{4nd_1d_2}{p}}\rangle$	$p = (2s-1)2d_1,\ p = (2w-1)d_2$ $d_2 \neq p,\ 2d_1 \neq p$

where

$$\mu_1 = \left(I,\ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_2 = \left(I,\ \left(\begin{bmatrix} \frac{1}{2p} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_3 = \left(I,\ \left(\begin{bmatrix} 0 \\ -\frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_4 = \left(I,\ \left(\begin{bmatrix} \frac{1}{2p} \\ -\frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_5 = \left(I,\ \left(\begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{1}{2d_1} & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_6 = \left(I,\ \left(\begin{bmatrix} \frac{1}{2p} \\ \frac{1}{4} - \frac{1}{4p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{1}{2d_1} & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_7 = \left(I,\ \left(\begin{bmatrix} 0 \\ \frac{1}{4} - \frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{1}{2d_1} & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_8 = \left(I,\ \left(\begin{bmatrix} \frac{1}{2p} \\ \frac{1}{4} - \frac{3}{4p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{1}{2d_1} & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_9 = \left(I,\ \left(\begin{bmatrix} 0 \\ \frac{1}{4} - \frac{3}{4p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{1}{2d_1} & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_{10} = \left(I, \left(\begin{bmatrix} \frac{1}{2p} \\ \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{1}{2d_1} & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_{11} = \left(I, \left(\begin{bmatrix} 0 \\ \frac{1}{4} - \frac{1}{4p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{1}{2d_1} & \frac{1}{d_2} \end{bmatrix}\right)\right),$$

$$\mu_{12} = \left(I, \left(\begin{bmatrix} \frac{1}{2p} \\ \frac{1}{4} - \frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{1}{2d_1} & \frac{1}{d_2} \end{bmatrix}\right)\right).$$

Note that π_4/N is abelian if and only if $N \supset [\pi_4, \pi_4] = \langle t_1^2,\ t_2^2,\ t_3^2,\ t_1t_2t_3\,\rangle$. Thus it is not hard to get Theorem 3.5 of [1] as a corollary of the above theorem.

In the following proposition, we show when affine conjugacy occurs between 4 types of normal nilpotent subgroups N_j $(j = 1, 2, 3, 4)$ of $\pi_{5,i}$ $(i = 1, 2, 3)$. It can be proved by applying the method in [1, Theorem 3.7] and Lemma 3.1.

Proposition 3.12 (π_5)**.** *Let N_j $(j = 1, 2, 3, 4)$ be a normal nilpotent subgroup of $\pi_{5,i}$ $(i = 1, 2, 3)$ in Lemma 3.1 and isomorphic to Γ_p. Then we have the following*:

(1) $N_1 \sim N_4$ *if and only if* $m = 0, d_1 = d_2 = p$.

(2) $N_2 \sim N_3$ *if and only if* $m = 0, d_1 = d_2$.

(3) $N_1 \nsim N_2$, $N_1 \nsim N_3$, $N_2 \nsim N_4$, $N_3 \nsim N_4$.

Note that in Lemma 3.1, the following conditions of a normal subgroup of $\pi_{5,i}$,

$$\frac{d_1}{d_2} + \frac{m^2}{d_1 d_2} \in \mathbb{Z},\ \frac{m}{d_1} \in \mathbb{Z},\ \frac{d_2}{d_1} \in \mathbb{Z}$$

are crucial to prove the following theorem. Let $d_2 = d_1 s$, $m = d_1 t$. Then we have

$$\frac{d_1}{d_2} + \frac{m^2}{d_1 d_2} \in \mathbb{Z} \Longleftrightarrow \frac{1 + t^2}{s} \in \mathbb{Z} \Longleftrightarrow 1 + t^2 \equiv 0 \pmod{s}.$$

Since $0 \le m < d_2$, if $s = 1$, then we must have $m = 0$. Also, if $s, t \in 2\mathbb{N}$, then $1 + t^2 \not\equiv 0 \pmod{s}$. So, if s is even, then t must be odd.

Theorem 3.13 (π_5)**.** *Table 5 gives a complete list of all free actions* (*up to topological conjugacy*) *of finite groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_{5,i}$ $(1 \le i \le 3)$.*

Table 5

Generators	AC classes of normal nilpotent subgroups	Conditions
${}^{\mu_i}\langle t_1, t_2, \hat{\alpha}\rangle$	$N_1 = \langle t_1^{d_1} t_2^{td_1},\ t_2^{sd_1},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	$0 \le t < s$ and if s is even, then t is odd
	$N_2 = \langle t_1^{d_1} t_2^{td_1},\ t_2^{sd_1} t_3^{\frac{Ksd_1^2}{2p}},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	$0 \le t < s$ and if s is even, then t is odd
	$N_3 = \langle t_1^{d_1} t_2^{td_1} t_3^{\frac{Ksd_1^2}{2p}},\ t_2^{sd_1},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	$0 \le t < s$, $s \ne 1$ and if s is even, then t is odd
	$N_4 = \langle t_1^{d_1} t_2^{td_1} t_3^{\frac{Ksd_1^2}{2p}},\ t_2^{sd_1} t_3^{\frac{Ksd_1^2}{2p}},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	$0 \le t < s$, if $s = 1$, then $d_1 \ne p$, and if s is even, then t is odd

where $K = 4n$ for the cases of $\pi_{5,2}$ and $\pi_{5,3}$, $K = 4n - 2$ for the case of $\pi_{5,1}$, $\hat{\alpha} = \alpha^{-1}t_3$ for the case of $\pi_{5,2}$, $\hat{\alpha} = \alpha$ for the cases of $\pi_{5,1}$ and $\pi_{5,3}$, and

$$\mu_1 = \left(I,\ \left(\begin{bmatrix} 0 \\ \frac{t}{2s} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{t}{sd_1} & \frac{1}{sd_1} \end{bmatrix}\right)\right),$$

$$\mu_2 = \left(I,\ \left(\begin{bmatrix} \frac{1}{2p} \\ \frac{t}{2s} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{t}{sd_1} & \frac{1}{sd_1} \end{bmatrix}\right)\right),$$

$$\mu_3 = \left(I,\ \left(\begin{bmatrix} 0 \\ \frac{t}{2s} - \frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{t}{sd_1} & \frac{1}{sd_1} \end{bmatrix}\right)\right),$$

$$\mu_4 = \left(I,\ \left(\begin{bmatrix} \frac{1}{2p} \\ \frac{t}{2s} - \frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{t}{sd_1} & \frac{1}{sd_1} \end{bmatrix}\right)\right).$$

Note that $\pi_{5,i}/N$ $(i = 1, 2, 3)$ is abelian if and only if $N \supset [\pi_{5,i}, \pi_{5,i}] = \langle t_1 t_2,\ t_2^2,\ t_3^K \rangle$. Therefore it is easy to obtain Theorem 3.7 of [1] as a corollary of the above theorem.

Now we shall find all possible finite groups acting freely (up to topological conjugacy) on the 3-dimensional nilmanifold $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_{6,i}$ $(i = 1, 2, 3, 4)$.

Let N be a normal nilpotent subgroup of $\pi_{6,i}$ and isomorphic to Γ_p. Then by Proposition 2.3,

$$N = \langle\, t_1^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{Kd_1d_2}{p}} \rangle, \quad \left(0 \le m < d_2,\ 0 \le \ell, r < \frac{Kd_1d_2}{p}\right),$$

where $K = 3n$ for $i = 1, 2$, $K = 3n - 2$ for $i = 3$ and $K = 3n - 1$ for $i = 4$.

We will begin by considering the following general situation. Let

$$N = \langle\, {t_1}^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{Kd_1d_2}{p}} \rangle, \quad N' = \langle\, {t_1}^{d_1} t_2^m t_3^{\ell'},\ t_2^{d_2} t_3^{r'},\ t_3^{\frac{Kd_1d_2}{p}} \rangle$$

be two normal nilpotent subgroups of $\pi_{6,i}$. Since N (or N') is a normal nilpotent subgroup of $\pi_{6,i}$, the following two relations

$$\alpha(t_1^{d_1} t_2^m t_3^\ell)\alpha^{-1} = (t_1^{d_1} t_2^m t_3^\ell)^{-\frac{m}{d_1}} (t_2^{d_2} t_3^r)^x (t_3^{\frac{Kd_1d_2}{p}})^y \in N,$$

$$\alpha^2(t_2^{d_2} t_3^r)(\alpha^2)^{-1} = (t_1^{d_1} t_2^m t_3^\ell)^{\frac{d_2}{d_1}} (t_2^{d_2} t_3^r)^{-\frac{m}{d_1}} (t_3^{\frac{Kd_1d_2}{p}})^z \in N$$

show that d_1 is a common divisor of m and d_2, and $x = \frac{d_1}{d_2} - \frac{m}{d_2} + \frac{m^2}{d_1 d_2} \in \mathbb{Z}$, which induce that $d_2 = (2s - 1)d_1$, $s \in \mathbb{N}$, and

$$\begin{aligned} y = &-\frac{pr}{K{d_2}^2} + \frac{prm}{Kd_1{d_2}^2} - \frac{prm^2}{K{d_1}^2{d_2}^2} - \frac{pm}{d_2} + \frac{pm^2}{d_1 d_2} - \frac{pm^3}{{d_1}^2 d_2} + \frac{p\ell m}{K{d_1}^2 d_2} \\ &+ \frac{pm^2}{2d_1d_2}\left(\frac{m}{d_1} - 1\right) + \frac{p\ell}{Kd_1d_2} - \frac{pm(m+1)}{2d_1d_2} \in \mathbb{Z}, \\ z = &-\frac{prm}{K{d_1}^2 d_2} - \frac{pm}{{d_1}^2} + \frac{p}{{d_1}^2}\left(d_1 m - \frac{\ell}{K}\right) + \frac{pm}{2d_1}\left(\frac{d_2}{d_1} - 1\right) + \frac{pr}{Kd_1d_2} \in \mathbb{Z}. \end{aligned}$$

Since d_2 is a divisor of p and $d_2=(2s-1)d_1$, $s\in\mathbb{N}$, we can get

$$-\frac{pr}{K{d_2}^2}+\frac{prm}{Kd_1{d_2}^2}-\frac{prm^2}{K{d_1}^2{d_2}^2}+\frac{p\ell m}{K{d_1}^2d_2}+\frac{p\ell}{Kd_1d_2}-\frac{pm(m+1)}{2d_1d_2}\in\mathbb{Z}, \tag{3.13}$$

$$-\frac{prm}{K{d_1}^2d_2}-\frac{p\ell}{K{d_1}^2}+\frac{pr}{Kd_1d_2}\in\mathbb{Z}. \tag{3.14}$$

If N is affinely conjugate to N', then there exists $\mu\in N_{\mathrm{Aff}(\mathcal{H})}(\pi_{6,i})$ satisfying either (3.1) or (3.2). In case (3.2), we have the following result.

Proposition 3.14. *Let N and N' be such normal nilpotent subgroups of $\pi_{6,i}$ whose sets of generators are*

$$N=\langle\, {t_1}^{d_1}t_2^m t_3^\ell,\, t_2^{d_2}t_3^r,\, t_3^{\frac{Kd_1d_2}{p}}\,\rangle,\quad N'=\langle\, {t_1}^{d_1}t_2^m t_3^{\ell'},\, t_2^{d_2}t_3^{r'},\, t_3^{\frac{Kd_1d_2}{p}}\,\rangle.$$

If $N\sim N'$, then $m=0$, $\ell=r$, $\ell'=r'$, $d_1=d_2$, and $\ell+\ell'=0$, $\frac{Kd_1}{3}$, $\frac{2Kd_1}{3}$, Kd_1, $\frac{4Kd_1}{3}$ or $\frac{5Kd_1}{3}$ for the cases of $\pi_{6,1}$ and $\pi_{6,2}$, $\ell+\ell'=0$ or Kd_1 for the cases of $\pi_{6,3}$ and $\pi_{6,4}$.

Proof. Recall that the normalizer $N_{\mathrm{Aff}(\mathcal{H})}(\pi_{6,i})$ of $\pi_{6,i}$ has been obtained [1, Theorem 3.8]:

$$\mu(x,y,z,u,v):=\left(\begin{bmatrix}1&x&z\\0&1&y\\0&0&1\end{bmatrix},\ \left(\begin{bmatrix}u\\v\end{bmatrix},\begin{bmatrix}a&b\\c&d\end{bmatrix}\right)\right)\in N_{\mathrm{Aff}(\mathcal{H})}(\pi_{6,i}),$$

where $z\in\mathbb{R}$ and if $ad-bc=1$, then

$$x=\frac{2r-s}{3},\quad y=\frac{r+s}{3}\quad (r,\, s\in\mathbb{Z}),$$

$$x+u=\frac{1}{2}ab+\frac{r'}{K},\ y+v=-\frac{1}{2}cd+\frac{s'}{K}\quad (r',\, s'\in\mathbb{Z}),$$

if $ad-bc=-1$, then

$$x=\frac{p+q}{3},\quad y=\frac{2p-q}{3}\quad (p,\, q\in\mathbb{Z}),$$

$$x+u=-\frac{1}{2}ab+\frac{r'}{K},\ y+v=\frac{1}{2}cd+\frac{s'}{K}\quad (r',\, s'\in\mathbb{Z}),$$

and

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathbb{Z}_6\rtimes\mathbb{Z}_2=\left\langle\begin{bmatrix}0&1\\-1&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\rangle.$$

Note that $\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]\in\mathrm{Aut}(\mathcal{H})$ can be evaluated respectively by the elements of $\mathbb{Z}_6\rtimes\mathbb{Z}_2$. More precisely, the values of $\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]\in\mathrm{Aut}(\mathcal{H})$ are $\left[\begin{smallmatrix}0\\\frac{1}{2}\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}-\frac{1}{2}\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}-\frac{1}{3}\\\frac{1}{3}\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}-\frac{1}{3}\\-\frac{1}{6}\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}\frac{1}{6}\\\frac{1}{3}\end{smallmatrix}\right]$.

From (3.6), we have $b = c = 1$ and so $d_1 = d_2$. Since $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \in \mathbb{Z}_6 \rtimes \mathbb{Z}_2$, the corresponding element $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in \mathrm{Aut}(\mathcal{H})$ is $\left[\begin{smallmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{smallmatrix}\right]$. Using these values, we can get

$$x - \frac{1}{3} = -\frac{\ell + r'}{Kd_2}, \quad y + \frac{1}{3} = \frac{\ell' + r}{Kd_1}. \tag{3.15}$$

Since $m = 0$, from (3.13) and (3.14), we have

$$-\frac{p\ell}{K{d_1}^2} + \frac{pr}{Kd_1d_2} = -\frac{p\ell}{Kd_1d_2} + \frac{pr}{Kd_1d_2} \in \mathbb{Z}. \tag{3.16}$$

Since $0 \le \frac{p\ell}{Kd_1d_2}, \frac{pr}{Kd_1d_2} < 1$, we have $\ell = r$ and $\ell' = r'$. Since $ad - bc = -1$, $x + y$ and $2x - y$ are integers, $3x$ and $3y$ must be integers. From (3.15), we obtain

$$3x = 1 - \frac{3(\ell + r')}{Kd_1} \in \mathbb{Z}, \quad 3y = \frac{3(\ell' + r)}{Kd_1} - 1 \in \mathbb{Z}.$$

Thus we have $\frac{3(\ell+r')}{Kd_1} = \frac{3(\ell'+r)}{Kd_1} = \frac{3(\ell+\ell')}{Kd_1} \in \mathbb{Z}$. Since $0 \le \frac{\ell}{Kd_1}, \frac{\ell'}{Kd_1} < \frac{d_2}{p} \le 1$ and $\frac{3(\ell+\ell')}{Kd_1} \in \mathbb{Z}$, we have

$$\frac{3(\ell + \ell')}{Kd_1} = 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5.$$

For all these values, we have $x + y \in \mathbb{Z}$, $2x - y \in \mathbb{Z}$. Also, we have

$$\frac{\ell + \ell'}{d_1} = 0, \quad \frac{K}{3}, \quad \frac{2K}{3}, \quad K, \quad \frac{4K}{3}, \quad \frac{5K}{3}.$$

Since $x - \frac{1}{3}$ and $y + \frac{1}{3}$ are multiples of $\frac{1}{K}$, $\frac{\ell+\ell'}{d_1}$ must be an integer. In $\pi_{6,1}$ and $\pi_{6,2}$, since $K = 3n$, we have

$$\frac{\ell + \ell'}{d_1} = 0, \quad \frac{K}{3}, \quad \frac{2K}{3}, \quad K, \quad \frac{4K}{3}, \quad \frac{5K}{3} \in \mathbb{Z}.$$

But in $\pi_{6,3}$ and $\pi_{6,4}$, since $K = 3n - 2$ or $3n - 1$, we have $\frac{\ell+\ell'}{d_1} = 0$ or K, respectively. $\square$

In case (3.1), we have the following lemma:

Lemma 3.15. *Let N and N' be such normal nilpotent subgroups of $\pi_{6,i}$ whose sets of generators are*

$$N = \langle\, {t_1}^{d_1} t_2^m t_3^\ell,\, t_2^{d_2} t_3^r,\, t_3^{\frac{Kd_1d_2}{p}} \,\rangle, \quad N' = \langle\, {t_1}^{d_1} t_2^m t_3^{\ell'},\, t_2^{d_2} t_3^{r'},\, t_3^{\frac{Kd_1d_2}{p}} \,\rangle.$$

If there exists $\mu \in N_{\mathit{Aff}(\mathcal{H})}(\pi_{6,i})$ satisfying the relations stated in (3.1), then

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right),$$

where $x = \frac{r-r'}{Kd_2}$ and $y = -\frac{(\ell-\ell')}{Kd_1} + \frac{m}{d_1}x$.

Proof. From (3.5), we have $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \mathbb{Z}_6 \rtimes \mathbb{Z}_2$. Therefore the corresponding element $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in \mathrm{Aut}(\mathcal{H})$ is $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. Using this, we obtain

$$x = \frac{r - r'}{Kd_2}, \quad y = -\frac{\ell - \ell'}{Kd_1} + \frac{m(r - r')}{Kd_1 d_2} = -\frac{\ell - \ell'}{Kd_1} + \frac{mx}{d_1}. \tag{3.17}$$ □

Remark. In case (3.1), since $ad - bc = 1$, $x + y$ and $-x + 2y$ are integers, $3x$ and $3y$ must be integers. By the same arguments used in the proof of Proposition 3.14 and (3.17), we obtain

$$\frac{3(r - r')}{Kd_2} = \pm 2, \ \pm 1, \quad 0, \qquad \frac{3(\ell - \ell')}{Kd_1} = \pm 2, \ \pm 1, \quad 0.$$

Assume $r \geq r'$. Then from (3.17) we have

$$x = \frac{r - r'}{Kd_2} = 0, \ \ \frac{1}{3}, \ \ \frac{2}{3}, \quad y = (\pm\frac{2}{3}, \ \ \pm\frac{1}{3}, \ \ 0) + \frac{m}{d_1}x.$$

Thus we can consider the following three cases: for $w \in \mathbb{N} \cup \{0\}$,

(I) When $m = 3wd_1$,

$$N_1 = \langle\, {t_1}^{d_1} t_2^{3wd_1} t_3^{\ell},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{Kd_1d_2}{p}} \,\rangle,$$

(II) When $m = (3w + 1)d_1$,

$$N_2 = \langle\, {t_1}^{d_1} t_2^{(3w+1)d_1} t_3^{\ell},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{Kd_1d_2}{p}} \,\rangle,$$

(III) When $m = (3w + 2)d_1$,

$$N_3 = \langle\, {t_1}^{d_1} t_2^{(3w+2)d_1} t_3^{\ell},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{Kd_1d_2}{p}} \,\rangle.$$

In the next theorem, we show when affine conjugacy occurs among three types of nilpotent subgroups N_j $(j = 1, 2, 3)$ of $\pi_{6,i}$ $(i = 1, 2)$.

Let

$$N_1^{\ell,r} = \langle\, {t_1}^{d_1} t_2^{3wd_1} t_3^{\ell},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{Kd_1d_2}{p}} \,\rangle, \quad N_2^{\ell,r} = \langle\, {t_1}^{d_1} t_2^{(3w+1)d_1} t_3^{\ell},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{Kd_1d_2}{p}} \,\rangle,$$

$$N_3^{\ell,r} = \langle\, {t_1}^{d_1} t_2^{(3w+2)d_1} t_3^{\ell},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{Kd_1d_2}{p}} \,\rangle.$$

Then we can obtain the following proposition which can be proved by using the method in Proposition 3.3.

Proposition 3.16 ($\pi_6 - a$)**.** *Let N_j $(j = 1, 2, 3)$ be a normal nilpotent subgroup of $\pi_{6,i}$ $(i = 1, 2)$ and isomorphic to Γ_p. Then we have the following:*

(1) $N_1^{\ell,r} \sim N_1^{\ell',r'}$ *if and only if* $(r - r', \ell - \ell') = (0, 0)$, $(\frac{Kd_2}{3}, \frac{Kd_1}{3})$, $(\frac{Kd_2}{3}, -\frac{2Kd_1}{3})$, $(\frac{2Kd_2}{3}, \frac{2Kd_1}{3})$, $(\frac{2Kd_2}{3}, -\frac{Kd_1}{3})$.

(2) $N_2^{\ell,r} \sim N_2^{\ell',r'}$ *if and only if* $(r - r', \ell - \ell') = (0, 0)$, $(\frac{Kd_2}{3}, \frac{2Kd_1}{3})$, $(\frac{Kd_2}{3}, -\frac{Kd_1}{3})$, $(\frac{2Kd_2}{3}, \frac{Kd_1}{3})$, $(\frac{2Kd_2}{3}, -\frac{2Kd_1}{3})$.

(3) $N_3^{\ell,r} \sim N_3^{\ell',r'}$ *if and only if* $(r - r', \ell - \ell') = (0, 0)$, $(\frac{Kd_2}{3}, 0), (\frac{2Kd_2}{3}, 0)$.

(4) $N_1 \nsim N_2$, $N_1 \nsim N_3$, $N_2 \nsim N_3$.

Proof. Assume $N_1^{\ell,r}$ is affinely conjugate to $N_1^{\ell',r'}$. Then there exists

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_{6,i})$$

satisfying the following two relations:

$$\mu({t_1}^{d_1}{t_2}^{3w}t_3^{\ell})\mu^{-1} = t_1^{d_1}{t_2}^{3w}t_3^{\ell'}, \quad \mu(t_2^{d_2}t_3^{r})\mu^{-1} = t_2^{d_2}t_3^{r'}.$$

By Lemma 3.15 and remark above, we obtain $\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right], \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right) = \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right)$,

$$(3.18)\quad x = \frac{r-r'}{Kd_2} = 0,\ \ \frac{1}{3},\ \frac{2}{3}, \qquad y = -\frac{\ell-\ell'}{Kd_1} + \frac{mx}{d_1} = (\pm\frac{2}{3},\ \ \pm\frac{1}{3},\ \ 0) + 3wx.$$

Thus we can consider the following three cases:

(i) When $x = 0$ $(r = r')$.

Since $x = 0$ and $x + y \in \mathbb{Z}$, we have $y = 0$ and so $\ell = \ell'$.

(ii) When $x = \frac{1}{3}$ $(r - r' = \frac{Kd_2}{3})$.

Since $x+y \in \mathbb{Z}$ and $-x+2y \in \mathbb{Z}$, from (3.18), we have $y = -\frac{1}{3}+w$ or $\frac{2}{3}+w$. Hence

$$(r-r', \ell-\ell') = (\frac{Kd_2}{3}, \frac{Kd_1}{3}), \qquad (\frac{Kd_2}{3},\quad -\frac{2Kd_1}{3}).$$

The converse is easy by using

$$\left(\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3}+w \\ 0 & 0 & 1 \end{bmatrix}, \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right) \right) \text{ or } \left(\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3}+w \\ 0 & 0 & 1 \end{bmatrix}, \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right) \right),$$

respectively.

(iii) When $x = \frac{2}{3}$ $(r - r' = \frac{2Kd_2}{3})$.

Since $x+y \in \mathbb{Z}$ and $-x+2y \in \mathbb{Z}$, from (3.18), we have $y = -\frac{2}{3}+2w,\ \frac{1}{3}+2w$. Hence

$$(r-r', \ell-\ell') = (\frac{2Kd_2}{3}, \frac{2Kd_1}{3}), \quad (\frac{2Kd_2}{3}, -\frac{Kd_1}{3}).$$

The converse is easy by using

$$\left(\begin{bmatrix} 1 & \frac{2}{3} & 0 \\ 0 & 1 & -\frac{2}{3}+2w \\ 0 & 0 & 1 \end{bmatrix}, \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right) \right) \text{ or } \left(\begin{bmatrix} 1 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3}+2w \\ 0 & 0 & 1 \end{bmatrix}, \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right) \right),$$

respectively. The proofs of (2) and (3) are similar to that of (1) and we omit their proofs. (4) is an immediate consequence of Theorem 3.2. □

Proposition 3.17 $(\pi_6 - b)$**.** *Let N_j $(j = 1, 2, 3)$ be a normal nilpotent subgroup of $\pi_{6,i}$ $(i = 3, 4)$ and isomorphic to Γ_p. If $r \neq r'$ or $\ell \neq \ell'$, then*

$$N_i^{\ell,r} \nsim N_j^{\ell',r'} \quad (1 \leq i, \quad j \leq 3).$$

Proof. Assume $N_i^{\ell,r}$ is affinely conjugate to $N_j^{\ell',r'}$ by conjugation of

$$\mu(x, y, z, u, v) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_{6,i}) \ (i = 3, 4).$$

By the same arguments in the proof of the preceding proposition and remark, we have

$$x = \frac{r-r'}{Kd_2} = 0,\ \frac{1}{3},\quad \frac{2}{3},\quad y = -\frac{\ell-\ell'}{Kd_1} + \frac{mx}{d_1} = (\pm\frac{2}{3},\quad \pm\frac{1}{3},\quad 0) + \frac{mx}{d_1}.$$

Recall that $K = 3n-2$ for the case of $\pi_{6,3}$, $K = 3n-1$ for the case of $\pi_{6,4}$ and

$$x + u = \frac{1}{2}ab + \frac{r'}{K},\quad y + v = -\frac{1}{2}cd + \frac{s'}{K}\quad (r',\ s' \in \mathbb{Z}).$$

Hence $\frac{r-r'}{d_2} \in \mathbb{Z}$ implies that $x = 0$ and $r = r'$. Therefore $\frac{\ell-\ell'}{d_1} \in \mathbb{Z}$ implies that $y = 0$ and $\ell = \ell'$. □

Theorem 3.18 (π_6)**.** *Tables* 6-1 *and* 6-2 *give a complete list of all free actions* (*up to topological conjugacy*) *of finite groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_{6,i}$* $(1 \le i \le 4)$.

Table 6-1

Generators	AC classes of normal nilpotent subgroups	Conditions ($s \in 2\mathbb{N}-1$)
${}^{\mu}\langle t_1, t_2, \hat{\alpha}\rangle$	$N_1 = \langle t_1^{d_1} t_2^{3wd_1} t_3^{\ell_1},\ t_2^{sd_1} t_3^{r_1},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	$(*)$
	$N_2 = \langle t_1^{d_1} t_2^{3wd_1} t_3^{\ell_2},\ t_2^{sd_1} t_3^{r_2},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	$(u,v) \neq (1,1), (1,-2), (2,2), (2,-1)$
	$N_3 = \langle t_1^{d_1} t_2^{(3w+1)d_1} t_3^{\ell_1},\ t_2^{sd_1} t_3^{r_1},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	
	$N_4 = \langle t_1^{d_1} t_2^{(3w+1)d_1} t_3^{\ell_2},\ t_2^{sd_1} t_3^{r_2},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	$(u,v) \neq (1,2), (1,-1), (2,1), (2,-2)$
	$N_5 = \langle t_1^{d_1} t_2^{(3w+2)d_1} t_3^{\ell_1},\ t_2^{sd_1} t_3^{r_1},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	
	$N_6 = \langle t_1^{d_1} t_2^{(3w+2)d_1} t_3^{\ell_2},\ t_2^{sd_1} t_3^{r_2},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	$(u,v) \neq (1,0), (2,0)$

where $u = \frac{3(r_1-r_2)}{Ksd_1}$, $v = \frac{3(\ell_1-\ell_2)}{Kd_1}$, $w \in \mathbb{N}\cup\{0\}$, $K = 3n$, $\hat{\alpha} = \alpha$ *for the case of* $\pi_{6,1}$, *and* $\hat{\alpha} = \alpha^{-1}t_3$ *for the case of* $\pi_{6,2}$.
Here $(*)$: *if* $s = 1, \ell_i = r_i\ (i = 1,2), (u,v) \neq (1,1)$, *then* $\ell_1 + \ell_2 \neq \frac{jKd_1}{3}$ $(0 \le j \le 5)$.

Table 6-2

Generators	AC classes of normal nilpotent subgroups	Conditions ($s \in 2\mathbb{N}-1$)
${}^{\mu}\langle t_1, t_2, \hat{\alpha}\rangle$	$N_1 = \langle t_1^{d_1} t_2^{3wd_1} t_3^{\ell_1},\ t_2^{sd_1} t_3^{r_1},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	$(*)$
	$N_2 = \langle t_1^{d_1} t_2^{3wd_1} t_3^{\ell_2},\ t_2^{sd_1} t_3^{r_2},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	$\ell_1 \neq \ell_2$ or $r_1 \neq r_2$
	$N_3 = \langle t_1^{d_1} t_2^{(3w+1)d_1} t_3^{\ell_1},\ t_2^{sd_1} t_3^{r_1},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	
	$N_4 = \langle t_1^{d_1} t_2^{(3w+1)d_1} t_3^{\ell_2},\ t_2^{sd_1} t_3^{r_2},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	$\ell_1 \neq \ell_2$ or $r_1 \neq r_2$
	$N_5 = \langle t_1^{d_1} t_2^{(3w+2)d_1} t_3^{\ell_1},\ t_2^{sd_1} t_3^{r_1},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	
	$N_6 = \langle t_1^{d_1} t_2^{(3w+2)d_1} t_3^{\ell_2},\ t_2^{sd_1} t_3^{r_2},\ t_3^{\frac{Ksd_1^2}{p}}\rangle$	$\ell_1 \neq \ell_2$ or $r_1 \neq r_2$

where $w \in \mathbb{N} \cup \{0\}$, $K = 3n - 2$ *for the case of* $\pi_{6,3}$, $K = 3n - 1$ *for the case of* $\pi_{6,4}$, $\hat{\alpha} = \alpha^{-1} t_3$ *for the case of* $\pi_{6,3}$, *and* $\hat{\alpha} = \alpha$ *for the case of* $\pi_{6,4}$.
Here $(*)$: *if* $s = 1, \ell_i = r_i (i = 1, 2), \ell_1 \neq \ell_2$, *then* $\ell_1 + \ell_2 \neq 0, Kd_1$.

$$\mu = \left(I, \left(\begin{bmatrix} \frac{r}{Ksd_1^2} \\ \frac{t}{2s} - \frac{\ell}{Ksd_1^2} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{t}{sd_1} & \frac{1}{sd_1} \end{bmatrix} \right) \right),$$

where $t = 3w$ *for* $j = 1, 2$, $t = 3w + 1$ *for* $j = 3, 4$, $t = 3w + 2$ *for* $j = 5, 6$, $\ell = \ell_1, r = r_1$ *for* $j = 1, 3, 5$, *and* $\ell = \ell_2, r = r_2$ *for* $j = 2, 4, 6$.

Note that $\pi_{6,i}/N$ $(i = 1, 2, 3, 4)$ is abelian if and only if $N \supset [\pi_{6,i}, \pi_{6,i}] = \langle t_2 t_1^{-1},\ t_1^{-1} t_2^{-2},\ t_3^{3n} \rangle = \langle t_1 t_2^{-1},\ t_2^3,\ t_3^{3n} \rangle$. Thus it is not hard to get Theorems 3.8 and 3.9 of [1] as corollaries of the above theorem by adding the abelian condition.

For the case of $\pi_{7,i}$ $(i = 1, 2, 3, 4)$, we can obtain the following result [11], by applying the same methods used in Proposition 3.12.

Proposition 3.19 (π_7). *Let* N_j $(j = 1, 2, 3, 4)$ *be a normal nilpotent subgroup of* $\pi_{7,i}$ $(i = 1, 2, 3, 4)$ *in Lemma 3.1 and isomorphic to* Γ_p. *Then we have the following*:
(1) $N_2 \sim N_3$ *if and only if* $m = 0$, $d_1 = d_2$.
(2) $N_1 \nsim N_2$, $N_1 \nsim N_3$, $N_1 \nsim N_4$, $N_2 \nsim N_4$, $N_3 \nsim N_4$.

Note that the following conditions of a normal subgroup of $\pi_{7,i}$ in Lemma 3.1

$$\frac{d_1}{d_2} + \frac{m(m - d_1)}{d_1 d_2} \in \mathbb{Z},\ \frac{m}{d_1} \in \mathbb{Z},\ \frac{d_2}{d_1} \in \mathbb{Z}$$

are critical to prove the next theorem. Let $d_2 = d_1 s$, $m = d_1 t$. Then we have

$$\frac{d_1}{d_2} + \frac{m(m - d_1)}{d_1 d_2} \in \mathbb{Z} \Longleftrightarrow \frac{t(t-1)+1}{s} \in \mathbb{Z} \Longleftrightarrow t(t-1) + 1 \equiv 0 \pmod{s}.$$

Since $0 \le m < d_2$, if $s = 1$, then we must have $m = 0$. Also, since $t(t-1) + 1$ is odd, if $s \in 2\mathbb{N}$, then $1 + t(t-1) \not\equiv 0 \pmod{s}$. So, s must be odd.

Theorem 3.20 (π_7). *Table 7 gives a complete list of all free actions* (*up to topological conjugacy*) *of finite groups* G *on* $\mathcal{N}_p$ *which yield an orbit manifold homeomorphic to* $\mathcal{H}/\pi_{7,i}$ $(1 \le i \le 4)$.

TABLE 7

Generators	AC classes of normal nilpotent subgroups	Conditions
${}^{\mu_i}\langle t_1, t_2, \hat{\alpha} \rangle$	$N_1 = \langle t_1^{d_1} t_2^{td_1},\ t_2^{sd_1},\ t_3^{\frac{Ksd_1^2}{p}} \rangle$	$s \in 2\mathbb{N} - 1,\ 0 \le t < s$
	$N_2 = \langle t_1^{d_1} t_2^{td_1},\ t_2^{sd_1} t_3^{\frac{Ksd_1^2}{2p}},\ t_3^{\frac{Ksd_1^2}{p}} \rangle$	$s \in 2\mathbb{N} - 1,\ 0 \le t < s$
	$N_3 = \langle t_1^{d_1} t_2^{td_1} t_3^{\frac{Ksd_1^2}{2p}},\ t_2^{sd_1},\ t_3^{\frac{Ksd_1^2}{p}} \rangle$	$s \in 2\mathbb{N} + 1,\ 0 \le t < s$
	$N_4 = \langle t_1^{d_1} t_2^{td_1} t_3^{\frac{Ksd_1^2}{2p}},\ t_2^{sd_1} t_3^{\frac{Ksd_1^2}{2p}},\ t_3^{\frac{Ksd_1^2}{p}} \rangle$	$s \in 2\mathbb{N} - 1,\ 0 \le t < s$

where $K = 6n$ for the cases of $\pi_{7,1}$ and $\pi_{7,3}$, $K = 6n-2$ for the case of $\pi_{7,2}$, $K = 6n-4$ for the case of $\pi_{7,4}$, $\hat{\alpha} = \alpha$ for the cases of $\pi_{7,1}$, $\pi_{7,2}$, $\hat{\alpha} = \alpha^{-1}t_3$ for the case of $\pi_{7,3}$ and $\pi_{7,4}$, and

$$\mu_1 = \left(I,\ \left(\begin{bmatrix} 0 \\ \frac{t}{2s} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{t}{sd_1} & \frac{1}{sd_1} \end{bmatrix}\right)\right),$$
$$\mu_2 = \left(I,\ \left(\begin{bmatrix} \frac{1}{2p} \\ \frac{t}{2s} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{t}{sd_1} & \frac{1}{sd_1} \end{bmatrix}\right)\right),$$
$$\mu_3 = \left(I,\ \left(\begin{bmatrix} 0 \\ \frac{t}{2s} - \frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{t}{sd_1} & \frac{1}{sd_1} \end{bmatrix}\right)\right),$$
$$\mu_4 = \left(I,\ \left(\begin{bmatrix} \frac{1}{2p} \\ \frac{t}{2s} - \frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{t}{sd_1} & \frac{1}{sd_1} \end{bmatrix}\right)\right).$$

Note that $\pi_{7,i}/N$ is abelian if and only if $N \supset [\pi_{7,i}, \pi_{7,i}] = \langle t_1,\ t_2,\ t_3^K\rangle$, where $K = 6n$ for $i = 1, 3$, $K = 6n-2$ for $i = 2$ and $K = 6n-4$ for $i = 4$. Thus Theorem 3.10 of [1] can be obtained immediately as a consequence of the above theorem.

Let $N = \langle {t_1}^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{nd_1d_2}{p}}\rangle$ be a normal nilpotent subgroup of π_1. We recall the following results [15] for the reader's conveniences which show the conditions of affine conjugacy to N for given d_1, d_2 and m, and corrects the missing one in [1, Theorem 3.11].

Proposition 3.21 (π_1)**.** *Let N and N' be normal nilpotent subgroups of π_1 whose sets of generators are*

$$N = \langle\, {t_1}^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{nd_1d_2}{p}}\ \rangle, \quad N' = \langle\, {t_1}^{d_1} t_2^m t_3^{\ell'},\ t_2^{d_2} t_3^{r'},\ t_3^{\frac{nd_1d_2}{p}}\ \rangle.$$

Then $N \sim N'$ is equivalent to either $r \equiv r' \pmod{d_2}$, $\ell \equiv (\ell' + \frac{m(r-r')}{d_2}) \pmod{d_1}$, or $m = 0$, $d_1 = d_2$ and d_1 is a divisor of $\ell + r'$ and $r + \ell'$.

The following theorem is easily obtained from the above proposition.

Theorem 3.22 (π_1)**.** *Table 1 gives a complete list of all free actions (up to topological conjugacy) of finite groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_1$.*

TABLE 1

Generators	AC classes of normal nilpotent subgroups	Conditions ($\frac{pm}{d_1d_2} \in \mathbb{Z}$)
${}^{\mu_i}\langle t_1, t_2, t_3\rangle$	$N_1 = \langle t_1^{d_1} t_2^m t_3^{\ell_1},\ t_2^{d_2} t_3^{r_1},\ t_3^{\frac{nd_1d_2}{p}}\rangle$	
	$N_2 = \langle t_1^{d_1} t_2^m t_3^{\ell_2},\ t_2^{d_2} t_3^{r_2},\ t_3^{\frac{nd_1d_2}{p}}\rangle$	$(*)$ or $(**)$

where

$$\mu_i = \left(I,\ \left(\begin{bmatrix} \frac{r_i}{nd_1d_2} \\ \frac{m}{2d_2} - \frac{\ell_i}{nd_1d_2} \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ -\frac{m}{d_1d_2} & \frac{1}{sd_2} \end{bmatrix}\right)\right)\ (i = 1, 2).$$

Here $(*)$: $r_1 \not\equiv r_2 \pmod{d_2}$ or $\ell_1 \not\equiv (\ell_2 + \frac{m(r_1-r_2)}{d_2}) \pmod{d_1}$.
$(**)$: $m \neq 0,\ d_1 \neq d_2,\ d_1 \nmid (\ell_1 + r_2),$ *or* $d_1 \nmid (r_1 + \ell_2)$.

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