East Asian Math. J.
Vol. 33 (2017), No. 4, pp. 431-451
http://dx.doi.org/10.7858/eamj.2017.030

Secondary Mathematics Teachers' Perceptions of Rate of Change

Jihwa Noh

ABSTRACT. This is a descriptive study with the intent of providing a rich characterization of teachers' perceptions of rate of change. The nature of teachers' perceptions and differences among teachers were examined by collecting data through a survey on teachers' conceptions of rate of change in terms of learning goals, prerequisites, and beliefs about teaching and learning of rate of change, and an interview individually assessing teachers' concept images and definitions. The participating 13 teachers were selected to provide a range of similar and contrasting levels of experiences based on the teachers' educational background and the number of years they had been teaching. Findings and implications of this study are discussed.

I. Background

Mathematics educators have identified the mathematics of change as an important strand of school mathematics , not only because of its critical role historically and in present day mathematics, the sciences, and the social sciences, but also because the concepts of the mathematics of change are rooted in everyday experiences of people-young and old (Stewart, 1990; NCTM, 2000).

Rate of change is one of the fundamental concepts of the mathematics of change (Noble et al., 2001). It is also identified as a key concept to the understanding of functions in mathematics and to the understanding of many ideas related to everyday life [for example, when describing qualitative change ("I grew taller over the summer") and

Received August 22, 2017; Revised August 25, 2017; Accepted August 26, 2017.
2010 Mathematics Subject Classification: 97B50, 97C30, 97D30
Key Words: Concept definition, Concept image, Conception, Rate of change

(pISSN 1226-6973, eISSN 2287-2833)

quantitative change ("I grew two inches in the last year")] (NCTM, 2000). NCTM(2000) recognizes it is an important goal in the development of algebraic thinking to be able to analyze change in various contexts by understanding relations and to model phenomena using appropriate tools. In particular, a conceptual understanding of rate of change is especially crucial for the study of calculus. Researchers claim that understanding of advanced topics such as calculus "develops from basic intuitions that children construct in their daily experiences with physical and symbolic change" and that calculus learning can be facilitated by earlier experiences that allow children to study and represent situations involving change (e.g., Herbert & Pierce, 2008). Rate of change can also be part of learning about rational numbers in presecondary school mathematics (Behr, Lesh, Post, & Silver, 1983; Wilhelm & Confrey, 2003). It can be recognized in algebra classes concerned with the slopes of lines or the consecutive differences or ratios in a table of numbers, which can show change over time for functions (Kalchman et al., 2001). And, rate of change can be recognized in calculus class as the derivative.

1. Student Knowledge of Rate of Change

The concept of rate of change can be introduced to students from elementary through high school in a variety of contexts. Distance traveled per second is an example of a simple way to think about rate of change that many elementary students can grasp. Traditionally the concept of rate of change is typically introduced via slope in middle school and it has been identified as a difficult concept even for high school and college students to understand (Monk, 1992; Ubuz, 2007). Several student misconceptions have been associated with the concept of slope. For example, when a linear function is written in the form $y=ax+b$, it may be difficult for students to consider the slope as a ratio if m is an integer. Another confusion often exists as to whether slope is computed as "x over y" or "y over x." Furthermore, students sometimes believe that the "order of the points" matters for computing slope $\frac{y_2-y_1}{x_2-x_1}$ or $\frac{y_1-y_2}{x_1-x_2}$ (Teuscher & Reys, 2010). There is also confusion between the roles of a and b when a function is written in the form $y=ax+b$

(Schoenfeld, Smith, & Arcavi, 1993). Reys and Reys(2009) presented several tasks involving rate of change to high school students and undergraduates. He identified several errors in students' reasoning. In their responses to a word problem involving the speed of a car, students were confused between average speed and constant speed. In a problem involving a difference table, students exhibited confusion between the existence of a number pattern in the table and the presence of a constant rate of change.

Several studies deal with knowledge of rate of change of precalculus students. Researchers at TERC (Technical Education Research Centers) used a device with a car moving on a track so that both the position and velocity of the car at each moment in time cold be graphed (Monk & Nermirovsky, 1994; Sullivan & Sullivan.III., 2006). Students were shown a velocity graph and asked to predict a corresponding position graph. The main result from these studies was that students assumed that the position graph should resemble the velocity graph. If the velocity graph increased, these students believed the position graph did too, and vice versa. For these students any change in the velocity graph meant the same kind of change in the position graph. These studies also reported students' errors in interpreting the sign of the velocity. Some students in the studies associated the sign of the velocity with the speed of the car-positive velocity means going faster and negative velocity means going slower.

A conceptual understanding of rate of change is especially crucial for the study of calculus and physics. Calculus typically begins with the study of derivatives and rates of change, using slopes of tangent lines to develop these concepts. Physics assumes the ability of students to interpret slopes as a functional relationship between two quantities (Carlson et al., 2002). Thus, in order to help students acquire a deep understanding of rate of change, it is important that secondary mathematics teachers have a robust mathematical understanding of rate of change and be able to utilize that knowledge when working with their students.

2. Teacher Knowledge of Rate of Change

Very few studies of teachers' knowledge related to rate of change exist. Stump(2001) examined both preservice and inservice secondary teachers' repertoires of representations of slope through paper-and-pencil surveys. This study showed that geometric (steepness) representations of slope were included in all teachers' responses and algebraic (as a parameter of a linear equation) and physical representations were mentioned more often that functional, trigonometric, or ratio representations. Differences were found between preservice and inservice teachers-inservice teachers made more references and had greater understanding of the trigonometric representations of slope.

Thompson and Thompson(1994, 1996) examined how one middle school teacher's mathematical knowledge was reflected in the language he used to teach the concept of speed to one student. The researchers found that the teacher's difficulty in speaking conceptually about rates had an effect on the student's understanding. While the student was oriented to how she was thinking about mathematical situations, the teacher was oriented to the calculations she employed. The teacher's use of calculational language interfered with his intention to facilitate the student's conceptual grasp of a situation. For example, in discussing the situation of trying to travel to an airport 120 miles away in 3.5 hours, the teacher said "We multiply 3.5 hours by his speed to go 120 miles," instead of something more situationally attuned, such as "He's going to go at some constant speed for 3.5 hours, and at the end of 3.5 hours he should have traveled 120 miles."

3. Concept Images and Definitions

Concept images and concept definitions are useful constructs for characterizing teachers' knowledge. A concept image is the total cognitive structure that is associated with a given concept. A concept definition is a set of words used to specify a given concept. Discussion of the concept image gives rise to an important distinction between the accepted formal mathematical definition for a concept and the way that an individual thinks about that concept. Usually, when the concept name is seen or heard, what is evoked is not the formal definition, but the concept image (Vinner, 1989). A person can hold a concept image for slope that does not seem to correspond directly to a formal mathematical definition, as found in the investigation of teachers' understanding of functions by Vinner and Dreyfus(1989).

Due to the complex nature and various uses of the idea of rate of change, including slope, situations embedding change lend themselves to a variety of representations such as equations, graphs, tables, and verbal descriptions. Multiple representations have increasingly found their way into the school curriculum as new technology facilitates more efficient constructions of them (NCTM, 2000).

As such, a regard for the multitude of ways that the same relationship can be represented is indispensable for teaching. A teacher whose concept image of slope highlights the associations between different representations is empowered to make the choice of which representations to employ (Dreyfus, 1991). For teachers, the ability and inclination to work with the concept of slope in a variety of formats is necessary for providing students with opportunities to explore and construct connections between alternative representations of situations embedding constant rate of change. Depending on how a relationship is displayed, different interpretations can be made. For example, a graph can provide insight into both global and local features (Leinhardt, Zaslavsky, & Stein, 1990), whereas a table of values may illuminate more local features. When combined, the information gleaned from diverse representations contributes to a deeper, more comprehensive understanding of the underlying situations of change.

II. Method

This study employed a survey given in person and individual interviews with all participants to address their range of knowledge of rate of change. All of the survey and interviews were conducted at the teachers' schools, usually in their classrooms. During the interviews, teachers were asked to think aloud as they completed the questions and responses were audiotaped and later transcribed. The researcher did not deviate from the questions, however, teachers were often asked to clarify the meaning of their responses or probed more deeply about their thinking. The length of the interviews varied from one hour to two hours. Any notes written by the teachers during the interview was collected for later analysis.

1. Participants

The participants were thirteen teachers from five middle schools and three high schools in Gyeongsang Province. The participants included 9 males and 4 females. The years of teaching experience for these teachers ranged from half a year to thirty two years, with an average of 11.5 years and a median of 8 years. Nine of the teachers held bachelor degrees in mathematics. Five held bachelor degrees in mathematics education. Four had received masters degrees in mathematics education and one(M2) was completing a masters degree in mathematics education at the time of the study. One had received a masters degree in mathematics. The participating teachers were grouped as: a) L-teachers for the teachers with a low number of years teaching, b) M-teachers for the teachers with a moderate number of years teaching; and, c) H-teachers for the teachers with a high number of years teaching. Background information for all 13 teachers is provided in Table II-1.

<Table II-1> Participant background information

Teachers		Degree(s) & Major(s)		Number of years teaching	Experience teaching Calculus
		Bachelor Major/Minor	Master		
L-teachers	L1	Math	None	0.5	No
	L2	Math Education	None	1	No
	L3	Math Education	None	2	No
	L4	Math/Physics	None	3.5	No
M-teachers	M1	Math Education	None	6.5	No
	M2	Math Education	Math Education	7	No
	M3	Math	Math Education	8	No
	M4	Math/Biology	None	12	No
	M5	Math	Math Education	13	No
H-teachers	H1	Math	Math Education	16	Yes
	H2	Math	Math Education	21	No
	H3	Math Education	Math	27	No
	H4	Math	None	32	Yes

2. Data Collection

The data collection consisted of two parts: A survey on perceptions of teaching and learning rate of change, and an interview on concept image and definition.

1) Conceptions of teaching and learning of rate of change

This survey consisted of two open-ended questions (Items 1 and 2) and 14 Likert scale items (Items 3-16). In order to gain additional insight into teachers' knowledge of rate of change, the two open-ended questions assessed teachers' thoughts about learning goals and prerequisites for students to understand rate of change. Teachers were asked to identify two or three of the most important ideas they want students to learn about rate of change (in Item 1) and concepts students need to know in order to understand the idea of rate of change (in Item2).

Items 3-16 were designed to identify important beliefs the teachers have about teaching and learning the idea of rate of change. Teachers were asked to indicate the extent of their agreement with each belief statement using a five-point Likert scale (Strongly disagree, Disagree, Neutral, Agree, Strongly agree). Researchers suggest that in order to understand teachers' thinking about a mathematical idea, either in planning for instruction or in teaching, it is important to understand their beliefs about the mathematical idea or their teaching as well as their knowledge (Cooney & Wilson, 1992). Teacher beliefs have the potential to affect the way teachers learn mathematics (which, in turn, affects what the teacher learns) and also the way that the teacher teaches mathematics. Thus, information on this section of the survey helped provide a clearer perspective on what characterized the teachers' knowledge of rate of change. For example, a teacher's reliance on an algebraic representation for teaching slope (e.g., "a" in an equation $y = ax + b$) may originate in her strong belief that students need to demonstrate a procedural understanding of slope before doing any investigation. Moreover, the teacher's own procedural-based understanding of slope may contribute to the development of such a belief. And the teacher's understanding of slope may be restricted by the way the idea of slope is presented in the curriculum he/she uses.

2) Concept image and definition.

This interview consisted of two questions assessing teachers' concept of rate of change as shown in their descriptions of concept images and concept definitions of rate of change:

1. In the context of algebra and function, what does "rate of change"

mean to you? How would you illustrate this idea?

2. How would you define rate of change? Can you give one or more examples to illustrate?

To determine the clarity of the questions, these two questions were pilot-tested with two high school teachers and three student teachers doing internships at a high school and changes were made as a result.

3. Data Analysis

The data gathered from the survey and the interview were used to provide a description of teachers' perceptions of rate of change in the context of algebra and functions. Responses were examined by participant, by question, and by relevant aspects of the concept of rate of change, with an emphasis on distinguishing general patterns or trends within the participants' responses individually and as a group.

III. Results

1. Conceptions of teaching and learning of rate of change

1) Learning goals

With the intention of gaining insight into teachers' mathematical content knowledge of rate of change, the first open-ended question asked teachers to identify two or three of the most important ideas they want students to learn about rate of change. A wide variety of ideas were listed and many of the ideas were mentioned by more than one teacher. (See Table III-1) There were no teachers who focused solely on computational procedures or formulas. Most teachers (11 out of 13) listed a goal that students be able to use and interpret multiple representations of rate of change. Of those eleven, nine of the teachers mentioned the ability to recognize patterns of change or rate of change shown in different representations and two teachers mentioned the ability to use multiple representations (e.g., tables, graphs and equations) to represent the relationship between two variables involving rate of change. Of those nine teachers, two teachers referred to tables, graphs and equations (L1,L2), and two teachers referred to only tables and graphs (M1,M5). The remaining five did not make reference to a particular representation, but made a more general statement such as "describing rate of change from various representations."

Ideas about the process of approximating the rate of change of a curve were listed by five teachers. The responses included "When a graph is curved, an estimate for rate of change is really the linear 'piece'. So it's like finding the slope of the line segment" (M5), and "How to find a rate of change at any given point as well as over an interval" (H2). All H-teachers, no L-teachers, and only one M-teacher considered the idea of the linear connection as an important idea about rate of change. "Understanding of derivatives" was mentioned by three teachers, with two H-teachers further elaborating that it is "knowing derivative functions of different function families and graphing the derivative functions."

The importance of understanding functions was identified by three M-teachers as understanding of linear relationships, understanding of dependent and independent variables, and understanding of function families (e.g., linear, quadratic, exponential and trigonometric) (2, 1 and 1 incidents, respectively). Two teachers included "developing graphing calculator skills" as an important idea for students in learning rate of change. Other ideas, such as ratio and applying real-life situations, each received a single mention. Table III-1 shows the ideas teachers listed and response rates by teacher experience category.

As illustrated in Table III-1, differences were found in the L-teachers and H-teachers' responses. While L-teachers and H-teachers both considered the ability to recognize patterns of change between two variables displayed in different representations to be one of the most important ideas about rate of change, no L- teachers related that idea to "non-constant rate of change," which is the idea mentioned by all H-teachers. Other ideas that were mentioned by L-teachers, but were not mentioned by any of the H-teachers, were using multiple representational forms to represent rate of change, applying real-life situations and graphing calculator use. All H-teachers, and some M-teachers, addressed ideas concerning the linear connection, derivative and functions. No L-teachers responded in these regards.

<Table III-1> Teacher responses identifying key ideas important for students' learning rate of change

Identified ideas	Teachers
Use and Interpretation of Multiple Representations	
1. Recognizing patterns of change represented in	L1,L2,L4,M1,M3,M4

multiple representations	M5,H1,H4
2. Using tables, graphs and equations to represent rate of change	L3,H3
Linear Connection	
1. Discriminating non-constant rate of change	M2,M5,H1,H2,H4
2. Knowing instantaneous vs. average rate of change	M2,H2,
3. Estimating rate of change for nonlinear functions is similar to finding the slope of a line	M5
Derivative	
1. Understanding of derivatives (with no details)	M5,H2,H4
2. Knowing derivative functions of different function families	M5
3. Graphing derivative functions	H4
Functions	
1. Understanding of linear relationship	L4,M3
2. Knowledge of dependent and independent variables	H3
3. Understanding of function families	H3
Others	
1. Developing graphing calculator skills	L3,H3
2. Understanding the idea of ratio	M4
3. Applying to real-world situations	L2

Note. Some teachers gave multiple ideas in their responses and those were marked.

2) Prerequisites for students' understanding of rate of change

Teachers were asked to identify concepts students need to know in order to understand the idea of rate of change. The most common responses involved experience with graphs and knowledge of dependent and independent variables (7 and 6 responses, respectively). "Experience with graphs" includes being able to make and interpret graphs and using graphing calculators as a way to make graphs. Knowledge of slope, solving equations, and function families were mentioned less often (5, 3 and 2 responses, respectively). "Knowledge of slope" includes being able to find the slope of a line (4 responses) and knowing that the slope indicates the constant rate of change of the line (1 response). Two LE teachers considered being able to make and read tables as prerequisite concepts for students' understanding as well as make and read graphs (L1,L2). Other concepts listed were arithmetic skills such as subtraction and fractions (L1) and ratios (M4). One teacher did not list any concept as a prerequisite, stating "not sure if there is any prerequisite concepts" (H2). Table III-2 presents the prerequisite concepts listed by teachers and response rates by teacher experience category.

<Table III-2> Teacher responses identifying key prerequisite concepts needed for students' learning of rate of change

Identified Concepts	Teachers
1. Being able to make and interpret graphs	L1,L2,M1,M2,M3,H3,H4
2. Knowledge of variables	L1,L2,L4,M1,H1,H3
3. Knowledge of slope	L3,M2,M3,M5,H4
4. Solving equations	L2,L3,H3
5. Being able to make and interpret tables	L1,L2
6. Knowledge of function families	M5,H1
7. Ratios	M4
8. Arithmetic skills	L1
9. Use of graphing calculators	H3
10. No concepts identified	H2

Note. Some teachers gave multiple ideas in their responses and those were marked.

3) Beliefs about teaching and learning of rate of change

Items 3 through 16 asked teachers to indicate the extent of their agreement with each belief statement using a five-point Likert scale (Strongly disagree, Disagree, Neutral, Agree, Strongly agree). For all teachers, rate of change was perceived as an important mathematics topic. All teachers indicated that basic ideas of rate of change and slope are not too difficult to learn. But two teachers indicated agreement with the item "Introducing slope as a rate of change makes it more difficult to understand." (L3 strongly agreed and H3 agreed.) However, they were not asked to explicate what they considered "basic ideas," so there may have been variation in how they interpreted that statement. All teachers felt confident with their ability to teach concepts of rate of change. They also believed that they needed to be aware of how students learn mathematics and that it was important that they be able to interpret what students are doing as they work on mathematics problems. A majority of teachers (12 of 13) agreed or strongly agreed that learning rate of change in context and through different representations is important (with L3 choosing "neutral" to the item), and that students should learn mathematics by being actively engaged in solving problems (with M4 choosing "neutral"). Also, most teachers (10 of 13) agreed or strongly agreed that working cooperatively is an effective way to learn mathematics (with L3, M3 and M4 choosing "neutral"). All but one teacher (L3) disagreed with the statement "Students need to demonstrate procedural understanding of rate of change before doing any investigation." Five teachers felt neutral or agreed with the statement "The best ways to teach how to solve complex problems is to decompose them into a sequence of basic skills." Three teachers (M3,M1,M4) chose "neutral" and two teachers (L3,L4) chose "agree."

2. Concept images and definitions

1) Concept images

In terms of the concept image, all but one teacher (H2) stated and illustrated their concept images of rate of change in more than one way and several themes were evident in teachers' responses. Table III-3 shows teacher responses classified (in the most specific category) by similar descriptions and mathematical content area (type).

<Table III-3> Teachers' concept images of rate of change

Description of concept images	Types[a]	Teachers
1. Slope of a line= $\frac{y_2 - y_1}{x_2 - x_1}$	Geometric	All
2. Rise over run	Geometric	L1,L3
3. For the line $y = mx + b$, m is the rate of change	Algebraic	L3,L4,M1,H4
4. Constant increase/addition	Algebraic	L1,L4,H3
5. How fast/much something changes over time	Functional	L2,M2,M3
6. Relationship between two variables	Functional	M1,M5,H1,H3
7. Derivative as the slope of the	Functional	H1,H4

tangent line of the curve at a point		
8. Derivative	Procedural	L2,M1,M3
9. Comparison, a ratio	Ratio	M4

Note. Some teachers gave multiple ideas in their responses and those were marked. aTypes were identified through direct references mentioned by teachers or interpreted by the interviewer.

As illustrated in Table III-3, all of the 13 teachers mentioned "slope of a line" illustrated by $\frac{y_2 - y_1}{x_2 - x_1}$ as a way to conceptualize rate of change. Functional concepts were the second most frequently mentioned type of concept image. Of the nine teachers who mentioned a function concept, four teachers stated their concept images as relationships between two variables using the generic terms, one (or one variable) vs. the other (or the other variable), while three teachers referred specifically to time as a variable. For example, one HE teacher (M2) said, "It's how much something changes over time. How much is water pouring out of a container over time, say per minute, or how much is a car traveling per hour." To illustrate this functional characteristic of their concept images many teachers drew a rough sketch of the graph of a function. The graphical examples that less experienced teachers used, however, were often restricted to those with a linear relationship. Some sample responses were:

> *A linear graph of some sort. As time passes the amount of pay changes in a job. It [the graph] will probably steadily increase. (L2)*

> *A linear function. For example, the amount of weight or force that's applied and that's going to make a difference in a stretch or in a distance. (H3)*

> *If you are looking at population of a city over time and asking them [students], as the years go by, what's happening to the population in this picture. And most of the time population is going to be an exponential growth. And so they'd see the curve increasing and hopefully seeing it not increasing straight because it's increasing at an increasing rate. (M1)*

> *I usually see it [rate of change] as a curve. Maybe a ball going through the air. A parabolic function where the rate of change is increasing. (H4)*

A total of five teachers mentioned the term "derivative" to illustrate their image of rate of change. However, they seemed to have different understandings of the idea of derivative. In responding to a follow-up question that probed their view of the derivative concept, three of these five teachers (L2,M3,M1) failed to provide a rationale for their response. These teachers did not seem to understand the concept of derivative beyond a procedural rule for calculating one. Thus their responses were classified as "derivative"(in description) and "procedural"(in type) because they used terminology without much understanding. For example, two of the teachers commented in the following manner:

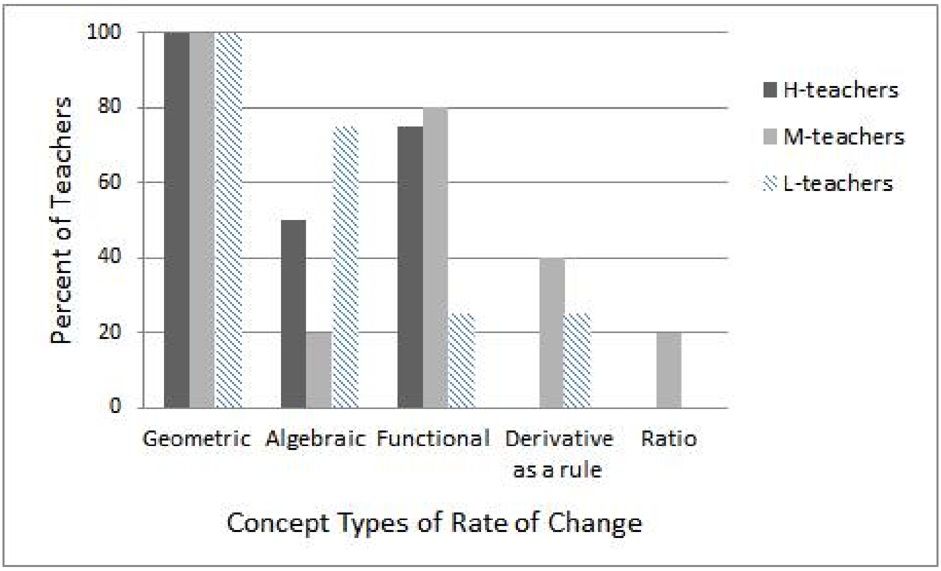
> *Derivative means rate of change. That's what I've heard from my calculus class in college. I don't remember it all but I do remember some easy ones. 2x is the derivative of x2 and x becomes 1. But this is it. This is all I remember about derivative. I know derivative means rate of change, but don't know how I can use it to tell you how I think about rate of change. Then I need to go back to the "rise over run" thing.(L2)*

> *In terms of rate of change, the first idea that comes to mind is actually the derivative of a function, which is more of a calculus idea. Why I brought that up is… well, I guess basically I was taught that rate of change is derivative, so I guess for me it was a fact that I remembered what I was taught in college. (M3)*

The remaining two teachers (H1,H4) demonstrated a more sophisticated understanding by stating that the derivative is a function of a point and the slope of the tangent line of the given function at that point. This suggests that their view of the derivative concept in relation to the rate of change concept is functional in nature, as compared to the other three teachers whose view of derivative was more procedural.

Algebraic types of description, referring to "the parameters in the equation $y = ax + b$" or "increment in a table of numerical data," were mentioned by seven teachers including one H-teacher. To exhibit their ideas, those seven teachers all described a linear model in which the rate of change is constant. Of all the teachers, one (M4) made reference to the idea of ratio by describing his concept image as "a comparison, a ratio of two measurements."

Other possible concepts that can be used to illustrate one's image of the rate of change idea are physical (e.g., the steepness of an incline such as pitch of a roof or grade of a road) or trigonometric (the tangent of the angle of inclination). Neither physical nor trigonometric concepts were mentioned by teachers in this study. As shown in Figure III-1, while geometric concepts dominated L-teachers' and M-teachers' concept image of rate of change, every H-teacher made reference to a characteristic of function as well as geometric concepts to illustrate their image of rate of change. Algebraic concepts were used more often by L-teachers and M-teachers than they were by H-teachers.



[Fig. III-1] Percent of teachers who mentioned multiple approaches in describing the concept of rate of change in their concept images

2) Concept definition

Unlike the variation in teachers' concept image of rate of change, all but one teacher (M4) defined the concept as "how fast/how much something changes over time," which is a "functional" definition of rate of change, as shown in Table III-4. Their selection of this definition as their formal definition of rate of change may point to an influence of using the textbook they were using (which introduces the term "rate of change" in situations that exhibit a functional relationship). The teacher who did not use a functional approach defined rate of change as a ratio of two measurements, which was also the approach used to illustrate his concept image of rate of change.

<Table III-4> Teachers' concept definitions of rate of change

Descriptions of concept definitions of	Types[a]	

rate of change		Teachers
1. How fast/how much something changes over time or over a certain period	Functional	All except M4
2. A ratio	Ratio	M4

Note. aTypes were identified through direct references mentioned by teachers or interpreted by the interviewer.

After giving a definition of rate of change, seven teachers also indicated that they may alter their definition slightly based on factors such as age of the students in their classroom and class appropriateness, as well as the demands of the task at hand (saying, for example, "it depends on what the student already knows and what problem they are working on"). However, they did not provide specific examples of these alterations. Some teachers seemed to defer to what was personally preferential. For example, teachers L2 and L4 indicated that they always try to do an algebraic approach first unless a graphical way seems more familiar or more applicable to the situation at hand. Teachers M5 and M2 expressed a preference for using a graphical approach to think about rate of change.

IV. Findings and Discussion

1. Overall

Rate of change was perceived by all teachers as an important and generally a not-too-difficult-to-learn mathematical topic. Many teachers viewed use and interpretation of multiple representations and the linear connection as key ideas in learning rate of change, although no L-teachers mentioned ideas concerning the linear connection. The ability to make and interpret representations was the most often identified prerequisite concept for learning rate of change. Not surprisingly, almost all teachers felt that learning in context and through different representations is important. Working cooperatively and being engaged in solving problems were viewed as effective ways to learn mathematics. All teachers indicated they were confident with their ability to teach concepts of rate of change and slope, and most teachers (11 of 13) felt that approaching slope as a rate of change did not make it more difficult to understand.

All teachers included a description of slope of a straight line in their image of rate of change, which is a geometric concept as a property of a graph of a line. Few teachers thought of rate of change as a derivative or ratio. Slightly more than half of the teachers used functional language to illustrate their image. It was a surprising finding that the functional representation of rate of change was missing from the descriptions of many teachers' concept images, because a functional interpretation is inherent in the term rate of change (e.g., change as a function of time). Physical or trigonometric representations of rate of change were virtually non-existent in teachers' concept images. This suggests that many teachers' mental images of rate of change are restricted to linear situations in that they likely think of rate of change as an attribute of slope, rather than slope as one representation among many that could be used to represent rate of change. This may be due to limitations in their own learning experience where some teachers might be less comfortable explicitly talking about rate of change, compared to talking about slope, since the term "rate of change" has only more recently become prominent in school mathematics curricula. Although teachers might be capable of making connections between various representations of rate of change, few have actually incorporated these representations into their concept images.

There did not seem to be a direct relationship between the representation used in their concept images and those used in their concept definitions. Unlike the various representations shown in their concept images, the use of functional language was much more prevalent in teachers' concept definitions.

2. Comparison by level of teaching experiences

In regards to their concept images, all HE teachers made reference to features of function as well as geometric concepts (such as rise over run). However, only three M-teachers and one L-teacher mentioned a functional characteristic to illustrate their concept image of rate of change. While these M-teachers and L-teachers were more likely to refer to the parameters in a linear equation (such as "a" in the equation $y = ax + b$), H-teachers incorporated the ideas of derivative and/or slope of a tangent line utilizing nonlinear functional situations. However, all but one M-teacher used a functional approach to define rate of change. It is puzzling that although most teachers preferred a description involving function characteristics for their formal definition of rate of change, many of them did not utilize a functional approach for an informal image of rate of change. This may suggest that their mental images of mathematical concepts are confined by contexts with which they are more familiar. Their formal definitions seemed to be more linked to how the concept is presented in the textbooks they have used, since the concept of rate of change is introduced and developed in various situations that exhibit a functional relationship in the textbooks they were using.

V. Limitations and Implications

1. Limitations

There are several limitations in this study, mainly due to the sample of teachers available for this study. The most prominent limitation may have been the inability to control for other factors (besides years of teaching) which might contribute to teachers' understanding, such as professional development experiences, opportunities to learn from students, experience teaching Calculus, etc. The varying degrees of these pre-experiences may have given rise to prominent knowledge differences among teachers. This study did not attempt to even out the teachers' pre-experiences, but rather used them to categorize the characterized perceptions of teachers.

This study was also limited by the sample size of teachers. Although this study employed some of the methods of quantitative analysis, larger sample sizes would have allowed for more robust statistical treatment of the data and possibly relationships could have been more firmly established between variables representing the various aspects of teachers' knowledge of rate of change. However, much of this study was qualitative in nature, and whereas qualitative research typically focuses intently on only a few subjects, this study was conducted with a relatively large number of participants in this regard. Thus, a limited amount of time was spent with each teacher. A more complete picture of teachers' perceptions might have been developed with further interviews.

2. Implications for Use and Further Study

Rate of change is an important concept both in secondary level mathematics and everyday life. This study provided a characterization of secondary mathematics teachers' perceptions of rate of change and identified some similarities and differences of their thinking.

Teachers in this study all demonstrated a richer perception of rate of change in linear relationships. This perception occurred in various representations. Further research could investigate the factors that contributed to teachers' perception and flexibility in viewing linear relationships exhibiting constant rate of change. Teachers in this study showed flexibility in describing rates of change using multiple representations. They were the least flexible with graphical representations. Further studies could investigate the factors that inhibit growth in graphical reasoning.

References

[1] Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational-number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91-126). New York: Academic Press.

[2] Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

[3] Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordrecht, The Netherlands: Kluwer.

[4] Herbert, S., & Pierce, R. (2008). An 'Emergent Model' for rate of change. *International Journal of Computers for Mathematical Learning, 13*(3), 231-249.

[5] Kalchman, M., Moss, J., & Case, R. (2001). *Psychological models for the development of mathematical understanding: Rational numbers and functions.* NY: Lawrence Erlbaum Associates, Inc.

[6] Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1-64.

[10] Monk. G. S. (1992). Students' understanding of function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of*

function: Aspects of epistemology and pedagogy (Vol. 25, pp. 175-193). Washington, DC: Mathematical Association of America.

[11] Monk, G. S., & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. *CBMS Issues in Mathematics Education, 4*, 139-168: American mathematical Society,

[7] National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

[8] Noble, T., Nemirovsky, R., Wright, T., & Tierney, C. (2001). Experiencing change: The mathematics of change in multiple environments. *Journal for Research in Mathematics Education, 32*(1), 85-108.

[9] Reys, R., & Reys, R. (2009). Two high school mathematics curricular paths—Which one to take? *Mathematics Teacher, 102(*8). 568-570.

[10] Schoenfeld, A. H., Smith, J. P., & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 4, pp. 55-175). Hillsdale, NJ: Lawrence Erlbaum.

[11] Stewart, I. (1990). Change. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 183-217). Washington, DC: National Academy Press.

[12] Stump, S. L. (2001). High school precalculus students' understanding of slope as measure. *School Science and Mathematics, 101* (2), 81-89.

[13] Sullivan, M., & Sullivan M. III. (2006). *Precalculus enhanced with graphing utilities.* 4th ed. Engle- wood Cliffs, NJ: Pearson Prentice Hall.

[14] Teuscher, D., & Reys, R. (2010). Slope, rate of change, and steepness: Do students understand these concepts? *Mathematics Teacher, 103*(7), 519-524.

[15] Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*(1), 2-24.

[16] Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2-3), 229-274.

[17] Ubuz, B. (2007). Interpreting a graph and constructing its derivative graph: Stability and change in students' conceptions. *International Journal of Mathematical Education in Science and Technology, 38*(5), 609-637.

[18] Vinner, S. (1989). The avoidance of visual considerations on calculus students. *Focus on Learning Problems in Mathematics, 11*(2), 149-156.

[19] Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*, 356-366.

[20] Wilhelm, J, & Confrey, J. (2003). Projecting rate of change in the context of motion onto the context of money. *International Journal of Mathematical Education in Science and Technology, 34*(6), 887-904.

Noh, Jihwa
Department of Mathematics Education
Pusan National University
Pusan, 46241 Korea
E-mail address: nohjihwa@gmail.com