CYLINDERS IN DEL PEZZO SURFACES WITH DU VAL SINGULARITIES

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Abstract. We consider del Pezzo surfaces with du Val singularities. We'll prove that a del Pezzo surface $X$ with du Val singularities has a $-K_X$-polar cylinder if and only if there exist tiger such that the support of this tiger does not contain anti-canonical divisor. Also we classify all del Pezzo surfaces $X$ such that $X$ has not any cylinders.

1. Introduction

A log del Pezzo surface is a projective algebraic surface $X$ with only quotient singularities and ample anti-canonical divisor $-K_X$. In this paper we assume that $X$ has only du Val singularities and we work over complex number field $\mathbb{C}$. Note that a del Pezzo surface with only du Val singularities is rational.

Definition 1.1. Let $X$ be a proper normal variety. Let $D$ be an effective $\mathbb{Q}$-divisor on $X$ such that $D \equiv -K_X$ and the pair $(X,D)$ is not log canonical. Such divisor $D$ is called non-log canonical special tiger (see [4]).

Remark 1.2. In this paper, a non-log canonical special tiger we will call a tiger.

Definition 1.3 (see. [5]). Let $M$ be a $\mathbb{Q}$-divisor on a projective normal variety $X$. An $M$-polar cylinder in $X$ is an open subset $U = X \setminus \text{Supp}(D)$ defined by an effective $\mathbb{Q}$-divisor $D$ in the $\mathbb{Q}$-linear equivalence class of $M$ such that $U \cong Z \times \mathbb{A}^1$ for some affine variety $Z$.

In this paper, we consider del Pezzo surfaces with du Val singularities over complex number field $\mathbb{C}$. Our interest is a connection between existence of a $-K_X$-polar cylinder in the del Pezzo surface and tigers on this surface.

The existence of a $H$-polar cylinder in $X$ is important due to the following fact.

Theorem 1.4 (see [6], Corollary 3.2). Let $Y$ be a normal algebraic variety over $\mathbb{C}$ projective over an affine variety $S$ with $\dim S \geq 1$. Let $H \in \text{Div}(Y)$ be an ample divisor on $Y$, and let $V = \text{Spec} A(Y,H)$ be the associated affine
quasicone over $Y$. Then $V$ admits an effective $G_a$-action if and only if $Y$ contains an $H$-polar cylinder.

There exists a classification of del Pezzo surfaces $X$ such that $X$ has a $-K_X$-polar cylinder (see [1], [2]). Also, in the papers [1], [2] the authors have proved that if a del Pezzo surface $X$ has not $-K_X$-polar cylinder, then all tigers contain a support at least one element of $|-K_X|$. Now we prove the inverse statement.

The main result of Section 3 is the followings.

**Theorem 1.5.** Let $X$ be a del Pezzo surface with du Val singularities. Then $X$ has a $-K_X$-polar cylinder if and only if there exist a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.

The main result of Section 4 is the followings.

**Theorem 1.6.** Let $X$ be a del Pezzo surface with du Val singularities. Then

- $X$ has not cylinders if $\rho(X) = 1$ and $X$ has one of the followings collections of singularities: $4A_2, 2A_1 + 2A_3, 2D_4$;
- In the rest cases there exist an ample divisor $H$ such that $X$ has a $H$-polarization.

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**2. Preliminary results**

We work over complex number field $\mathbb{C}$. We employ the following notation:

- $(-n)$-curve is a smooth rational curve with self intersection number $-n$.
- $K_X$: the canonical divisor on $X$.
- $\rho(X)$: the Picard number of $X$.

**Theorem 2.7** (Riemann–Roch, see, for example, [3], Theorem 1.6, Ch. 5). Let $D$ be a divisor on the surface $X$. Then

$$\chi(D) = \frac{1}{2} D(D - K_X) + \chi(\mathcal{O}_X).$$

**Theorem 2.8** (Kawamata–Viehweg Vanishing Theorem, see, for example, [7], Theorem 5-2-3). Let $X$ be a non-singular projective variety, $A$ an ample $\mathbb{Q}$-divisor such that the fractional part $[A]$ of $A$ has the support with only normal crossings. Then

$$H^p(X, K_X + [A]) = 0, \quad p > 0.$$  

Let $X$ be a del Pezzo surface with du Val singularities. Let $d$ be the degree of $X$, i.e., $d = K_X^2$.

**Theorem 2.9** (see [1], Theorem 1.5). Let $X$ be a del Pezzo surface of degree $d$ with at most du Val singularities.

1. The surface $X$ does not admit a $-K_X$-polar cylinder when
(1) \( d = 1 \) and \( X \) allows only singular points of types \( A_1, A_2, A_3, D_4 \) if any.

(2) \( d = 2 \) and \( X \) allows only singular points of types \( A_1 \) if any.

(3) \( d = 3 \) and \( X \) allows no singular point.

II. The surface \( X \) has a \( -K_X \)-polar cylinder if it is not one of the del Pezzo surfaces listed in I.

3. The proof of Theorem 1.5

In the papers [1] and [2] authors have classified del Pezzo surfaces \( X \) such that \( X \) has a \( -K_X \)-polar cylinder. Moreover, they prove that if a del Pezzo surface \( X \) has not a \( -K_X \)-polar cylinder, then every tiger on \( X \) contains an element of \( | -K_X | \). So, we need prove that if a del Pezzo surface \( X \) has a \( -K_X \)-polar cylinder, then there exist a tiger such that the support of this tiger does not contain any elements of \( | -K_X | \).

Lemma 3.10. Let \( X \) be a del Pezzo surface with du Val singularities and let \( d \) be the degree of \( X \). Assume that \( d \geq 7 \). Then \( X \) has a \( -K_X \)-polar cylinder and there exist a tiger such that the support of this tiger does not contain any elements of \( | -K_X | \).

Proof. By Theorem 2.9, we see that \( X \) has a \( -K_X \)-polar cylinder. Now, we construct a tiger such that the support of this tiger does not contain any elements of \( | -K_X | \). Consider \( | -2K_X | \). By Theorem 2.7 and Theorem 2.8, \( \dim | -2K_X | = -2(K_X \cdot -2K_X - K_X) = 3d \). Let \( P \) be an arbitrary smooth point on \( X \). Consider a set \( \Omega \) of elements \( L \in | -2K_X | \) such that \( \mult_P L \geq 5 \). Then \( \Omega \) is a linear subsystem of the linear system \( | -2K_X | \). Note that \( \dim \Omega = 3d - 15 \geq 6 \) for \( d \geq 7 \). Hence, \( \Omega \) is not empty. Let \( N \in \Omega \) be a general element of the linear system \( \Omega \).

Note that \( N \) does not contain a support of anti-canonical divisor. Indeed, assume that there exist an element \( M_1 \in | -K_X | \) such that \( \text{Supp} M_1 \subseteq \text{Supp} N \). Then \( N = M_1 + M_2 \), where \( M_2 \in | -K_X | \). We see that \( \dim | -K_X | = -K_X \cdot (-K_X - K_X) = d \). Therefore, \( \mult_P M_1 \leq 3 \) and \( \mult_P M_2 \leq 3 \). Hence, we may assume that \( \mult_P M_1 = 2 \), \( \mult_P M_2 = 3 \). Let \( \tilde{M}_1 \) be the linear subsystem of \( | -K_X | \) such that \( \tilde{M}_1 \) consist of elements with multiply two in the point \( P \). Let \( \tilde{M}_2 \) be the linear subsystem of \( | -K_X | \) such that \( \tilde{M}_2 \) consist of elements with multiply three in the point \( P \). Then

\[
\dim | \tilde{M}_1 + \tilde{M}_2 | = \dim | \tilde{M}_1 | + \dim | \tilde{M}_2 | = (d - 3) + (d - 6) = 2d - 9.
\]

Note that \( 3d - 15 > 2d - 9 \) for \( d \geq 7 \). Hence, a general element \( N \) of the linear system \( \Omega \) does not contain a support of anti-canonical divisor. Then \( \frac{1}{2} N \) is a tiger such that the support of this tiger does not contain any elements of \( | -K_X | \).

4. Lemma 3.11. Let \( X \) be a del Pezzo surface with du Val singularities and let \( d \) be the degree of \( X \). Assume that \( d = 4, 6 \). Then \( X \) has a \( -K_X \)-polar cylinder.
and there exists a tiger such that the support of this tiger does not contain any elements of $| - K_X |$.

Proof. By Theorem 2.9, we see that $X$ has a $-K_X$-polar cylinder. Now, we construct a tiger such that the support of this tiger does not contain any elements of $| - K_X |$. Let $f : \tilde{X} \to X$ be the minimal resolution. Let $E$ be a $(-1)$-curve on $X$ and $E' = f(E)$. Put $-3K_{\tilde{X}} \sim 2E + F$. Then $-3K_{\tilde{X}} \cdot E = 2E^2 + F \cdot E$. Since $K_{\tilde{X}} \cdot E = -1$ and $E^2 = -1$, we see that $F \cdot E = 5$. We have $-3K_{\tilde{X}}^2 = 2E \cdot K_{\tilde{X}} + F \cdot K_{\tilde{X}}$. Since $K_{\tilde{X}} \cdot E = -1$ and $K_{\tilde{X}}^2 = d$, we see that $F \cdot K_{\tilde{X}} = -(3d - 2)$. We obtain $-3K_{\tilde{X}} \cdot F = 2E \cdot F + F^2$. Since $F \cdot E = 5$ and $F \cdot K_{\tilde{X}} = -(3d - 2)$ we see that $F^2 = 9d - 16$. Hence, by Theorem 2.7 and Theorem 2.8, $\dim | F | = 6d - 9$. Let $P'$ be a general smooth point on $E'$ and $P = f(P)$. Consider a set $\Omega$ of elements $L \in | F |$ such that $\mult_P L \geq 5$. Note that $\dim | \Omega | = 6d - 9 - 15 = 6d - 24 \geq 0$ for $d \geq 4$, i.e., $\Omega$ is non-empty. We see that $\Omega$ contains an element $N$ such that $N + E$ does not contain a support of anti-canonical divisor. Indeed, assume that for all $N \in \Omega$ there exist $M_1 \in | - K_X |$ such that $\Supp M_1 \subseteq \Supp(N + E)$. Then $N + 2E = M_1 + M_2$, where $M_2 \in | - 2K_X |$. We have the following cases.

Case 1. $M_1 = 2E + F_1, M_2$ does not contain the curve $E$. Hence, $F_1 \cdot E = 3, F_1 \cdot K_X = -(d - 2), F_1^2 = d - 8 \leq -2$, a contradiction.

Case 2. $M_1 = E + F_1, M_2 = E + F_2$. Then $F_1 \cdot E = 2, F_1 \cdot K_X = -(d - 1), F_1^2 = d - 3, F_2 \cdot E = 3, F_2 \cdot K_X = -(2d - 1), F_2^2 = 4d - 5$. Hence, $\dim | F_1 | = d - 2$, $\dim | F_2 | = 3d - 3$. Note that the multiplicities $F_1$ and $F_2$ in the point $P$ are equaled 2 and 3 correspondingly. Let $\tilde{F}_1$ be the linear subsystem of $| F_1 |$ such that the multiplicity of elements of $\tilde{F}_1$ is equaled two in the point $P$, let $\tilde{F}_2$ be the linear subsystem of $| F_2 |$ such that the multiplicity of elements of $\tilde{F}_2$ is equaled three in the point $P$. Then $\dim | \tilde{F}_1 | = d - 5$. Hence, $d = 6$. Note that

$$\dim | \tilde{F}_1 + \tilde{F}_2 | = \dim | \tilde{F}_1 | + \dim | \tilde{F}_2 | = (d - 5) + (3d - 9) = 4d - 14 = 10.$$ 

On the other hand, $\dim | \Omega | = 6d - 24 = 12 > 10$. Therefore, a general element $N \in \Omega$ does not contain $\Supp(-K_X) \setminus \Supp(E)$.

Case 3. $M_2 = 2E + F_2, M_1$ does not contain the curve $E$. Then $F_2 \cdot E = 4, F_2 \cdot K_X = -(2d - 2), F_2^2 = 4d - 12$. Hence, $\dim | F_2 | = 3d - 7, \dim | M_1 | = d$. Note that the multiplicities $M_1$ and $F_2$ in the point $P$ are equal to 1 and 4 correspondingly. Let $\tilde{M}_1$ be the set of elements of the linear system $| - K_X |$ that pass through the point $P$, let $\tilde{F}_2$ be the set of elements of the linear system $| F_2 |$ that have multiplicity four in the point $P$. Note that $\tilde{F}_1$ and $\tilde{M}_2$ are the linear system. Then $\dim | \tilde{F}_2 | = 3d - 17$. Hence, $d = 6$. Note that

$$\dim | \tilde{M}_1 + \tilde{F}_2 | = \dim | \tilde{M}_1 | + \dim | \tilde{F}_2 | = (d - 1) + (3d - 17) = 4d - 18 = 6.$$ 

On the other hand, $\dim | \Omega | = 6d - 24 = 12 > 6$. Therefore, a general element $N \in \Omega$ does not contain any elements of $| - K_X - E |$.

So, a general element $N \in \Omega$ does not contain any elements of $| - K_X - E |$. Denote this element by $N$. Note that $\mult_P (2E + N) \geq 7$. Then $\frac{1}{2}f(N) + \frac{1}{2}E'$
Lemma 3.12. Let $X$ be a del Pezzo surface with du Val singularities and let $d$ be the degree of $X$. Assume that $d = 5$. Then $X$ has a $-K_X$-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. 

Proof. By Theorem 2.9, we see that $X$ has a $-K_X$-polar cylinder. Now, we construct a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. Consider $|-4K_X|$. By Theorem 2.7 and Theorem 2.8, we see that $\dim |-4K_X| = 50$. Let $P$ be an arbitrary smooth point on $X$. Consider a set $\Omega$ of elements $L \in |-4K_X|$ such that $\text{mult}_P L \geq 9$. Then $\Omega$ is the linear subsystem of the linear system of $|-4K_X|$. Note that $\dim |\Omega| = 50 - 45 = 5$. Hence, $\Omega$ is non-empty. Let $N \in \Omega$ be a general element of the linear system $\Omega$. We see that $N$ does not contain a support of anti-canonical divisor. Indeed, assume that there exists an element $M_1 \in |-K_X|$ such that $\text{Supp} M_1 \subseteq \text{Supp} N$. Then $N = M_1 + M_2$, where $M_2 \in |-3K_X|$. Note that $\dim |-K_X| = 5$, $\dim |-3K_X| = 30$. Put $d_1 = \text{mult}_P M_1$ and $d_2 = \text{mult}_P M_2$. Since

$$\dim |-K_X| - \frac{d_1 \cdot (d_1 + 1)}{2} = 5 - \frac{d_1 \cdot (d_1 + 1)}{2} \geq 0$$

and

$$\dim |-3K_X| - \frac{d_2 \cdot (d_2 + 1)}{2} = 30 - \frac{d_2 \cdot (d_2 + 1)}{2} \geq 0,$$

we see that $\text{mult}_P M_1 \leq 2$ and $\text{mult}_P M_1 \leq 7$. Hence, $\text{mult}_P M_1 = 2$, $\text{mult}_P M_2 = 7$. Let $\tilde{M}_1$ be the set of elements of the linear system $|-K_X|$ that have multiply 2 in the point $P$, let $\tilde{M}_2$ be the set of elements of the linear system $|-3K_X|$ that have multiply 7 in the point $P$. Note that $\tilde{M}_1$ and $\tilde{M}_2$ are the linear system. Then $\dim |\tilde{M}_1| = 5 - 3 = 2 \dim |\tilde{M}_2| = 30 - 28 = 2$. Hence,

$$\dim |\tilde{M}_1 + \tilde{M}_2| = 4 < 5 = \dim |\Omega|.$$ 

So, a general element $N$ of $\Omega$ does not contain the support of anti-canonical divisor. Then $\frac{1}{2} N$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. 

Lemma 3.13. Let $X$ be a del Pezzo surface with du Val singularities and let $d$ be the degree of $X$. Assume that $d \geq 3$ and there exists a singular point of type $A_1$. Then $X$ has a $-K_X$-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. 

Proof. Let $X$ be a del Pezzo surface with du Val singularities, and let $P$ be a singular point of type $A_1$. By Lemmas 3.10, 3.11 and 3.12 we may assume that $d = 3$. Let $f : \bar{X} \to X$ be the minimal resolution of singularities of $X$, and let $D = \sum_{i=1}^{g} D_i$ be the exceptional divisor of $f$, where $D_i$ is a $(-2)$-curve. We may assume that $P = f(D_1)$. By Theorem 2.9, we see that $X$
has a \(-K_X\)-polar cylinder. Now, we construct a tiger such that the support of this tiger does not contain any elements of \(|-K_X|\). Consider \(-4K_X\). Put \(-4K_X \sim 3D_1 + F\). Then \(F \cdot D_1 = 6\), \(F \cdot K_X = -12\), \(F^2 = 30\). Hence, \(\dim |F| = 21\). Let \(Q\) be a point on \(D_1\). Note that there exists an element \(N \in |F|\) such that \(\text{mult}_Q N = 6\). Now, we prove that \(N + D_1\) does not contain the support of anti-canonical divisor. Indeed, assume that for all \(N \in \Omega\) there exists an element \(M_1 \in |-K_X|\) such that \(\text{Supp}(M_1) \subseteq \text{Supp}(N + D_1)\). Then \(N + 3D_1 = M_1 + M_2\), where \(M_2 \in |-3K_X|\). So, we have the following four cases.

Case 1. \(M_2 = 3D_1 + F_2\), \(M_1\) does not contain the curve \(D_1\). Then \(F_2 \cdot D_1 = 6\), \(F_2 \cdot K_X = -9\), \(F_2^2 = 9\). Hence, \(\text{dim}|F_2| = 9\). Therefore, \(\text{mult}_Q F_2 \leq 3\). Since \(M_1\) does not meet \(D_1\), we have a contradiction.

Case 2. \(M_1 = D_1 + F_1\), \(M_2 = 2D_1 + F_2\). Then \(F_1 \cdot D_1 = 2\), \(F_1 \cdot K_X = -d\), \(F_2^2 = 1\). Therefore, \(\text{dim}|F_1| = 2\). Hence, \(\text{mult}_Q F_2 \leq 1\), a contradiction.

Case 3. \(M_1 = 2D_1 + F_1\), \(M_2 = D_1 + F_2\). Then \(F_1 \cdot D_1 = 4\), \(F_1 \cdot K_X = -3\), \(F_2^2 = -5\), a contradiction.

Case 4. \(M_1 = 3D_1 + F_2\), \(M_2\) does not contain the curve \(D_1\). Then \(F_1 \cdot D_1 = 6\), \(F_1 \cdot K_X = -3\), \(F_2^2 = -15\), a contradiction.

So, \(\text{Supp}(N + D_1)\) does not contain the support of anti-canonical divisor. Note that \(\text{mult}_Q(3D_1 + N) = 9\). Then \(\frac{1}{4} f(N)\) is a tiger such that the support of this tiger does not contain any elements of \(|-K_X|\).

**Lemma 3.14.** Let \(X\) be a del Pezzo surface with du Val singularities and let \(d\) be the degree of \(X\). Assume that \(d \geq 2\) and there exists a singular point of type \(A_2\) or \(A_3\). Then \(X\) has a \(-K_X\)-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of \(|-K_X|\).

**Proof.** As above, we may assume that \(d = 2\) or \(d = 3\). By Theorem 2.9, we see that \(X\) contains \(-K_X\)-polar cylinder. Let \(f : \bar{X} \to X\) be the minimal resolution of singularities of \(X\), and let \(D = \sum_{i=1}^n D_i\) be the exceptional divisor of \(f\), where \(D_i\) is a \((-2)\)-curve. Consider two cases.

Case 1. There exists a point \(P \in X\) such that \(P\) of type \(A_2\). We may assume that \(D_1\) and \(D_2\) correspond to \(P\). So, \(D_1 \cdot D_2 = 1\). Let \(Q\) be the point of intersection of \(D_1\) and \(D_2\). Consider \(-2K_X\). Put \(-2K_X \sim 2D_1 + 2D_2 + F\). Then \(F \cdot D_i = F \cdot D_j = 2\), \(F \cdot K_X = -2d\), \(F^2 = 4d = 8\). Hence, \(\dim|F| = 3d - 4\). Consider the set \(\Omega\) of elements \(L \in |F|\) such that \(Q \in L\). Then \(\dim \Omega = 3d - 4 - 1 = 3d - 5\). Put \(-K_X \sim D_1 + D_2 + \bar{F}\). Then \(\bar{F} \cdot D_i = \bar{F} \cdot D_j = 1\), \(\bar{F} \cdot K_X = -d\), \(\bar{F}^2 = -d - 2\). Hence, \(\bar{|F|} = d - 1\). Consider the set \(\bar{\Omega}\) of elements \(L \in |\bar{F}|\) such that \(Q \in L\). Then \(\dim \bar{\Omega} = d - 2\). Note that \(\dim \bar{\Omega} = 3d - 5 > \dim \Omega = d - 2\). So, there exists an element \(N \in \Omega\) such that \(f(N)\) does not contain the support of anti-canonical divisor. Note that \(\text{mult}_Q(2D_1 + 2D_2 + N) \geq 5\). Then \(\frac{1}{4} f(N)\) is a tiger such that the support of this tiger does not contain any elements of \(|-K_X|\).

Case 2. There exists a point \(P \in X\) such that \(P\) of type \(A_3\). We may assume that \(D_1\), \(D_2\) and \(D_3\) correspond to \(P\). So, \(D_1 \cdot D_2 = D_2 \cdot D_3 = 1\). Let
Let $X$ be a del Pezzo surface with du Val singularities and let $d$ be the degree of $X$. Assume that $d \geq 2$ and there exists a singular point of type $D_4$. Then $X$ has a $-K_X$-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. \hfill \Box

**Lemma 3.15.** Let $X$ be a del Pezzo surface with du Val singularities and let $d$ be the degree of $X$. Assume that $d \geq 2$ and there exists a singular point of type $D_4$. Then $X$ has a $-K_X$-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. 

**Proof.** Let $X$ be a del Pezzo surface with du Val singularities, and let $P$ be a singular point of type $D_4$. By Theorem 2.9, we see that $X$ has a $-K_X$-polar cylinder. Let $f : X \rightarrow \overline{X}$ be the minimal resolution of singularities of $X$, and let $\mathcal{D} = \sum_{i=1}^{n} D_i$ be the exceptional divisor of $f$, where $D_i$ is a $(2)$-curve. We may assume that $D_1, D_2, D_3$ and $D_4$ correspond to $P$. Moreover, $D_1$ is the central component. Put $-3K_X \sim 4D_1 + 3D_2 + 2D_3 + 2D_4 + F$. Then $F \cdot D_1 = 1$, $F \cdot D_2 = 2$, $F \cdot D_3 = 0$ $F \cdot D_4 = 0$ $F \cdot K_X = -3d$, $F^2 = 2d - 3 > 0$ for $d \geq 2$. Note that $4D_1 + 3D_2 + 2D_3 + 2D_4 + F$ does not admit representation as $M_1 + M_2$, where $M_1 \in |-K_X|$ and $M_2 \in |-2K_X|$. Let $N$ be an element of $|F|$. Note that the support $\mathcal{N} = 4D_1 + 3D_2 + 2D_3 + 2D_4$ does not contain any elements of $|-K_X|$. So, $f(N)$ does not contain the support of anti-canonical divisor. Note that mult$_f(2D_1 + 3D_2 + D_3 + N) \geq 5$. Then $\frac{1}{2}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. \hfill \Box

**Lemma 3.16.** Let $X$ be a del Pezzo surface with du Val singularities and let $d$ be the degree of $X$. Assume that there exists a singular point of type $A_k$, where $k = 4, 5, 6, 7, 8$. Then $X$ has a $-K_X$-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. 

**Proof.** Let $X$ be a del Pezzo surface with du Val singularities, and let $P$ be a singular point of type $A_k$. By Theorem 2.9, we see that $X$ has a $-K_X$-polar cylinder. Let $f : X \rightarrow \overline{X}$ be the minimal resolution of singularities of $X$, and let $\mathcal{D} = \sum_{i=1}^{n} D_i$ be the exceptional divisor of $f$, where $D_i$ is a $(2)$-curve. We may assume that $D_1, D_2, \ldots, D_k$ correspond to $P$. Moreover, $D_i, D_{i+1} = 1$ for $i = 1, 2, \ldots, k - 1$. Consider the following cases.

**Case 1.** $k = 4$. Put $-2K_X \sim D_1 + 2D_2 + 2D_3 + D_4 + F$. Let $Q$ be the intersection of $D_2$ and $D_3$. We obtain $F \cdot D_1 = F \cdot D_4 = 0$, $F \cdot D_2 = F \cdot D_3 = 1$, $F \cdot K_X = -2d$, $F^2 = 4d - 4$. Then $|F| = 3d - 2$. So, there exists an element $N \in |F|$ such that $N$ passes through $Q$. Note that $D_1 + 2D_2 + 2D_3 + D_4 + N$ does not admit representation as $M_1 + M_2$, where $M_1, M_2 \in |-K_X|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that mult$_f(D_1 + 2D_2 + 2D_3 + D_4 + N) \geq 5$. Then $\frac{1}{2}f(N)$ is
a tiger such that the support of this tiger does not contain any elements of \(|-K_X|\).

Case 2. \(k = 5\). Put \(-3K_X \sim D_1 + 2D_2 + 3D_3 + 3D_4 + 2D_5 + F\). Let \(Q\) be the intersection of \(D_3\) and \(D_4\). We obtain \(F \cdot D_1 = F \cdot D_2 = 0, F \cdot D_5 = F \cdot D_4 = F \cdot D_3 = 1, F \cdot K_X = -3d, F^2 = 9d - 8\). Then \(\dim |F| = 6d - 4\). So, there exists an element \(N \in |F|\) such that \(N\) passes through \(Q\). Note that \(D_1 + 2D_2 + 3D_3 + 3D_4 + 2D_5 + N\) does not admit representation as \(M_1 + M_2\), where \(M_1 \in |-K_X|\) and \(M_2 \in |-2K_X|\). Hence, \(f(N)\) does not contain the support of anti-canonical divisor. Note that \(\text{mult}_Q(D_1 + 2D_2 + 3D_3 + 3D_4 + 2D_5 + N) \geq 7\). Then \(\frac{1}{2}f(N)\) is a tiger such that the support of this tiger does not contain any elements of \(|-K_X|\).

Case 3. \(k = 6\). Put \(-3K_X \sim D_1 + 2D_2 + 3D_3 + 3D_4 + 2D_5 + D_6 + F\). Let \(Q\) be the intersection of \(D_3\) and \(D_4\). We obtain

\[F \cdot D_1 = F \cdot D_2 = F \cdot D_3 = F \cdot D_6 = 0,\]
\[F \cdot D_4 = F \cdot D_5 = 1, F \cdot K_X = -3d, F^2 = 9d - 6.\]

Then \(\dim |F| = 6d - 3\). So, there exists an element \(N \in |F|\) such that \(N\) passes through \(Q\). Note that \(D_1 + 2D_2 + 3D_3 + 3D_4 + 2D_5 + N\) does not admit representation as \(M_1 + M_2\), where \(M_1 \in |-K_X|\) and \(M_2 \in |-2K_X|\). Hence, \(f(N)\) does not contain the support of anti-canonical divisor. Note that \(\text{mult}_Q(D_1 + 2D_2 + 3D_3 + 3D_4 + 2D_5 + D_6 + N) \geq 7\). Then \(\frac{1}{2}f(N)\) is a tiger such that the support of this tiger does not contain any elements of \(|-K_X|\).

Case 4. \(k = 7\). Put

\[-4K_X \sim D_1 + 2D_2 + 3D_3 + 4D_4 + 4D_5 + 3D_6 + 2D_7 + F.\]

Let \(Q\) be the intersection of \(D_4\) and \(D_5\). We obtain

\[F \cdot D_1 = F \cdot D_2 = F \cdot D_3 = F \cdot D_6 = 0,\]
\[F \cdot D_4 = F \cdot D_5 = F \cdot D_7 = -4d, F^2 = 16d - 10.\]

Then \(\dim |F| = 10d - 5\). So, there exists an element \(N \in |F|\) such that \(N\) passes through \(Q\). Note that \(D_1 + 2D_2 + 3D_3 + 4D_4 + 4D_5 + 3D_6 + 2D_7 + N\) does not admit representation as \(M_1 + M_2\), where \(M_1 \in |-K_X|\) and \(M_2 \in |-3K_X|\). Hence, \(f(N)\) does not contain the support of anti-canonical divisor. Note that

\[\text{mult}_Q(D_1 + 2D_2 + 3D_3 + 4D_4 + 4D_5 + 3D_6 + 2D_7 + N) \geq 9.\]

Then \(\frac{1}{2}f(N)\) is a tiger such that the support of this tiger does not contain any elements of \(|-K_X|\).

Case 5. \(k = 8\). Put

\[-4K_X \sim D_1 + 2D_2 + 3D_3 + 4D_4 + 4D_5 + 3D_6 + 2D_7 + D_8 + F.\]

Let \(Q\) be the intersection of \(D_4\) and \(D_5\). We obtain

\[F \cdot D_1 = F \cdot D_2 = F \cdot D_3 = F \cdot D_6 = F \cdot D_7 = F \cdot D_8 = 0,\]
\[F \cdot D_4 = F \cdot D_5 = 1, F \cdot K_X = -4d, F^2 = 16d - 8.\]
Then \( \dim |F| = 10d - 4 \). So, there exists an element \( N \in |F| \) such that \( N \) passes through \( Q \). Note that \( D_1 + 2D_2 + 3D_3 + 4D_4 + 4D_5 + 3D_6 + 2D_7 + D_8 + N \) does not admit representation as \( M_1 + M_2 \), where \( M_1 \in | - K_X | \) and \( M_2 \in | - 3K_X | \). Hence, \( f(N) \) does not contain the support of anti-canonical divisor. Note that

\[
\text{mult}_Q(D_1 + 2D_2 + 3D_3 + 4D_4 + 4D_5 + 3D_6 + 2D_7 + D_8 + N) \geq 9.
\]

Then \( \frac{1}{2} f(N) \) is a tiger such that the support of this tiger does not contain any elements of \( | - K_X | \).

\[ \square \]

**Lemma 3.17.** Let \( X \) be a del Pezzo surface with du Val singularities and let \( d \) be the degree of \( X \). Assume that there exists a singular point of type \( D_k \), where \( k = 5, 6, 7, 8 \). Then \( X \) has a \( -K_X \)-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of \( | - K_X | \).

**Proof.** Let \( X \) be a del Pezzo surface with du Val singularities, and let \( P \) be a singular point of type \( D_k \). By Theorem 2.9, we see that \( X \) has a \( -K_X \)-polar cylinder. Let \( f : \bar{X} \to X \) be the minimal resolution of singularities of \( X \), and let \( D = \sum_{i=1}^n D_i \) be the exceptional divisor of \( f \), where \( D_i \) is a \((-2)\)-curve. We may assume that \( D_1, D_2, \ldots, D_k \) correspond to \( P \). Moreover, \( D_i \) is the central component, \( D_1, D_2 \) meet only \( D_3 \), and \( D_1 \cdot D_{i+1} = 1 \) for \( i = 3, 4, \ldots, k - 1 \).

Consider the following cases.

**Case 1.** \( k = 5 \). Put \(-2K_X \sim 2D_1 + 2D_2 + 3D_3 + 2D_4 + D_5 + F \). Then \( F \cdot D_1 = F \cdot D_2 = 1 \), \( F \cdot D_3 = F \cdot D_4 = F \cdot D_5 = 0 \), \( F \cdot K_X = -2d \), \( F^2 = 4d - 4 \). Then \( \dim |F| = 3d - 2 \). So, there exists an element \( N \in |F| \). Note that \( 2D_1 + 2D_2 + 3D_3 + 2D_4 + D_5 + N \) does not admit representation as \( M_1 + M_2 \), where \( M_1, M_2 \in | - K_X | \). Hence, \( f(N) \) does not contain the support of anti-canonical divisor. Note that \( \text{mult}_Q(2D_1 + 2D_2 + 3D_3 + 2D_4 + D_5 + N) \geq 5 \), where \( Q \) is the intersection of \( D_3 \) and \( D_4 \). Then \( \frac{1}{2} f(N) \) is a tiger such that the support of this tiger does not contain any elements of \( | - K_X | \).

**Case 2.** \( k = 6 \). Put \(-2K_X \sim 2D_1 + 2D_2 + 4D_3 + 3D_4 + 2D_5 + D_6 + F \). Then \( F \cdot D_3 = 1 \), \( F \cdot D_4 = 0 \) for \( i \neq 3 \), \( F \cdot K_X = -2d \), \( F^2 = 4d - 4 \). Then \( \dim |F| = 3d - 2 \). So, there exists an element \( N \in |F| \). Note that \( 2D_1 + 2D_2 + 4D_3 + 3D_4 + 2D_5 + D_6 + N \) does not admit representation as \( M_1 + M_2 \), where \( M_1, M_2 \in | - K_X | \). Hence, \( f(N) \) does not contain the support of anti-canonical divisor. Note that \( \text{mult}_Q(2D_1 + 2D_2 + 4D_3 + 3D_4 + 2D_5 + D_6 + N) \geq 7 \), where \( Q \) is the intersection of \( D_3 \) and \( D_4 \). Then \( \frac{1}{2} f(N) \) is a tiger such that the support of this tiger does not contain any elements of \( | - K_X | \).

**Case 3.** \( k = 7 \). Put

\[-3K_X \sim 3D_1 + 3D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7 + F.\]

Then \( F \cdot D_3 = F \cdot D_7 = 1 \), \( F \cdot D_i = 0 \) for \( i \neq 3, 7 \), \( F \cdot K_X = -3d \), \( F^2 = 9d - 8 \). Then \( \dim |F| = 6d - 4 \). So, there exists an element \( N \in |F| \). Note that \( 3D_1 + 3D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7 + N \) does not admit representation as \( M_1 + M_2 \), where \( M_1 \in | - K_X | \) and \( M_2 \in | - 2K_X | \). Hence, \( f(N) \) does not contain the support of anti-canonical divisor. Note that \( \text{mult}_Q(3D_1 + 3D_2 + \)


Let $X$ be a del Pezzo surface with du Val singularities and let $d$ be the degree of $X$. Assume that there exists a singular point of type $E_k$, where $k = 6, 7, 8$. Then $X$ has a $-K_X$-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of $| - K_X |$.

**Proof.** Let $X$ be a del Pezzo surface with du Val singularities, and let $P$ be a singular point of type $D_k$. By Theorem 2.9, we see that $X$ has a $-K_X$-polar cylinder. Let $f : X \to X$ be the minimal resolution of singularities of $X$, and let $D = \sum_{i=1}^n D_i$ be the exceptional divisor of $f$, where $D_i$ is a $(-2)$-curve. We may assume that $D_1, D_2, \ldots, D_k$ correspond to $P$. Moreover, $D_k$ is the central component, $D_1$ meets only $D_k$, $D_2$ meets $D_1$ and $D_3$, $D_2$ meets only $D_3$, and $D_i \cdot D_{i+1} = 1$ for $i = 3, 4, \ldots, k - 1$. Consider the following cases.

**Case 1.** $k = 6$. Put $-2K_X \sim 2D_1 + 2D_2 + 3D_3 + 4D_4 + 3D_5 + 2D_6 + D_7 + F$.

Then $f \cdot D_1 = 1$, $f \cdot D_i = 0$ for $i \geq 2$, $f \cdot K_X = -2d$, $F^2 = 4d - 2$. Then dim $| F | = 3d - 1$. So, there exists an element $N \in | F |$. Note that $2D_1 + 2D_2 + 3D_3 + 4D_4 + 3D_5 + 2D_6 + D_7 + N$ does not admit representation as $M_1 + M_2$, where $M_1, M_2 \in | - K_X |$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that mult$_Q(2D_1 + 2D_2 + 3D_3 + 4D_4 + 3D_5 + 2D_6 + D_7 + N) \geq 5$, where $Q$ is the intersection of $D_4$ and $D_5$. Then $\frac{1}{2}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $| - K_X |$.

**Case 2.** $k = 7$. Put $-2K_X \sim 2D_1 + 2D_2 + 3D_3 + 4D_4 + 3D_5 + 2D_6 + D_7 + F$.

Then $f \cdot D_2 = 1$, $f \cdot D_i = 0$ for $i \neq 2$, $f \cdot K_X = -2d$, $F^2 = 4d - 2$. Then dim $| F | = 3d - 1$. So, there exists an element $N \in | F |$. Note that $2D_1 + 2D_2 + 3D_3 + 4D_4 + 3D_5 + 2D_6 + D_7 + N$ does not admit representation as $M_1 + M_2$, where $M_1, M_2 \in | - K_X |$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that mult$_Q(2D_1 + 2D_2 + 3D_3 + 4D_4 + 3D_5 + 2D_6 + D_7 + N) \geq 7$, where $Q$ is the intersection of $D_4$ and $D_5$. Then $\frac{1}{2}f(N)$
is a tiger such that the support of this tiger does not contain any elements of $|−K_X|$. 

Case 3. $k = 8$. Put

$$-2K_{\bar{X}} \sim 3D_1 + 2D_2 + 4D_3 + 6D_4 + 5D_6 + 4D_7 + 2D_8 + F.$$ 

Then $F \cdot D_i = 1$, $F \cdot D = 0$ for $i \neq 8$, $F \cdot K_{\bar{X}} = -2d$, $F^2 = 4d^2 - 2$. Then $\dim |F| = 3d - 1$. So, there exists an element $N \in |F|$. Note that $3D_1 + 2D_2 + 4D_3 + 6D_4 + 5D_6 + 4D_7 + 2D_8 + N$ does not admit representation as $M_1 + M_2$, where $M_1, M_2 \in |−K_{\bar{X}}|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that $mult_Q(2D_1 + 2D_2 + 3D_3 + 4D_4 + 3D_5 + 2D_6 + D_7 + N) \geq 7$, where $Q$ is the intersection of $D_4$ and $D_5$. Then $\frac{1}{2}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|−K_X|$. □

So, Theorem 1.5 follows from Lemmas 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, and 3.18.

4. The proof of theorem 1.6

Assume that $\rho(X) = 1$. Then $X$ has a $H$-polar cylinder if and only if $X$ has a $−K_X$-polar cylinder, where $H$ is an arbitrary ample divisor. On the other hand, there exists a classification of del Pezzo surfaces $X$ such that $X$ has a $−K_X$-polar cylinder (see [1]). By a classification of a del Pezzo surface $X$ has not cylinders if $X$ has one of the following collections of singularities: $4A_2, 2A_1 + 2A_3, 2D_4$. So, we may assume that $\rho(X) > 1$.

Let $f : \bar{X} \to X$ be the minimal resolution of singularities of $X$, and let $D = \sum_{i=1}^{n}D_i$ be the exceptional divisor of $f$, where $D_i$ is a $(−2)$-curve.

Lemma 4.19. Assume that there exists a $\mathbb{P}^1$-fibration $g : \tilde{X} \to \mathbb{P}^1$ such that at most one irreducible component of the exceptional divisor $D$ not contained in any fiber of $g$. Moreover, this component is an 1-section. Then there exists an ample divisor $H$ such that $X$ has a $H$-polar cylinder.

Proof. Let $F$ be a unique exception curve not contained in any fiber of $g$ (if there exist no such component, then $F$ is an arbitrary 1-section). Put

$$-K_{\bar{X}} \sim_{\mathbb{Q}} 2F + \sum a_i E_i.$$ 

Note that all $E_i$ are contained in fibers of $g$. Consider an ample divisor $H = -K_{\bar{X}} + mC$, where $C$ is a fiber of $g$, $m$ is a sufficiently large number. Then there exists a divisor $\tilde{H} \sim_{\mathbb{Q}} H$ such that

$$\tilde{H} = 2F + \sum b_i \tilde{E}_i,$$

where $b_i > 0$ and the set of $\tilde{E}_i$ contains all irreducible curves in singular fibers of $g$. Then

$$\tilde{X} \setminus \text{Supp}(\tilde{H}) \cong \mathbb{A}^1 \times (\mathbb{P}^1 \setminus \{p_1, \ldots, p_k\}).$$
where \( p_1, \ldots, p_k \) correspond to singular fibers of \( g \). So, \( \bar{X} \) has a \( H \)-polarization. Hence, \( X \) has a \( f(H) \)-polarization. \( \square \)

Run MMP for \( X \). We obtain

\[
X = X_1 \to X_2 \to \cdots \to X_n. 
\]

Assume that \( X_n = \mathbb{P}^1 \). Consider the composition of the minimal resolution and MMP, We have a \( \mathbb{P}^1 \)-fibration \( g: \bar{X} \to \mathbb{P}^1 \). Note that all exception curves of \( f \) are contained in fibers of \( g \). Hence, by Lemma 4.19, we see that there exists an ample divisor \( H \) such that \( X \) has a \( H \)-polar cylinder.

So, we may assume that \( X_n \) is a del Pezzo surface with \( \rho(X_n) = 1 \) and du Val singularities.

**Lemma 4.20.** Assume that \( X_n \) has a \( -K_{X_n} \)-polar cylinder. Then there exists an ample divisor \( H \) such that \( X \) has a \( H \)-polar cylinder.

**Proof.** Put \( h: X \to X_n \). Assume that \( h \) contracts extremal rays in points \( p_1, p_2, \ldots, p_m \). Let \( M \) be an anti-canonical divisor such that \( X_n \setminus \text{Supp}(M) \cong Z \times \mathbb{A}^1 \). Let \( \phi: X_n \setminus \text{Supp}(M) \to Z \) be the projection on first factor. Let \( C_1, C_2, \ldots, C_k \) be the fibers of \( \phi \) such that \( C_1, C_2, \ldots, C_k \) contain \( p_1, p_2, \ldots, p_m \), and let \( \bar{C}_1, \bar{C}_2, \ldots, \bar{C}_k \) be the closure of \( C_1, C_2, \ldots, C_k \) on \( X_n \). Since \( \rho(X_n) = 1 \), we see that \( \bar{C}_i \sim_{Q} -a_i K_{X_n} \). Consider the divisor

\[
L = M + m_1 \bar{C}_1 + m_2 \bar{C}_2 + \cdots + m_k \bar{C}_k, 
\]

where \( m_1, m_2, \ldots, m_k \) are sufficiently large numbers. Note that the divisor \( L \sim_{Q} -\alpha K_{X_n} \). Let \( L \) be the proper transform of the divisor \( L \). Consider \( H = \bar{L} + \sum \epsilon_i E_i \), where \( E_i \) are irreducible components of the exceptional divisor of \( h \) and \( \epsilon_i \) are positive numbers. Note that for sufficiently large \( m_i \) and for sufficiently small \( \epsilon_i \), the divisor \( H \) is ample. Moreover, \( X \setminus \text{Supp}(H) \cong (Z \setminus \{q_1, \ldots, q_k\}) \times \mathbb{A}^1 \), where \( q_1, \ldots, q_k \) are \( k \) points on \( Z \). So, \( X \) has a \( H \)-polar cylinder. \( \square \)

Let \( X \) be a del Pezzo surface with du Val singularities. Assume that \( \rho(X) = 1 \). Then \( X \) has a \( H \)-polar cylinder if and only if \( X \) has a \( -K_X \)-polar cylinder, where \( H \) is an arbitrary ample divisor. On the other hand, there exists a classification of del Pezzo surfaces \( X \) such that \( X \) has a \( -K_X \)-polar cylinder (see [1]). By a classification of a del Pezzo surface \( X \) has not cylinders if \( X \) has one of the following collections of singularities: \( 4A_2, 2A_1 + 2A_3, 2D_4 \). So, we may assume that \( \rho(X) > 1 \) and \( X \) has not cylinders. Run MMP for \( X \). We obtain

\[
X = X_1 \to X_2 \to \cdots \to X_n. 
\]

By Lemma 4.19 we may assume that \( X_n \) is a del Pezzo surface with \( \rho(X_n) = 1 \) and du Val singularities. By Lemma 4.20 we see that \( X_n \) is a del Pezzo surface with one of the following collection of singularities: \( 4A_2, 2A_1 + 2A_3, 2D_4 \). On the other hand, the surface \( X \) has a smaller degree than \( X_n \). But degree of \( X_n \) is equal to one. So, \( X = X_n \). On the other hand, \( \rho(X_n) = 1 \), a contradiction.
This completes the proof of Theorem 1.6.

References


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