REMARK ON A SUMMATION FORMULA FOR THE SERIES $4\text{F}_3(1)$

**Junesang Choi**, **Yashoverdhan Vyas** and **Arjun K. Rathie**

**Abstract.** We aim to prove a known summation formula for the series $4\text{F}_3(1)$ by mainly using a similar method as in [2], which is different from that in [3]. The method of proof here as well as that in [2] is potentially useful in getting some other summation formulas for $p\text{F}_q$.

1. Introduction

Throughout this paper, let $p\text{F}_q$ denote the generalized hypergeometric series (see, for details, e.g., [6], [7], [8, Section 1.5]). We begin by recalling the following two summation formulas for the series $3\text{F}_2$ and $4\text{F}_3$ (see, e.g., [7, p. 245])

\[
3\text{F}_2 \left[ \begin{array}{c}
\frac{1}{2}a, 1 + \frac{1}{2}a, b; \\
\frac{1}{2}a, 1 + a - b; \\
\end{array} \right] - 1 = \frac{\Gamma \left( \frac{1}{2}a + \frac{1}{2} \right) \Gamma (1 + a - b)}{\Gamma (1 + a) \Gamma \left( \frac{1}{2} + \frac{1}{2}a - b \right)} \tag{1.1}
\]

and

\[
4\text{F}_3 \left[ \begin{array}{c}
\frac{1}{2}a, a + \frac{1}{2}a, b, c; \\
\frac{1}{2}a, 1 + a - b, 1 + a - c; \\
\end{array} \right] = \frac{\Gamma \left( \frac{1}{2}a + \frac{1}{2} \right) \Gamma (1 + a - b) \Gamma (1 + a - c) \Gamma \left( \frac{1}{2} + \frac{1}{2}a - b - c \right)}{\Gamma (1 + a) \Gamma \left( \frac{1}{2}a - b + \frac{1}{2} \right) \Gamma \left( \frac{1}{2}a - c + \frac{1}{2} \right) \Gamma (1 + a - b - c)} \tag{1.2}
\]

For our present investigation, we also recall the following two summation formulas due to Kim et al. [3]:

\[
3\text{F}_2 \left[ \begin{array}{c}
a, b, 1 + d; \\
1 + a - b, 1 + a - d; \\
\end{array} \right] - 1 = \left( 1 - \frac{a}{2d} \right) \frac{\Gamma (1 + \frac{1}{2}a) \Gamma (1 + a - b)}{\Gamma (1 + a) \Gamma \left( 1 + \frac{1}{2}a - b \right)} + \frac{a}{2d} \cdot \frac{\Gamma \left( \frac{1}{2}a + \frac{1}{2} \right) \Gamma (1 + a - b)}{\Gamma (1 + a) \Gamma \left( \frac{1}{2}a - b + \frac{1}{2} \right)} \tag{1.3}
\]
and
\[
\begin{align*}
&\quad \quad 4F_3\left[ \begin{array}{c} a, b, c, d + 1; \\ 1 + a - b, 1 + a - c, d; \\ 1 \end{array} \right] \\
&= \left(1 - \frac{a}{2d}\right) \frac{\Gamma(1 + \frac{1}{2}a) \Gamma(1 + a - b) \Gamma(1 + a - c) \Gamma(1 + \frac{1}{2}a - b - c)}{\Gamma(1 + a) \Gamma(1 + a - b - c) \Gamma(1 + \frac{1}{2}a - b) \Gamma(1 + \frac{1}{2}a - c)} \\
&\quad + \frac{a}{2d} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}a) \Gamma(1 + a - b) \Gamma(1 + a - c) \Gamma(\frac{1}{2} + \frac{1}{2}a - b - c)}{\Gamma(1 + a) \Gamma(1 + a - b - c) \Gamma(\frac{1}{2} + \frac{1}{2}a - b) \Gamma(\frac{1}{2} + \frac{1}{2}a - c)}.
\end{align*}
\]
(1.4)

Remark 1. The identities (1.1) and (1.2) are obvious special cases of (1.3) and (1.4), respectively. Taking the limit in (1.4) as $c \to \infty$ yields (1.3).

Setting $b = -n$ ($n \in \mathbb{N}_0$) in (1.3) and (1.4), respectively, we obtain the following interesting identities:
\[
\begin{align*}
&\quad \quad 3F_2\left[ \begin{array}{c} -n, b, 1 + d; \\ 1 + a + n, d; \\ -1 \end{array} \right] \\
&= \left(1 - \frac{a}{2d}\right) \frac{(1 + a)_n}{(1 + \frac{1}{2}a)_n} + \frac{a}{2d} \frac{(1 + a)_n}{(\frac{1}{2}a + \frac{1}{2})}_n 
\end{align*}
\]
(1.5)

and
\[
\begin{align*}
&\quad \quad 4F_3\left[ \begin{array}{c} -n, a, b, d + 1; \\ 1 + a + n, 1 + a - b, d; \\ 1 \end{array} \right] \\
&= \left(1 - \frac{a}{2d}\right) \frac{(1 + a)_n (1 + \frac{1}{2}a - c)_n}{(1 + \frac{1}{2}a)_n (1 + a - c)_n} \\
&\quad + \frac{a}{2d} \frac{(1 + a)_n (\frac{1}{2} + \frac{1}{2}a - c)_n}{(\frac{1}{2}a + \frac{1}{2})_n (1 + a - c)_n}.
\end{align*}
\]
(1.6)

Here and in the following, let $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{Z}_{\leq 0}$ be the sets of complex numbers, positive integers and non-positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Kim et al. [3] established the result (1.3) with the help of classical Kummer’s summation theorem and its contiguous results in [5] and established the result (1.4) with the help of classical Dixon’s summation theorem and its contiguous result in [4]. Very recently, Choi et al. [2] have proved an extended Watson’s summation theorem for the series $4F_3(1)$ in [3] by mainly using a known summation formula for $3F_2(1/2)$. Here, similarly as in [2], we aim to prove (1.4) by mainly using (1.3).

2. Derivation of (1.4)

Let $\mathcal{L}$ be the left side of (1.4). Expressing $4F_3$ as the series, we obtain
\[
\mathcal{L} = \sum_{k=0}^{\infty} \frac{(-1)^k (a)_k (b)_k (1 + d)_k}{(1 + a - b)_k (d)_k k!} \left\{ \frac{(-1)^k (c)_k}{(1 + a - c)_k} \right\},
\]
(2.1)
where \((\lambda)_n\) is the Pochhammer symbol defined (for \(\lambda \in \mathbb{C}\)) by (see [8, p. 2 and pp. 4-6]):

\[
(\lambda)_n := \begin{cases} 
1 & (n = 0) \\
\lambda(\lambda + 1)\ldots(\lambda + n - 1) & (n \in \mathbb{N}) 
\end{cases}
\]

\[
= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-),
\]

where \(\Gamma\) is the familiar Gamma function.

Using the following identity (cf., [6, p. 69, Exercise 5])

\[
\binom{-k}{a + k; 1} = \frac{(-1)^k (c)_k}{(1 + a - c)_k} \quad (k \in \mathbb{N}_0)
\]

in (2.1), we have

\[
\mathcal{L} = \sum_{k=0}^{\infty} \frac{(-1)^k (a)_k (b)_k (1 + d)_k}{(1 + a - b)_k (d)_k k!} \binom{-k}{a + k; 1}.
\]

Expressing \(\binom{-k}{a + k, 1}\) as the series, we get

\[
\mathcal{L} = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(-1)^k (a)_k (b)_k (1 + d)_k}{(1 + a - b)_k (d)_k (1 + a - c)_m k! m!}.
\]

which, upon using the identities

\[
(\alpha)_k (\alpha + k)_m = (\alpha)_{k+m} \quad (\alpha \in \mathbb{C}; \ k, m \in \mathbb{N}_0)
\]

and

\[
(-k)_m = \frac{(-1)^m k!}{(k - m)!},
\]

yields

\[
\mathcal{L} = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(-1)^{k+m} (a)_{k+m} (b)_k (1 + d)_k}{(1 + a - b)_k (1 + a - c)_m (d)_k m! (k - m)!}.
\]

Applying the following formal manipulation of double series (see, e.g., [1], [6, p. 57, Lemma 10(2)])

\[
\sum_{k=0}^{\infty} \sum_{m=0}^{k} A(m, k) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} A(m, k + m),
\]

we obtain

\[
\mathcal{L} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^k (a)_{k+2m} (b)_{k+m} (1 + d)_{k+m}}{(1 + a - b)_{k+m} (d)_{k+m} (1 + a - c)_m m! k!}.
\]
Using (2.4) in (2.6), we get
\[
L = \sum_{m=0}^{\infty} \frac{(a)_{2m} (b)_m (1 + d)_m}{(1 + a - b)_m (1 + a - c)_m (d)_m m!} 
\times \sum_{k=0}^{\infty} \frac{(-1)^k (a + 2m)_k (b + m)_k (1 + d + m)_k}{(1 + a - b + m)_k (d + m)_k k!},
\]
which, upon expressing the inner series as \(3F_2\), gives
\[
L = \sum_{m=0}^{\infty} \frac{(a)_{2m} (b)_m (1 + d)_m}{(1 + a - b)_m (1 + a - c)_m (d)_m m!} 
\times 3F_2 \left[ \frac{a + 2m, b + m, 1 + d + m;}{1 + a - b + m, d + m;} - 1 \right].
\]
(2.7)
Finally, using (1.3) to evaluate the \(3F_2\) in (2.7), after some simplification, we find that the resulting right side of (2.7) leads to the right side of (1.4).

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References


JUNESANG CHOI
DEPARTMENT OF MATHEMATICS, DONGGUK UNIVERSITY, GYEONGJU 38066, REPUBLIC OF KOREA
E-mail address: junesang@mail.dongguk.ac.kr

YASHOVERDHAHNY VYAS
DEPARTMENT OF MATHEMATICS, SCHOOL OF ENGINEERING, SIR PADAMPAT SINGHANIA UNIVERSITY, BHATTEWAR,UDAIPUR, 313601, RAJASTHAN STATE, INDIA
E-mail address: yashoverdhan.vyas@spsu.ac.in
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Arjun K. Rathie
Department of Mathematics, School of Physical sciences, Central University of Kerala, Periye P.O., Kasaragod-671316, Kerala, India
E-mail address: akrathie@cukerala.ac.in