HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITELY TRANS-SASAKIAN MANIFOLD WITH A QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract. Jin [10] studied lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection. We study further the geometry of this subject. The object of this paper is to study the geometry of half lightlike submanifolds of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection.

1. Introduction

A linear connection $\nabla$ on a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ is said to be a \textit{quarter-symmetric connection} if its torsion tensor $\tilde{T}$ satisfies

$$\tilde{T}(\tilde{X}, \tilde{Y}) = \theta(\tilde{Y})J\tilde{X} - \theta(\tilde{X})J\tilde{Y},$$

where $J$ is a $(1, 1)$-type tensor field and $\theta$ is a $1$-form associated with a smooth vector field $\zeta$ by $\theta(X) = \tilde{g}(X, \zeta)$. Moreover, if this connection $\nabla$ is metric, \textit{i.e.}, $\tilde{\nabla}\tilde{g} = 0$, then $\tilde{\nabla}$ is called a \textit{quarter-symmetric metric connection}. The notion of quarter-symmetric metric connection was introduced by Yano-Imai [14]. The geometry of lightlike hypersurface of an indefinite trans-Sasakian manifolds with a quarter-symmetric metric connection was studied by Jin [10]. Throughout this paper, denote by $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ the smooth vector fields on $\tilde{M}$.

Let $M$ be a submanifold of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ of codimension 2 with the tangent bundle $TM$ and the normal bundle $TM^\perp$. Denoted by $\text{Rad}(TM) = TM \cap TM^\perp$ the radical distribution. Then $M$ is called

1. \textit{half lightlike submanifold} if $\text{rank}\{\text{Rad}(TM)\} = 1$,
2. \textit{coisotropic submanifold} if $\text{rank}\{\text{Rad}(TM)\} = 2$.

Half lightlike submanifold was introduced by Duggal-Bejancu [4] and later, studied by Duggal-Jin [5]. Its geometry is more general than that of lightlike hypersurface or coisotropic submanifold. Much of its theory will be immediately generalized in a formal way to general lightlike submanifolds.

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The notion of trans-Sasakian manifold, of type \((\alpha,\beta)\), was introduced by Oubina [13]. Sasakian, Kenmotsu and cosymplectic manifolds are important kinds of trans-Sasakian manifold such that

\[
\alpha = 1, \quad \beta = 0; \quad \alpha = 0, \quad \beta = 1; \quad \alpha = \beta = 0,
\]

respectively. We say that a trans-Sasakian manifold \(\tilde{M}\) is an indefinite trans-Sasakian manifold if \(\tilde{M}\) is a semi-Riemannian manifold.

In this paper, we study half lightlike submanifolds of an indefinite trans-Sasakian manifold \(\tilde{M} \equiv (\tilde{M}, J, \zeta, \theta, \tilde{g})\) with a quarter-symmetric metric connection, in which the tensor field \(J\) and the 1-form \(\theta\), defined by (1.1), are identical with the structure tensor field \(J\) and the structure 1-form \(\theta\) of the indefinite trans-Sasakian structure \((J, \theta, \zeta, \tilde{g})\) on \(\tilde{M}\), respectively.

**Remark 1.** Denote by \(\tilde{\nabla}\) the Levi-Civita connection of \(\tilde{M}\) with respect to the semi-Riemannian metric \(\tilde{g}\). Due to [9], it is known that a linear connection \(\nabla\) on \(\tilde{M}\) is a quarter-symmetric metric connection if and only if \(\nabla\) satisfies

\[
\tilde{\nabla}_X \tilde{Y} = \tilde{\nabla}_X \tilde{Y} - \theta(\tilde{X})J\tilde{Y}. \tag{1.2}
\]

### 2. Preliminaries

An odd-dimensional semi-Riemannian manifold \((\tilde{M}, \tilde{g})\) is called an indefinite trans-Sasakian manifold if there exist a structure set \(\{J, \zeta, \theta, \tilde{g}\}\), a Levi-Civita connection \(\tilde{\nabla}\) and two smooth functions \(\alpha\) and \(\beta\), where \(J\) is a \((1,1)\)-type tensor field, \(\zeta\) is a vector field, and \(\theta\) is a 1-form such that

\[
\begin{align*}
J^2 \tilde{X} &= -\tilde{X} + \theta(\tilde{X})\zeta, \quad \theta(\zeta) = 1, \quad \theta(\tilde{X}) = \epsilon \tilde{g}(\tilde{X}, \zeta), \\
\theta \circ J &= 0, \quad \tilde{g}(J\tilde{X}, J\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}) - \epsilon \theta(\tilde{X})\theta(\tilde{Y}), \\
(\tilde{\nabla}_X J)\tilde{Y} &= \alpha\{\tilde{g}(\tilde{X}, \tilde{Y})\zeta - \epsilon \theta(\tilde{Y})\tilde{X}\} \\
&\quad + \beta\{\tilde{g}(J\tilde{X}, \tilde{Y})\zeta - \epsilon \theta(\tilde{Y})J\tilde{X}\},
\end{align*}
\]

where \(\epsilon\) denotes \(\epsilon = 1\) or \(-1\) according as \(\zeta\) is spacelike or timelike, respectively.

\(\{J, \zeta, \theta, \tilde{g}\}\) is called an indefinite trans-Sasakian structure of type \((\alpha, \beta)\).

In the entire discussion of this paper, we shall assume that the structure vector field \(\zeta\) is a spacelike one, i.e., \(\epsilon = 1\), without loss of generality.

Replacing the Levi-Civita connection \(\tilde{\nabla}\) by the quarter-symmetric metric connection \(\nabla\) given by (1.2), the last equation of (2.1) is reduced to

\[
(\nabla_X J)\tilde{Y} = \alpha\{\tilde{g}(\tilde{X}, \tilde{Y})\zeta - \theta(\tilde{Y})\tilde{X}\} + \beta\{\tilde{g}(J\tilde{X}, \tilde{Y})\zeta - \theta(\tilde{Y})J\tilde{X}\}. \tag{2.2}
\]

Replacing \(\tilde{Y}\) by \(\zeta\) to (2.2) and using \(J\zeta = 0\) and \(\theta(\nabla_X \zeta) = 0\), we obtain

\[
\nabla_X \zeta = -\alpha J\tilde{X} + \beta(1 - \theta(X)\zeta). \tag{2.3}
\]

Let \((M, g)\) be a half lightlike submanifold of an indefinite trans-Sasakian manifold \(\tilde{M}\) equipped with the radical distribution \(\text{Rad}(TM)\), a screen distribution \(S(TM)\) and a coscreen distribution \(S(TM^\perp)\) such that

\[
TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM), \quad TM^\perp = \text{Rad}(TM) \oplus_{\text{orth}} S(TM^\perp).
\]
Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$. Also denote by $(2.1)_i$ the $i$-th equation of the six equations in $(2.1)$. We use the same notations for any others. Let $\xi$ be a section of $\text{Rad}(TM)$. Assume that $L$ is a unit spacelike basis vector field of $S(TM\perp)$, without loss of generality. Consider the orthogonal complementary distribution $S(TM\perp)$ to $S(TM)$ in $TM$. Certainly $\xi$ and $L$ belong to $\Gamma(S(TM\perp))$. Thus we have

$$S(TM\perp) = S(TM\perp) \oplus_{\text{orth}} S(TM\perp)\perp,$$

where $S(TM\perp)$ is the orthogonal complementary to $S(TM\perp)$ in $S(TM\perp)$. It is known [5] that, for any null section $\xi$ of $\text{Rad}(TM)$, there exists a uniquely defined null vector field $N \in \Gamma(S(TM\perp))$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \quad \forall X \in \Gamma(S(TM)).$$

Denote by $\text{ltr}(TM)$ the vector subbundle of $S(TM\perp)$ locally spanned by $N$. Then we show that $S(TM\perp) = \text{Rad}(TM) \oplus \text{ltr}(TM)$. We call $N$, $\text{ltr}(TM)$ and $\text{tr}(TM) = S(TM\perp) \oplus_{\text{orth}} \text{ltr}(TM)$ the null transversal vector field, lightlike transversal vector bundle and transversal vector bundle of $M$ with respect to the screen distribution $S(TM)$, respectively.

Denote by $X$, $Y$ and $Z$ the vector fields on $M$, unless otherwise specified. As the tangent bundle $\bar{TM}$ of the ambient manifold $\bar{M}$ is satisfied

$$\bar{TM} = TM \oplus \text{tr}(TM) = TM \oplus \text{ltr}(TM) \oplus_{\text{orth}} S(TM\perp),$$

the Gauss and Weingarten formulae of $M$ are given respectively by

\begin{align*}
\bar{\nabla}_XY &= \nabla_XY + B(X,Y)N + D(X,Y)L, \\
\bar{\nabla}_XN &= -A_XX + \tau(X)N + \rho(X)L, \\
\bar{\nabla}_XL &= -A_LX + \lambda(X)N,
\end{align*}

(2.4) \hfill (2.5) \hfill (2.6)

where $\nabla$ is the linear connection on $M$, $B$ and $D$ are the local second fundamental forms of $M$, $A_X$ and $A_L$ are the shape operators, and $\tau$, $\rho$ and $\lambda$ are 1-forms on $TM$. Let $P$ be the projection morphism of $TM$ on $S(TM)$ and $\eta$ a 1-form such that $\eta(X) = \bar{g}(X, N)$. As $TM = S(TM) \oplus_{\text{orth}} \text{Rad}(TM)$, the Gauss and Weingarten formulae of $S(TM)$ are given respectively by

\begin{align*}
\nabla_XPY &= \nabla^*_XPY + C(X, PY)\xi, \\
\nabla_X\xi &= -A^*_\xi X - \tau(X)\xi,
\end{align*}

(2.7) \hfill (2.8)

where $\nabla^*$ is the linear connection on $S(TM)$, $C$ is the local screen second fundamental form of $S(TM)$, $A^*_\xi$ is the shape operator.

From the facts that $B(X,Y) = \bar{g}(\nabla_XY, \xi)$ and $D(X,Y) = \bar{g}(\nabla_XY, L)$, we show that $B$ and $D$ are independent of the choice of $S(TM)$ and satisfy

$$B(X, \xi) = 0, \quad D(X, \xi) = -\lambda(X). \quad (2.9)$$
The local second fundamental forms are related to their shape operators by
\[ B(X,Y) = g(A^*_X X,Y), \quad \bar{g}(A^*_X X,N) = 0, \quad (2.10) \]
\[ C(X,PY) = g(A^*_X X,PY), \quad \bar{g}(A^*_X X,N) = 0, \quad (2.11) \]
\[ D(X,Y) = g(A^*_X X,Y) - \lambda(X)\eta(Y), \quad \bar{g}(A^*_X X,N) = \rho(X). \quad (2.12) \]

### 3. Structure equations on \( M \)

Călin [2] proved that if \( \zeta \) is tangent to \( M \), then it belongs to \( S(TM) \) which we assume. It is known [7] that, for any half lightlike submanifold \( M \) of an indefinite trans-Sasakian manifold \( \bar{M} \), \( J(Rad(TM)) \), \( J(ltr(TM)) \) and \( J(S(TM^\perp)) \) are vector subbundles of \( S(TM) \), of rank 1. There exist two non-degenerate almost complex distributions \( H_o \) and \( H \) with respect to \( J \) such that
\[ S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^\perp)) \oplus_{orth} H_o, \]
\[ H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o. \]

In this case, the tangent bundle \( TM \) is decomposed as follow:
\[ TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\perp)). \quad (3.1) \]

Consider two local null vector fields \( U \) and \( V \), a local unit spacelike vector field \( W \) on \( S(TM) \), and their 1-forms \( u, v \) and \( w \) defined by
\[ U = -JN, \quad V = -J\xi, \quad W = -JL, \quad (3.2) \]
\[ u(X) = g(X,V), \quad v(X) = g(X,U), \quad w(X) = g(X,W). \quad (3.3) \]

Let \( S \) be the projection morphism of \( TM \) on \( H \) and \( F \) the tensor field of type \((1,1)\) globally defined on \( M \) by \( F = J \circ S \). Then \( JX \) is expressed as
\[ JX = FX + u(X)N + w(X)L. \quad (3.4) \]

Applying \( J \) to (3.4) and using (2.1) and (3.2), we have
\[ F^2 X = -X + u(X)U + w(X)W + \theta(X)\zeta. \quad (3.5) \]

In the following, we say that \( F \) is the structure tensor field of \( M \).

Substituting (3.4) into (2.3) and using (2.4), we see that
\[ \nabla_X \zeta = -\alpha FX + \beta(X - \theta(X)\zeta), \quad (3.6) \]
\[ B(X,\zeta) = -\alpha u(X), \quad D(X,\zeta) = -\alpha w(X). \quad (3.7) \]

Applying \( \bar{\nabla}_X \) to \( \bar{g}(\zeta,N) = 0 \) and using (2.3), (2.5) and (2.11), we have
\[ C(X,\zeta) = -\alpha v(X) + \beta \eta(X). \quad (3.8) \]

Substituting (2.4) and (3.4) into (1.1) and then, comparing the tangent, lightlike transversal and co-screen components, we obtain
\[ T(X,Y) = \theta(Y)FX - \theta(X)FY, \quad (3.9) \]
\[ B(X,Y) - B(Y,X) = \theta(Y)u(X) - \theta(X)u(Y), \quad (3.10) \]
\[ D(X,Y) - D(Y,X) = \theta(Y)w(X) - \theta(X)w(Y). \quad (3.11) \]
where $T$ is the torsion tensor with respect to $\nabla$. From (3.10) and (3.11), we see that $B$ and $D$ are never symmetric. Replacing $Y$ by $\xi$ to (2.10) and using (2.9)\(_1\), (3.10) and the fact that $S(TM)$ is non-degenerate, we obtain

\[ A_2^\varpi \xi = 0. \]  

(3.12)

Applying $\nabla_X$ to (3.2) $\sim$ (3.4) by turns and using (2.4), (2.5), (2.6), (2.9) $\sim$ (2.10), (2.12) and (3.2) $\sim$ (3.4), we have

\[ B(X, U) = C(X, V), \quad B(X, W) = D(X, V), \quad C(X, W) = D(X, U), \]  

(3.13)

\[ \nabla_X U = F(A_N X) + \tau(X)U + \rho(X)W - \{\alpha \eta(X) + \beta \nu(X)\}, \]  

(3.14)

\[ \nabla_X V = F(A_N X) - \tau(X)V - \lambda(X)W - \beta \nu(X), \]  

(3.15)

\[ \nabla_X W = F(A_N X) + \lambda(X)U - \beta \omega(X), \]  

(3.16)

\[ (\nabla_X F)(Y) = u(Y)A_N X + w(Y)A_L X - B(X, Y)U - D(X, Y)V \]  

+ \(\alpha\{g(X, Y)\zeta - \theta(Y)X\} + \beta\{f(X, Y)\zeta - \theta(Y)V\}, \]  

(3.17)

\[ (\nabla_X u)(Y) = -u(Y)\tau(X) - w(Y)\lambda(X) - \beta \theta(Y)u(X) - B(X, FV), \]  

(3.18)

\[ (\nabla_X v)(Y) = v(Y)\tau(X) + w(Y)\rho(X) - \theta(Y)\{\alpha \eta(X) + \beta \nu(X)\} \]  

- \(g(A_N X, FV). \)  

(3.19)

4. Recurrent and Lie recurrent structure tensors

**Definition 1.** The structure tensor field $F$ of $M$ is said to be recurrent [8] if there exists a smooth 1-form $\varpi$ on $M$ such that

\[(\nabla_X F)Y = \varpi(X)FV.\]

**Definition 2.** A half lightlike submanifold $M$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be statical [6] if $\nabla_X L \in \Gamma(S(TM))$ for any $X \in \Gamma(TM)$.

**Remark 2.** From (2.6) and (2.12), we show that Definition 2 is equivalent to the conditions: $\lambda = 0$ and $\rho = 0$. The condition $\lambda = 0$ is equivalent to the conception: $M$ is irrotational, i.e., $\nabla_X \xi \in \Gamma(TM)$ [12]. The condition $\rho = 0$ is equivalent to the conception: $M$ is solenoidal, i.e., $A_L X \in \Gamma(S(TM))$ [11].

**Theorem 4.1.** Let $M$ be a half lightlike submanifold of an indefinite trans-Sasakian manifold $M$ with a quarter-symmetric metric connection. If $F$ is recurrent, then the following six statements are satisfied:

1. $F$ is parallel with respect to the induced connection $\nabla$ on $M$,
2. $M$ is an indefinite cosymplectic manifold, i.e., $\alpha = \beta = 0$,
3. $M$ is statical, i.e., $\lambda = 0$ and $\rho = 0$,
4. $W$ is parallel vector field with respect to the connection $\nabla$,
5. $H, J(tr(TM))$ and $J(S(TM^\perp))$ are parallel distributions on $M$,
6. $M$ is locally a product manifold $C_\upsilon \times C_\omega \times M^2$, where $C_\upsilon$ is a null curve tangent to $J(tr(TM))$, $C_\omega$ is a spacelike curve tangent to $J(S(TM^\perp))$, and $M^2$ is a leaf of the distributions $H$. 

Proof. Denote by \( \mu, \nu \) and \( \sigma \) the 1-forms on \( M \) such that
\[
\mu(X) = B(X, U) = C(X, V), \quad \sigma(X) = D(X, W),
\]
\[
\nu(X) = B(X, W) = D(X, V).
\]

(1) As \( F \) is recurrent, from the above definition and (3.17), we get
\[
\varpi(X)FY = u(Y)A_N X + w(Y)A_L X - B(X, Y)U - D(X, Y)W - B(X, Y)U = D(X, Y)W
\]
\[
+ \alpha\{g(X, Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX, Y)\zeta - \theta(Y)F\zeta\}.
\]
Replacing \( Y \) by \( \xi \) and using (2.9) and the fact that \( F\xi = -V \), we get
\[
-\varpi(X)V = \lambda(X)W + \beta u(X)\zeta.
\]
Taking the scalar product with \( U \) to (4.2), we obtain \( \varpi = 0 \). Thus \( F \) is parallel with respect to the connection \( \nabla \).

(2) Taking the scalar product with \( \zeta \) to (4.2), we get \( \beta = 0 \). Taking the scalar product with \( U \) to (4.1) satisfying \( \varpi = \beta = 0 \), we get
\[
u(Y)g(A_N X, U) + w(Y)g(A_L X, U) - \alpha \theta(Y)v(X) = 0.
\]
Replacing \( Y \) by \( \zeta \) to this equation, we have \( \alpha = 0 \). As \( \alpha = \beta = 0 \), \( \bar{M} \) is an indefinite cosymplectic manifold.

(3) Taking the scalar product with \( W \) to (4.2) and with \( N \) to (4.1), we have
\[
\lambda(X) = 0, \quad \rho(X) = \bar{g}(A_L X, N) = 0.
\]
As \( \lambda = 0 \), \( M \) is irrotational. As \( \rho = 0 \), \( M \) is solenoidal. Thus \( M \) is statical.

(4) Taking \( Y = U \) and \( Y = W \) to (4.3) by turns, we have
\[
g(A_N X, U) = C(X, U) = 0, \quad g(A_L X, U) = 0.
\]
Taking the scalar product with \( V \) and \( W \) to (4.1) by turns, we have
\[
B(X, Y) = u(Y)\mu(X) + w(Y)\nu(X), \quad D(X, Y) = w(Y)\sigma(X),
\]
due to (4.5)\(_2\). Replacing \( Y \) by \( V \) to the two equations of (4.6), we have
\[
B(X, V) = 0, \quad \nu(X) = B(X, W) = D(X, V) = 0.
\]
Taking \( Y = U \) and \( Y = W \) to (4.1) and using (4.5)\(_2\) and (4.7)\(_2\), we get
\[
A_N X = \mu(X)U, \quad A_L X = \sigma(X)W.
\]
Using (4.7)\(_2\) and the fact that \( S(TM) \) is non-degenerate, (4.6)\(_1\) reduces
\[
A_N^\ast X = \mu(X)V.
\]
Substituting (4.8)\(_1\) into (3.14) and (4.8)\(_2\) into (3.16), and using the facts that \( \lambda = \rho = \alpha = \beta = 0 \) and \( FU = FW = 0 \), we have
\[
\nabla_X U = \tau(X)U, \quad \nabla_X W = 0.
\]
From (4.10)\(_2\), we see that \( W \) is parallel vector field with respect to \( \nabla \).
(5) From (4.10), we see that both $J(ltr(TM))$ and $J(S(TM^\perp))$ are parallel distributions on $M$ with respect to the connection $\nabla$, that is,
\[ \nabla_X U \in \Gamma(J(ltr(TM))), \quad \nabla_X W \in \Gamma(J(S(TM^\perp))). \]

On the other hand, taking $Y \in \Gamma(H)$ to (4.1), we have
\[ B(X, Y) = 0, \quad D(X, Y) = 0, \quad \forall X \in \Gamma(TM), \quad \forall Y \in \Gamma(H). \quad (4.11) \]
By straightforward calculations from (2.8), (2.10), (3.4), (3.15), (3.16), (4.7), (4.11) and the facts that $\lambda = 0$ and $FZ \in \Gamma(H_o)$ for $Z \in \Gamma(H_o)$, we have
\[ g(\nabla_X \xi, V) = -B(X, V) = 0, \quad g(\nabla_X \xi, W) = -\nu(X) = 0, \]
\[ g(\nabla_X V, Y) = 0, \quad g(\nabla_X V, W) = -\lambda(X) = 0, \]
\[ g(\nabla_X Z, V) = B(X, FZ) = 0, \quad g(\nabla_X Z, W) = D(X, FZ) = 0. \]
for all $X \in \Gamma(TM)$ and $Z \in \Gamma(H_o)$, or equivalently, we get
\[ \nabla_X Y \in \Gamma(H), \quad \forall X \in \Gamma(TM), \quad \forall Y \in \Gamma(H). \]
Thus $H$ is a parallel distribution on $M$ with respect to $\nabla$.

(6) As $J(ltr(TM))$, $J(S(TM^\perp))$ and $H$ are parallel distributions and satisfied (3.1), by the decomposition theorem of de Rham [3], $M$ is locally a product manifold $\mathcal{C}_u \times \mathcal{C}_w \times M^5$, where $\mathcal{C}_u$ is a null curve tangent to $J(ltr(TM))$, $\mathcal{C}_w$ is a spacelike curve tangent to $J(S(TM^\perp))$, and $M^5$ is a leaf of $H$. □

**Definition 3.** The structure tensor field $F$ of $M$ is said to be *Lie recurrent* [8] if there exists a smooth 1-form $\vartheta$ on $M$ such that
\[ (\mathcal{L}_X F) Y = \vartheta(X) F Y, \]
where $\mathcal{L}_X$ denotes the Lie derivative on $M$ with respect to $X$. The structure tensor field $F$ is called *Lie parallel* if $\mathcal{L}_X F = 0$.

**Theorem 4.2.** Let $M$ be a half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. If $F$ is Lie recurrent, then the following four statements are satisfied:

1. $F$ is Lie parallel,
2. $\alpha = 0$, i.e., $\bar{M}$ is not an indefinite Sasakian manifold,
3. the 1-forms $\vartheta$ and $\tau$ satisfy $d\vartheta = 0$ and $\tau = -\beta \vartheta$ on $M$,
4. the shape operator $A^*_F V$ satisfies
\[ A^*_F V = 0, \quad A^*_F U = 0. \]

**Proof.**

(1) As $(\mathcal{L}_X F)Y = [X, FY] - F[X, Y]$, using (3.9) and (3.17), we get
\[ \vartheta(X) FY = -\nabla_{FY} X + F \nabla_Y X - \vartheta(Y)\{X - \vartheta(X) \xi\} \]
\[ + u(Y) A_X X + w(Y) A_L X \]
\[ - \{B(X, Y) - \vartheta(Y) u(X)\} U - \{D(X, Y) - \vartheta(Y) w(X)\} W \]
\[ + \alpha\{g(X, Y) \zeta - \vartheta(Y) X\} + \beta\{\bar{g}(JX, Y) \zeta - \vartheta(Y) FX\}, \quad (4.12) \]
by (3.5). Taking $Y = \xi$ to this equation and using (2.9), we have
\[ -\vartheta(X)V = \nabla_YX + F\nabla_{\xi}X + \lambda(X)W + \beta u(X)\zeta. \]  
(4.13)

Taking the scalar product with $V$, $W$ and $\zeta$ to (4.13) by turns, we have
\[ u(\nabla_YX) = 0, \quad w(\nabla_YX) = -\lambda(X), \quad \vartheta(\nabla_YX) = -\beta u(X). \]  
(4.14)

Replacing $Y$ by $V$ to (4.12) and using the fact that $\vartheta(V) = 0$, we have
\[ \vartheta(X)\xi = -\nabla_{\xi}X + F\nabla_YX - B(X,V)U - D(X,V)W + \alpha u(X)\zeta. \]  
(4.15)

Applying $F$ to this equation and using (3.5) and (4.14), we obtain
\[ \vartheta(X)V = \nabla_YX + F\nabla_{\xi}X + \lambda(X)W + \beta u(X)\zeta. \]

Comparing this equation with (4.13), we get $\vartheta = 0$. Thus $F$ is Lie parallel.

(2) Taking the scalar product with $\zeta$ to (4.15) with $\vartheta = 0$, we have
\[ g(\nabla_{\xi}X, \zeta) = \alpha u(X). \]

Replacing $X$ by $U$ to this equation and using (3.14), we obtain $\alpha = 0$.

(3) Applying $\nabla_X$ to $\theta(\bar{Y}) = \bar{g}(\bar{Y}, \zeta)$ and using (1.1) and (2.3), we obtain
\[ d\theta(\bar{X}, \bar{Y}) = \alpha \bar{g}(\bar{X}, J\bar{Y}), \]

due to the fact $\bar{\nabla}$ is metric. As $\alpha = 0$, we see that $d\theta = 0$.

Taking $X = W$ to (4.12) and using (2.12), (3.5), (3.10) and (3.11), we get
\[ u(Y)A_{\bar{X}}W + w(Y)A_{\bar{X}}W - A_{\bar{X}}Y - F(A_{\bar{X}}FY) \]
\[ - \lambda(FY)U - \vartheta(Y)W = 0. \]  
(4.16)

Taking the scalar product with $N$ and using (2.11)$_2$ and (2.12)$_1,2$, we have
\[ D(FY, U) = w(Y)\rho(W) - \rho(Y). \]  
(4.17)

Replacing $Y$ by $V$ and using (2.9)$_2$, we get $\rho(V) = \lambda(U)$, while taking $X = U$ to (4.14)$_2$ and using (3.14), we have $\rho(V) = -\lambda(U)$. Thus, $\rho(V) = \lambda(U) = 0$.

Taking $\bar{Y} = \xi$ to (4.16), we have $A_{\bar{X}}\xi = F(A_{\bar{X}}V) + \lambda(V)U$. Multiplying this by $V$ and using (2.9), (2.12) and (3.11), we get $\lambda(V) = 0$. Therefore,
\[ \rho(V) = 0, \quad \lambda(U) = 0, \quad \lambda(V) = 0. \]  
(4.18)

Taking the scalar product with $N$ to (4.12) and using (2.12)$_2$, we have
\[ - \bar{g}(\nabla_{FY}X, N) + g(\nabla_YX, U) + w(Y)\rho(X) \]
\[ - \vartheta(Y)\{\eta(X) + \beta v(X)\} = 0. \]  
(4.19)

Replacing $X$ by $\xi$ to (4.19) and using (2.8) and (2.10)$_1,2$, we have
\[ B(X, U) + \vartheta(X) - w(X)\rho(\xi) = \tau(FX). \]  
(4.20)

Replacing $X$ by $U$ and using (3.13)$_1$ and the fact that $FU = 0$, we get
\[ C(U, V) = B(U, U) = 0. \]  
(4.21)
Replacing $X$ by $V$ to (4.19) and using (2.10), (3.15) and $\rho(V) = 0$, we have
\[ B(FX, U) + \tau(X) + \beta\theta(X) = 0. \]

Taking $X = U$, $X = W$ and $X = \zeta$ to this equation by turns, we get
\[ \tau(U) = 0, \quad \tau(W) = 0, \quad \tau(\zeta) = -\beta. \]  \hspace{1cm} (4.22)

Replacing $Y$ by $\xi$ to (4.17) and using (3.11), we obtain
\[ D(U, V) = \rho(\xi). \]  \hspace{1cm} (4.23)

Taking $X = U$ to (4.12) and using (2.11), (3.5) and (3.10) $\sim$ (3.14), we get
\[ u(Y)A_\xi U + w(Y)A_\zeta U - \theta(Y)U \]
\[ - F(A_\xi FY) - A_\xi Y - \tau(FY)U - \rho(FY)W = 0. \]  \hspace{1cm} (4.24)

Taking the scalar product with $V$ and using (3.13), (4.21) and (4.23), we get
\[ B(X, U) + \theta(X) - w(X)\rho(\xi) = -\tau(FX). \]

Comparing this equation with (4.20), we obtain $\tau(FX) = 0$. Replacing $X$ by $FY$ and using (3.5) and (4.22), we have $\tau = -\beta\theta$ on $M$.

(4) Replacing $Y$ by $W$ to (4.24) and using $FW = 0$, we have $A_\xi U = A_\zeta W$.

Taking the scalar product with $U$ and using (3.13)$_3$, we have
\[ C(W, U) = C(U, W). \]

Taking the scalar product with $W$ to (4.24), we have
\[ \rho(FY) = -C(Y, W) + u(Y)C(U, W) + w(Y)D(U, W). \]

Taking the scalar product with $U$ to (4.16) and using (3.13)$_3$, we have
\[ \rho(FY) = C(Y, W) - u(Y)C(U, W) - w(Y)D(U, W). \]

From the last two equations, we obtain $\rho(FY) = 0$. It follows that $\rho(\xi) = 0$.

As $\tau(X) = \beta\theta(X)$, we have $\tau(V) = \tau(\xi) = 0$. Taking $X = \xi$ to (4.13) and using (3.12), we obtain $A_\xi^* V = 0$. From (3.10) and (4.20), we have $B(U, X) = 0$, i.e., $g(A_\xi^* U, X) = 0$. As $S(TM)$ is non-degenerate, we obtain $A_\xi^* U = 0$. \hspace{1cm} \Box

5. Indefinite generalized Sasakian space forms

Definition 4. An indefinite trans-Sasakian manifold $(\tilde{M}, J, \zeta, \theta, \bar{g})$ is called an indefinite generalized Sasakian space form, denote it by $\tilde{M}(f_1, f_2, f_3)$, if there exist three smooth functions $f_1$, $f_2$ and $f_3$ on $\tilde{M}$ such that
\[
\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = f_1\{\bar{g}(\tilde{Y}, \tilde{Z})\tilde{X} - \bar{g}(\tilde{X}, \tilde{Z})\tilde{Y}\} \]
\[ + f_2\{\bar{g}(\tilde{X}, J\tilde{Z})\tilde{Y} - \bar{g}(\tilde{Y}, J\tilde{Z})\tilde{X} + 2\bar{g}(\tilde{X}, J\tilde{Y})J\tilde{Z}\} \]
\[ + f_3\{\theta(\tilde{X})\theta(\tilde{Z})\tilde{Y} - \theta(\tilde{Y})\theta(\tilde{Z})\tilde{X} \]
\[ + \bar{g}(\tilde{X}, \tilde{Z})\theta(\tilde{Y})\zeta - \bar{g}(\tilde{Y}, \tilde{Z})\theta(\tilde{X})\zeta\}, \]
where $\tilde{R}$ is the curvature tensor of the Levi-Civita connection $\tilde{\nabla}$ on $\tilde{M}$.
Remark 3. The notion of generalized Sasakian space form \(\bar{M}(f_1, f_2, f_3)\) was introduced by Alegre et al. [1]. Indefinite Sasakian, Kenmotsu and cosymplectic space forms are important kinds of generalized Sasakian space forms such that

\[
f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c+1}{4}; \quad f_1 = \frac{c-3}{4}, f_2 = f_3 = \frac{c+1}{4}; \quad f_1 = f_2 = f_3 = \frac{c}{4}
\]

respectively, where \(c\) is a constant \(J\)-sectional curvature of each space forms.

Let \(\bar{R}\) be the curvature tensor of the quarter-symmetric metric connection \(\nabla\) on \(\bar{M}\). By directed calculations from (1.1) and (1.2), we see that

\[
\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \bar{R}(\bar{X}, \bar{Y})\bar{Z} - \{(\nabla_X \theta)(Y) - (\nabla_Y \theta)(X)\}\bar{J}Z. \tag{5.2}
\]

Denote by \(R\) and \(R^*\) the curvature tensors of the induced connections \(\nabla\) and \(\nabla^*\) on \(M\) and \(S(TM)\) respectively. Using the local Gauss-Weingarten formulae, we have the Gauss-Codazzi equations for \(M\) and \(S(TM)\) such that

\[
\bar{R}(X, Y)Z = R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \tag{5.3}
+ D(X, Z)A_L Y - D(Y, Z)A_L X
+ \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)
+ \tau(X)B(Y, Z) - \tau(Y)B(X, Z)
+ \lambda(X)D(Y, Z) - \lambda(Y)D(X, Z)
- \theta(X)B(FY, Z) + \theta(Y)B(FX, Z)\}N,
+ \{(\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z)
+ \rho(X)B(Y, Z) - \rho(Y)B(X, Z)
- \theta(X)D(FY, Z) + \theta(Y)D(FX, Z)\}L,
\]

\[
R(X, Y)PZ = R^*(X, Y)PZ + C(X, PZ)A_\xi Y - C(Y, PZ)A_\xi X \tag{5.4}
+ \{(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ)
- \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ)
- \theta(X)C(FY, PZ) + \theta(Y)C(FX, PZ)\}\xi.
\]

\[
R(X, Y)\xi = -\nabla^*_X (A^*_\xi Y) + \nabla^*_Y (A^*_\xi X) + A^*_\xi [X, Y]
- \tau(X)A^*_\xi Y + \tau(Y)A^*_\xi X
+ \{C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X, Y)\}\xi. \tag{5.5}
\]

Comparing the tangential and lightlike transversal components of two equations of (5.3) and (5.2) and using (3.4), we obtain

\[
R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} \tag{5.6}
+ f_2\{\bar{g}(X, JZ)FY - \bar{g}(Y, JZ)FX + 2\bar{g}(X, JY)FZ\}
+ f_3\{[\theta(X)Y - \theta(Y)X]\theta(Z) + [g(X, Z)\theta(Y) - g(Y, Z)\theta(X)]\zeta\}
- \{(\nabla_X \theta)(Y) - (\nabla_Y \theta)(X)\}\bar{J}Z
\]
Proof. Applying (2.11) Then \( \alpha \) Sasakian space form \( \bar{\nabla} \theta \) (3.4), (3.7), we have

\[
\begin{align*}
(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z) \\
+ \lambda(X)D(Y, Z) - \lambda(Y)D(X, Z) - \theta(X)B(FY, Z) + \theta(Y)B(FX, Z) \\
+ \{(\bar{\nabla}_X \theta)(Y) - (\bar{\nabla}_Y \theta)(X)\}u(Z)
= f_2\{u(Y)\bar{g}(X, JZ) - u(X)\bar{g}(Y, JZ) + 2u(Z)\bar{g}(X, JY)\},
\end{align*}
\]

Taking the scalar product with \( N \) to (5.3) and then, substituting (5.4) and (5.2) into the left and right terms and using (2.12)_4, we obtain

\[
\begin{align*}
(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) - \tau(X)C(Y, PZ) \\
+ \tau(Y)C(X, PZ) - \rho(X)D(Y, PZ) + \rho(Y)D(X, PZ) \\
- \theta(X)C(FY, PZ) + \theta(Y)C(FX, PZ) \\
+ \{(\bar{\nabla}_X \theta)(Y) - (\bar{\nabla}_Y \theta)(X)\}v(PZ)
= f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\
+ f_2\{v(Y)\bar{g}(X, JPZ) - v(X)\bar{g}(Y, JPZ) + 2v(PZ)\bar{g}(X, JY)\} \\
+ f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ).
\end{align*}
\]

**Theorem 5.1.** Let \( M \) be a half lightlike submanifold of an indefinite generalized Sasakian space form \( M(f_1, f_2, f_3) \) with a quarter-symmetric metric connection. Then \( \alpha, \beta, f_1, f_2 \) and \( f_3 \) satisfy \( \beta = 0, \alpha \) is a constant on \( M \) and

\[
f_1 - f_2 = \alpha^2, \quad f_1 - f_3 = \alpha(\alpha + 1).
\]

**Proof.** Applying \( \nabla_Y \) to (3.13)\_1: \( B(X, U) = C(X, V) \) and using (2.1), (2.10)\_1, 2, (2.11)\_1, 2, (3.4), (3.7)\_1, (3.8), (3.14) and (3.15), we have

\[
(\nabla_X B)(Y, U)
= (\nabla_X C)(Y, V) - 2\tau(X)C(Y, V) - \lambda(X)C(Y, W) - \rho(X)B(Y, W) \\
- \alpha^2 u(Y)\eta(X) - \beta^2 u(X)\eta(Y) + \alpha\beta\{u(X)v(Y) - u(Y)v(X)\} \\
- g(A^X_C, F(A^X_C, V)) - g(A^X_C, F(A^X_C, X)).
\]

Substituting this equation into (5.7) with \( Z = U \) and using (3.13)\_2, 3, we get

\[
\begin{align*}
(\nabla_X C)(Y, V) - (\nabla_Y C)(X, V) - \tau(X)C(Y, V) \\
+ \tau(Y)C(X, V) - \rho(X)D(Y, V) + \rho(Y)D(X, V) \\
- \theta(X)C(FY, V) + \theta(Y)C(FX, V) \\
+ (\bar{\nabla}_X \theta)(Y) - (\bar{\nabla}_Y \theta)(X) \\
+ (\alpha^2 - \beta^2)\{u(Y)\eta(X) - u(X)\eta(Y)\} \\
+ 2\alpha\beta\{u(X)v(Y) - u(Y)v(X)\}
= f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}.
\end{align*}
\]

Comparing this equation with (5.8) such that \( PZ = V \), we obtain

\[
\{f_1 - f_2 - \alpha^2 + \beta^2\}u(Y)\eta(X) - u(X)\eta(Y)\]
\[
= 2\alpha\beta\{u(Y)v(X) - u(X)v(Y)\}.
\]
Applying $\nabla_X$ to $\eta(Y) = \bar{g}(Y, N)$ and using (2.4) and (2.5) we have

\[(\nabla_X \eta)(Y) = -g(A_{\alpha}X, Y) + \tau(X)\eta(Y).\]

Applying $\nabla_Y$ to (3.8) and using (2.11), (3.6), (3.8), (3.19) and $\alpha\beta = 0$, we have

\[(\nabla_X C)(Y, \zeta) = -(X\alpha)v(Y) + (\alpha\beta)\eta(Y) + \alpha^2\theta(Y)\eta(X) + \beta^2\theta(X)\eta(Y)
+ \alpha\{g(A_{\alpha}X, FY) + g(A_{\alpha}Y, FX) - v(Y)\tau(X) - w(Y)\rho(X)\}
- \beta\{g(A_{\alpha}X, Y) + g(A_{\alpha}Y, X) - \tau(X)\eta(Y)\}.

Substituting this equation and (3.8) into (5.8) such that $PZ = \zeta$, we get

\[
\begin{aligned}
&\{X\beta + [f_1 - f_3 - (\alpha^2 - \beta^2) - \alpha\theta(X)]\eta(Y)
- \{Y\beta + [f_1 - f_3 - (\alpha^2 - \beta^2) - \alpha\theta(Y)]\eta(X)
= \{X\alpha + \beta\theta(X)\}v(Y) - \{Y\alpha + \beta\theta(Y)\}v(X).
\end{aligned}
\]

Taking $X = \zeta$, $Y = \xi$ and $X = U$, $Y = V$ to this by turns, we obtain

\[f_1 - f_3 = (\alpha^2 - \beta^2) + \alpha - \zeta \beta, \quad U\alpha = 0.\]

Applying $\nabla_Y$ to (3.7) and using (3.6) and (3.18), we have

\[(\nabla_X B)(Y, \zeta) = -(X\alpha)v(Y) - \beta B(Y, X)
+ \alpha\{u(Y)\tau(X) + w(Y)\lambda(X) + B(X, FY) + B(Y, FX)\}.\]

Substituting this into (5.7) such that $Z = \zeta$ and using (3.7) and (3.10), we get

\[
\{X\alpha + \beta\theta(X)\}u(Y) = \{Y\alpha + \beta\theta(X)\}u(X).
\]

Taking $Y = U$ and using the fact that $U\alpha = 0$, we have $X\alpha + \beta\theta(X) = 0$.

Assume that $\beta \neq 0$. Then $X\alpha \neq 0$ due to $X\alpha = -\beta\theta(X)$. Applying $\nabla_X$ to $\alpha\beta = 0$ and using the fact that $X\alpha = -\beta\theta(X)$, we obtain

\[\alpha X\beta = \beta^2\theta(X).\]

Multiplying $\beta$ to this result, we get $\beta = 0$. It is a contradiction to $\beta \neq 0$. Thus $\beta = 0$. Therefore, $\alpha$ is a constant, $f_1 - f_2 = \alpha^2$ and $f_1 - f_3 = \alpha(\alpha + 1).$  

**Definition 5.** (1) A screen distribution $S(TM)$ is called **totally umbilical** [5] in $M$ if there exists smooth function $\gamma$ such that $A_{\alpha} = \gamma P$, or equivalently,

\[C(X, PY) = \gamma g(X, Y).\]

In case $\gamma = 0$, we say that $S(TM)$ is **totally geodesic** in $M$.

(2) A lightlike submanifold $M$ is called **screen conformal** [6] if there exists non-vanishing smooth function $\varphi$ on $U$ such that $A_{\alpha} = \varphi A_{\alpha}^*$, or equivalently,

\[C(X, PY) = \varphi B(X, PY).\]  

**Theorem 5.2.** Let $M$ be a half lightlike submanifold of $\bar{M}(f_1, f_2, f_3)$ with a quarter-symmetric metric connection. If one of the following four statements
(1) $F$ is recurrent,
(2) $F$ is Lie recurrent,
(3) $S(TM)$ is totally umbilical,
(4) $M$ is screen conformal,

is satisfied, then $M(f_1, f_2, f_3)$ is a flat manifold with an indefinite cosymplectic structure. In case (1), $M$ is also flat. In case (3), $S(TM)$ is totally geodesic.

Proof. (1) By Theorem 4.1, we get (4.8), (4.10) and the results: $\alpha = \beta = 0$ and $\lambda = \rho = 0$. Since $\alpha = \beta = 0$, we have $f_1 = f_2 = f_3$ by Theorem 5.1.

Taking the scalar product with $U$ to (4.8)$_{1, 2}$, we get

$$C(X, U) = 0, \quad D(X, U) = 0.$$  

Applying $\nabla_X$ to $C(Y, U) = 0$ and using (4.10)$_1$, we obtain

$$(\nabla_X C)(Y, U) = 0.$$  

Substituting the last equations into (5.8) with $PZ = U$, we have

$$(f_1 + f_2)\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = V$ and $Y = \xi$ to this result, we obtain $f_1 + f_2 = 0$. Therefore, we see that $f_1 = f_2 = f_3 = 0$. Thus $\bar{M}(f_1, f_2, f_3)$ is flat.

As $f_1 = f_2 = f_3 = 0$, (5.6) is reduced to

$$R(X, Y)Z = B(Y, Z)A_N X - B(X, Z)A_N Y$$
$$+ D(Y, Z)A_L X - D(X, Z)A_L Y.$$

Using this, (2.10), (2.12), (4.8), (4.12) and the fact that $\lambda = 0$, we obtain

$$R(X, Y)Z = \{\mu(Y)\mu(X) - \mu(X)\mu(Y)\}u(Z)U$$
$$+ \{\sigma(Y)\sigma(X) - \sigma(X)\sigma(Y)\}w(Z)W = 0,$$

for all $X, Y, Z \in \Gamma(TM)$. Therefore $R = 0$ and $M$ is also flat.

(2) By Theorem 4.2 and 5.1, we get $\alpha = 0$ and $\beta = 0$. Thus $\bar{M}$ is an indefinite cosymplectic manifold. Since $\alpha = 0$, we have $f_1 = f_2 = f_3$ by Theorem 5.1. Also, since $\beta = 0$, by (3) of Theorem 4.2, we see that $\tau = 0$. Taking the scalar product with $N$ to (5.6) with $Z = \xi$ and then, comparing this result with the radical component of (5.5) and using (2.9) and (2.12), we have

$$C(Y, A^*_N X) - C(X, A^*_N Y)$$
$$= f_2\{u(Y)v(X) - u(X)v(Y)\} + \lambda(X)\rho(Y) - \lambda(Y)\rho(X).$$

Taking $X = U$ and $Y = V$ to this and using (4.18) and the result (4) in Theorem 4.2, we get $f_2 = 0$. Thus $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat.

(3) Assume that $S(TM)$ is totally umbilical. Then (3.8) is reduced to

$$\gamma\theta(X) = -\alpha v(X) + \beta \eta(X).$$

Replacing $X$ by $V$, $\xi$ and $\zeta$ to this equation by turns, we have $\alpha = 0$, $\beta = 0$ and $\gamma = 0$ respectively. Since $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold. As $\gamma = 0$, $S(TM)$ is totally geodesic.
As \( \alpha = 0, f_1 = f_2 = f_3 \) by Theorem 5.1. Taking \( PZ = U \) to (5.8) with \( C = 0 \) and using the facts that \( D(X, Uk) = C(X, W) = 0 \), we get
\[
(f_1 + f_2)\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.
\]
Taking \( X = \xi \) and \( Y = V \) to this equation, we get \( f_1 + f_2 = 0 \). Thus \( f_1 = f_2 = f_3 = 0 \) and \( M(f_1, f_2, f_3) \) is flat.

(4) Replacing \( Y \) by \( \zeta \) to (5.9) and using (3.7)_1 and (3.8), we have
\[
\alpha v(X) - \beta \eta(X) = \alpha \varphi u(X).
\]
Taking \( X = V \) and \( X = \xi \) to this equation by turns, we obtain \( \alpha = 0 \) and \( \beta = 0 \) respectively. As \( \alpha = \beta = 0 \), \( M \) is an indefinite cosymplectic manifold. Since \( \alpha = 0 \), we have \( f_1 = f_2 = f_3 \) by Theorem 5.1.

Applying \( \nabla_X \) to \( C(Y, PZ) = \varphi B(Y, PZ) \), we have
\[
(\nabla_X C)(Y, PZ) = (X \varphi)B(Y, PZ) + \varphi(\nabla_X B(Y, PZ)).
\]
Substituting this equation into (5.8) and using (5.7), we have
\[
\{X \varphi - 2 \varphi \tau(X)\}B(Y, PZ) - \{Y \varphi - 2 \varphi \tau(Y)\}B(X, PZ)
\]
\[
- \{\rho(X) + \varphi \lambda(X)\}D(Y, PZ) + \{\rho(Y) + \varphi \lambda(Y)\}D(X, PZ)
\]
\[
+ \{(\nabla_X \theta)(Y) - (\nabla_Y \theta)(X)\}g(\omega, PZ)
\]
\[
= f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}
\]
\[
+ f_2\{g(\omega, Y)\bar{g}(X, JPZ) - g(\omega, X)\bar{g}(Y, JPZ) + 2g(\omega, PZ)\bar{g}(X, JY)
\]
\[
+ f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ).
\]
where \( \omega = U - \varphi \). From (3.13)_1 and (5.9); (3.13)_2,3 and (5.9), we get
\[
B(X, \omega) = 0, \quad D(X, \omega) = 0.
\]
Applying \( \nabla_X \) to \( \theta(\xi) = 0 \) and \( \theta(V) = 0 \) by turns and using (2.4), (2.8), (2.10), (3.15) and the fact that \( \alpha = \beta = 0 \), we have
\[
(\nabla_X \theta)(\xi) = B(X, \xi) = 0, \quad (\nabla_X \theta)(V) = \beta \varphi u(X) = 0.
\]
Replacing \( PZ \) by \( \omega \) to (5.10) and using (5.11), we obtain
\[
- 2 \varphi\{(\nabla_X \theta)(Y) - (\nabla_Y \theta)(X)\}
\]
\[
= (f_1 + f_2)\{g(\omega, Y)\eta(X) - g(\omega, X)\eta(Y)\} - 4\varphi f_2 \bar{g}(X, JY)\}
\]
Taking \( X = \xi \) and \( Y = V \) to this equation and using (5.12), we get \( f_1 + f_2 = 0 \). Therefore, \( f_1 = f_2 = f_3 = 0 \) and \( M(f_1, f_2, f_3) \) is flat. \( \square \)

References


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