GRADED INTEGRAL DOMAINS AND PRÜFER-LIKE DOMAINS

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Abstract. Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be an integral domain graded by an arbitrary torsionless grading monoid $\Gamma$, $\bar{R}$ be the integral closure of $R$, $H$ be the set of nonzero homogeneous elements of $R$, $C(f)$ be the fractional ideal of $R$ generated by the homogeneous components of $f \in RH$, and $N(H) = \{f \in R \mid C(f)_+ = R\}$. Let $RH$ be a UFD. We say that a nonzero prime ideal $Q$ of $R$ is an upper to zero in $R$ if $Q = fRH \cap R$ for some $f \in R$ and that $R$ is a graded UMT-domain if each upper to zero in $R$ is a maximal $t$-ideal. In this paper, we study several ring-theoretic properties of graded UMT-domains. Among other things, we prove that if $R$ has a unit of nonzero degree, then $R$ is a graded UMT-domain if and only if every prime ideal of $R_{N(H)}$ is extended from a homogeneous ideal of $R$, if and only if $R_{N(H)}$ is a Prüfer domain, if and only if $R$ is a UMT-domain.

0. Introduction

Prüfer-$v$-multiplication domains (PeMD) are one of the most important research topics in “Multiplicative Ideal Theory” because many essential non-Noetherian integral domains (e.g., Krull domains, Prüfer domains, GCD domains) are PeMDs and an integral domain $D$ is a PeMD if and only if $D[X]$, the polynomial ring over $D$, is a PeMD. It is known that $D$ is a PeMD if and only if $D$ is an integrally closed UMT-domain; hence UMT-domains can be considered as non-integrally closed PeMDs. UMT-domains were introduced by Houston and Zafrullah [34] and studied in greater detail by Fontana, Gabelli, and Houston [26] and Chang and Fontana [17]. In this paper, we study UMT-domain properties of graded integral domains.

This section consists of three subsections. In Section 0.1, we review the definitions related to the $t$-operation and in Section 0.2, we review those of

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graded integral domains; so the reader who is familiar with these two notions can skip to Section 0.3 where we give the motivation and results of this paper.

0.1. The $t$-operation

Let $D$ be an integral domain with quotient field $K$. An overring of $D$ means a subring of $K$ containing $D$. Let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of $D$. For $I \in \mathbf{F}(D)$, let $I^{-1} = \{ x \in K \mid xI \subseteq D \}$, $I_v = (I^{-1})^{-1}$, and $I_t = \bigcup \{ J_v \mid J \in \mathbf{F}(D) \text{ is finitely generated and } J \subseteq I \}$. An $I \in \mathbf{F}(D)$ is called a $t$-ideal (resp., $v$-ideal) if $I_t = I$ (resp., $I_v = I$). A $t$-ideal (resp., $v$-ideal) is a maximal $t$-ideal (resp., maximal $v$-ideal) if it is maximal among proper integral $t$-ideals (resp., $v$-ideals). Let $\text{t-Max}(D)$ (resp., $\text{v-Max}(D)$) be the set of maximal $t$-ideals (resp., $v$-ideals) of $D$. It may happen that $v$-$\text{Max}(D) = \emptyset$ even though $D$ is not a field as in the case of a rank-one nondiscrete valuation domain $D$.

However, it is well known that $t$-$\text{Max}(D) \neq \emptyset$ if $D$ is not a field; each maximal $t$-ideal is a prime ideal; each proper $t$-ideal is contained in a maximal $t$-ideal; each prime ideal minimal over a $t$-ideal is a $t$-ideal; and $D = \bigcap_{P \in \text{t-Max}(D)} D_P$.

We mean by $t$-$\dim(D) = 1$ that $D$ is not a field and each prime $t$-ideal of $D$ is a maximal $t$-ideal of $D$. Clearly, if $\dim(D) = 1$ (i.e., $D$ is one-dimensional), then $t$-$\dim(D) = 1$.

An $I \in \mathbf{F}(D)$ is said to be $t$-invertible if $(II^{-1})_t = D$, and $D$ is a Prüfer $v$-multiplication domain (PvMD) if each nonzero finitely generated ideal of $D$ is $t$-invertible. Let $T(D)$ (resp., $\text{Prin}(D)$) be the group of $t$-invertible fractional $t$-ideals (resp., nonzero principal fractional ideals) of $D$ under the $t$-multiplication $I \ast J = (IJ)_t$. It is obvious that $\text{Prin}(D) \subseteq T(D)$. The $t$-class group of $D$ is the abelian group $C(D) = T(D)/\text{Prin}(D)$. It is clear that if $D$ is a Krull domain (resp., Prüfer domain), then $C(D)$ is the divisor class (resp., ideal class) group of $D$. Let $\{ D_\alpha \}$ be a set of integral domains such that $D = \bigcap_\alpha D_\alpha$. We say that the intersection $D = \bigcap_\alpha D_\alpha$ is locally finite if each nonzero nonunit of $D$ is a unit of $D_\alpha$ for all but a finite number of $D_\alpha$.

Let $\{ X_\alpha \}$ be a nonempty set of indeterminates over $D$, $D[\{ X_\alpha \}]$ be the polynomial ring over $D$, and $\text{cp}(f)$ (simply $c(f)$) be the fractional ideal of $D$ generated by the coefficients of a polynomial $f \in K[\{ X_\alpha \}]$. It is known that if $I$ is a nonzero fractional ideal of $D$, then $(ID[\{ X_\alpha \}])_t = I^{-1}D[\{ X_\alpha \}]$, $(ID[\{ X_\alpha \}])_v = I_vD[\{ X_\alpha \}]$, and $(ID[\{ X_\alpha \}])_t = I_tD[\{ X_\alpha \}]$ [32, Lemma 4.1 and Proposition 4.3]; so $I$ is a (prime) $t$-ideal of $D$ if and only if $ID[\{ X_\alpha \}]$ is a (prime) $t$-ideal of $D[\{ X_\alpha \}]$.

0.2. Graded integral domains

Let $\Gamma$ be a (nonzero) torsionless grading monoid, that is, $\Gamma$ is a torsionless commutative cancellative monoid (written additively), and $\langle \Gamma \rangle = \{ a-b \mid a, b \in \Gamma \}$ be the quotient group of $\Gamma$; so $\langle \Gamma \rangle$ is a torsionfree abelian group. It is well known that a cancellative monoid $\Gamma$ is torsionless if and only if $\Gamma$ can be given a total order compatible with the monoid operation [39, page 123]. By a $(\Gamma-)
graded integral domain \( R = \bigoplus_{a \in \Gamma} R_a \), we mean an integral domain graded by \( \Gamma \). That is, each nonzero \( x \in R_a \) has degree \( \alpha \), i.e., \( \deg(x) = \alpha \), and \( \deg(0) = 0 \). Thus, each nonzero \( f \in R \) can be written uniquely as \( f = x_{\alpha_1} + \cdots + x_{\alpha_n} \) with \( \deg(x_{\alpha_n}) = \alpha_1 \) and \( \alpha_1 < \cdots < \alpha_n \). Since \( R \) is an integral domain, we may assume that \( R_\alpha \neq \{0\} \) for all \( \alpha \in \Gamma \).

A nonzero \( x \in R_\alpha \) for every \( \alpha \in \Gamma \) is said to be homogeneous. Let \( H \) be the saturated multiplicative set of nonzero homogeneous elements of \( R \), i.e., \( H = \bigcup_{\alpha \in \Gamma} (R_\alpha \setminus \{0\}) \). Then \( R_H \), called the homogeneous quotient field of \( R \), is a graded integral domain whose nonzero homogeneous elements are units. Hence, \( R_H \) is a completely integrally closed GCD-domain \([1, \text{Proposition 2.1]}\) and \( R_H \) is a \((\Gamma)\)-graded integral domain. We say that an overring \( T \) of \( R \) is a homogeneous overring of \( R \) if \( T = \bigoplus_{\alpha \in \Gamma} (T \cap (R_H)_\alpha) \); so \( T \) is a \((\Gamma)\)-graded integral domain such that \( R \subseteq T \subseteq R_H \). Clearly, if \( \Lambda = \{ \alpha \in \Gamma \mid T \cap (R_H)_\alpha \neq \{0\} \} \), then \( \Lambda \) is a torsionless grading monoid such that \( \Gamma \subseteq \Lambda \subseteq (\Gamma) \) and \( T = \bigoplus_{\alpha \in \Lambda} (T \cap (R_H)_\alpha) \). The integral closure of \( R \) is a homogeneous overring of \( R \) by Lemma 1.6. Also, \( R_S \) is a homogeneous overring of \( R \) for a multiplicative set \( S \) of nonzero homogeneous elements of \( R \) (with \( \deg(\frac{1}{x}) = \deg(a) - \deg(b) \) for \( a \in H \) and \( b \in S \)).

For a fractional ideal \( A \) of \( R \) with \( A \subseteq R_H \), let \( A^* \) be the fractional ideal of \( R \) generated by homogeneous elements in \( A \). It is easy to see that \( A^* \subseteq A \); and if \( A \) is a prime ideal, then \( A^* \) is a prime ideal. The \( A \) is said to be homogeneous if \( A^* = A \). A homogeneous ideal (resp., homogeneous t-ideal) of \( R \) is called a homogeneous maximal ideal (resp., homogeneous maximal t-ideal) if it is maximal among proper homogeneous ideals (resp., homogeneous t-ideals) of \( R \). It is known that a homogeneous maximal ideal need not be a maximal ideal, while a homogeneous maximal t-ideal is a maximal t-ideal \([8, \text{Lemma 2.1]}\). Also, it is easy to see that each proper homogeneous ideal (resp., homogeneous t-ideal) of \( R \) is contained in a homogeneous maximal ideal (resp., homogeneous maximal t-ideal) of \( R \).

For \( f \in R_H \), let \( C_R(f) \) denote the fractional ideal of \( R \) generated by the homogeneous components of \( f \). For a fractional ideal \( I \) of \( R \) with \( I \subseteq R_H \), let \( C_R(I) = \sum_{f \in I} C_R(f) \). It is clear that both \( C_R(f) \) and \( C_R(I) \) are homogeneous fractional ideals of \( R \). If there is no confusion, we write \( C(f) \) and \( C(I) \) instead of \( C_R(f) \) and \( C_R(I) \). Let \( N(H) = \{ f \in R \mid C(f)_\alpha = R \} \) and \( S(H) = \{ f \in R \mid C(f) = R \} \). It is well known that if \( f, g \in R_H \), then \( C(f)^n C(g) = C(f)^a C(fg) \) for some integer \( n \geq 1 \) \([39]\); so \( N(H) \) and \( S(H) \) are saturated multiplicative subsets of \( R \) and \( S(H) \subseteq N(H) \). Let \( \Omega \) be the set of maximal t-ideals \( Q \) of \( R \) with \( Q \cap H \neq \emptyset \), i.e., \( \Omega = \{ Q \in t\text{-Max}(R) \mid Q \text{ is homogeneous} \} \) \([8, \text{Lemma 2.1]}\). As in \([9]\), we say that \( R \) satisfies property (\#) if \( C(I)_H = R \) implies \( I \cap N(H) \neq \emptyset \) for all nonzero ideals \( I \) of \( R \); equivalently, \( \text{Max}(R_{N(H)}) = \{ Q_{N(H)} \mid Q \in \Omega \} \) \([9, \text{Proposition 1.4]}\). It is known that \( R \) satisfies property (\#) if \( R \) is one of the following integral domains: (i) \( R \) contains a unit of nonzero degree, (ii) \( R = D[\Gamma] \) is the monoid domain of \( \Gamma \) over an integral domain \( D \), (iii)
$R$ contains a homogeneous prime element of nonzero degree, (iv) $R = D[[X]]$ is the polynomial ring over $D$, or (v) the intersection $\bigcap_{Q \in \Omega} R_Q$ is locally finite [9, Example 1.6 and Lemma 2.2].

We say that $R$ is a graded-Prüfer domain if each nonzero finitely generated homogeneous ideal of $R$ is invertible. Clearly, invertible ideals are $t$-invertible, and hence a graded-Prüfer domain is a PeMD [1, Theorem 6.4] but need not be a Prüfer domain [9, Example 3.6]. The reader can refer to [10] or [42] for more on graded-Prüfer domains.

0.3. Motivation and results

Let $X$ be an indeterminate over $D$ and $D[X]$ be the polynomial ring over $D$. A nonzero prime ideal $Q$ of $D[X]$ is called an upper to zero in $D[X]$ if $Q \cap D = (0)$. We say that $D$ is a UMT-domain if each upper to zero in $D[X]$ is a maximal $t$-ideal of $D[X]$. (UMT stands for Upper to zero is a Maximal $T$-ideal.) A quasi-Prüfer domain is a UMT-domain in which every maximal ideal is a $t$-ideal; equivalently, its integral closure is a Prüfer domain [25, Chapter VI]. The most important properties of UMT-domains are that (i) $D$ is a UMT-domain if and only if every prime ideal of $D[X]_{N_v}$, where $N_v = \{ f \in D[X] \mid c(f)_v = D \}$, is extended from $D$ and (ii) $D$ is an integrally closed UMT-domain if and only if $D$ is a PeMD [34, Theorem 3.1 and Proposition 3.2]. A subring $D[X^2, X^3] = D + X^2D[X]$ of $D[X]$ over a PeMD $D$ is an easy example of a non-integrally closed UMT-domain. In many cases, UMT-domains are used like: $D[X]$ (or $D[X]_{N_v}$) has a ring-theoretic property $(P)$ if and only if $D$ is a UMT-domain with property $(P)$. For example, $t$-dim$(D[X]) = 1$ if and only if $D$ is a UMT-domain with $t$-dim$(D) = 1$; and $D[X]_{N_v}$ is a pseudo-valuation domain if and only if $D$ is a pseudo-valuation UMT-domain [13, Lemma 3.7]. (A quasi-local domain $D$ with maximal ideal $M$ is a pseudo-valuation domain if and only if $D$ has a unique valuation overring with maximal ideal $M$ [31, Theorem 2.7].) For more results on UMT-domains, see, for example, [22,23,41,44] including a survey article [33].

Clearly, $Q$ is an upper to zero in $D[X]$ if and only if $Q = fK[X] \cap D[X]$ for some prime element $0 \neq f \in K[X]$, if and only if either $Q = XD[X]$ or $Q = fK[X, X^{-1}] \cap D[X]$ for some prime element $0 \neq f \in K[X]$. Note that $D[X] = \bigoplus_{n \geq 0} DX^n$ is an $N_0$-graded integral domain, where $N_0$ is the additive monoid of nonnegative integers, and if $H$ is the set of nonzero homogeneous elements of $D[X]$, then $D[X]_H = K[X, X^{-1}]$ and $K[X, X^{-1}]$ is a unique factorization domain (UFD). In [19, Section 2], the notion of “upper to zero” was generalized to graded integral domains as follows: Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a (nontrivial) graded integral domain graded by an arbitrary torsionless grading monoid $\Gamma$ and $H$ be the set of nonzero homogeneous elements of $R$. Assume that $R_H$ is a UFD. Then a nonzero prime ideal $Q$ of $R$ is called an upper to zero in $R$ if $Q = fR_H \cap R$ for some $f \in R_H$. Thus, $Q$ is an upper to zero in $D[X]$ as the original definition if and only if either $Q = XD[X]$ or $Q$ is an upper to zero in
$D[X]$ as a prime ideal of the $\mathbb{N}_0$-graded integral domain $D[X] = \bigoplus_{n \geq 0} DX^n$.

As a graded integral domain analog, in [19, Theorem 2.5], it was shown that if $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded integral domain with a unit of nonzero degree such that $R_H$ is a UFD, then $R$ is a Prüfer domain if and only if $R$ is integrally closed and each upper to zero in $R$ is a maximal $t$-ideal. In this paper, we further study some ring-theoretic properties of graded integral domains $R$ such that $R_H$ is a UFD and each upper to zero in $R$ is a maximal $t$-ideal.

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a $\Gamma$-graded integral domain. In Section 1, we introduce the notion of graded UMT-domains, and we then study general properties of both UMT-domains and graded UMT-domains. For example, we prove that UMT-domains are graded UMT-domains, and $R$ is a graded UMT-domain if and only if $Q$ is homogeneous for all nonzero prime ideals $Q$ of $R$ with $C(Q) \not\subseteq R$, and $R$ is an graded UMT-domain if and only if every prime ideal of $R_{N[H]}$ is extended from a homogeneous ideal of $R$, and $R$ is a weakly Krull domain if and only if $R_{N[H]}$ is a weakly Krull domain. We study in Section 3 graded UMT-domains with a unit of nonzero degree. Among other things, we prove that if $R$ has a unit of nonzero degree, then $R$ is a graded UMT-domain if and only if $R$ is a UMT-domain, if and only if the integral closure of $R_{N[H]}$ is a graded-Prüfer domain for all homogeneous maximal $t$-ideals $Q$ of $R$, if and only if the integral closure of $R_{N[H]}$ is a Prüfer domain. Finally, in Section 4, we use the $D + XK[X]$ construction to give several counterexamples of the results in Sections 2 and 3. Assume that $D \subseteq K$, and let $R = D + XK[X]:= \{ f \in K[X] \mid f(0) \in D \}$. Then $R$ is an $\mathbb{N}_0$-graded integral domain such that $R_H = K[X, X^{-1}]$ is a UFD. We show that $R$ is a graded UMT-domain, and $R$ is a UMT-domain if and only if $D$ is a UMT-domain. Thus, if $D$ is not a UMT-domain, then $R = D + XK[X]$ is a graded UMT-domain but not a UMT-domain. We also give examples which show that the results of Section 3 do not hold without assuming that $R$ has a unit of nonzero degree.

1. UMT-domains and graded UMT-domains

Let $\Gamma$ be a nonzero torsionless grading monoid, $\langle \Gamma \rangle = \{ a - b \mid a, b \in \Gamma \}$ be the quotient group of $\Gamma$, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a nontrivial $\Gamma$-graded integral domain, and $H$ be the set of nonzero homogeneous elements of $R$. Throughout this paper, $R_H$ is always assumed to be a UFD.

We begin this section with examples of graded integral domains $R$ such that $R_H$ is a UFD.

Example 1.1. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then $R_H$ is a UFD if one of the following conditions is satisfied.

1. [7, Proposition 3.5] $\langle \Gamma \rangle$ satisfies the ascending chain condition on its cyclic subgroups.
2. $R = D[[X_\alpha]]$ is the polynomial ring over an integral domain $D$. 
(3) [38, Section A.1.4.] \( (\Gamma) = \mathbb{Z} \) is the additive group of integers.

(4) \( R = D[\Gamma] \) is the monoid domain of \( \Gamma \) over \( D \) such that \( (\Gamma) \) satisfies the ascending chain condition on its cyclic subgroups.

Let \( \bar{D} \) be the integral closure of an integral domain \( D \). For easy reference, we recall from [37, Theorem 44] that (i) (Lying Over) if \( P \) is a prime ideal of \( D \), then there is a prime ideal \( Q \) of \( \bar{D} \) with \( Q \cap D = P \); (ii) (Going Up) if \( P_1 \subseteq P_2 \) are prime ideals of \( D \) and \( Q_1 \) is a prime ideal of \( \bar{D} \) with \( Q_1 \cap D = P_1 \), then there exists a prime ideal \( Q_2 \) of \( \bar{D} \) such that \( Q_1 \subseteq Q_2 \) and \( Q_2 \cap D = P_2 \); and (iii) (Incomparable) if \( P \subseteq Q \) are prime ideals of \( \bar{D} \) with \( P \cap D = Q \cap D \), then \( P = Q \).

The next result appears in [26, Theorem 1.5], but we include it because our proof is easy and direct without using other results.

**Theorem 1.2.** An integral domain \( D \) is a UMT-domain if and only if the integral closure of \( D_P \) is a Prüfer domain for all \( P \in t\text{-Max}(D) \).

**Proof.** Let \( \bar{D} \) be the integral closure of \( D \). Hence, \( \bar{D}_P \) is the integral closure of \( D_P \) for a prime ideal \( P \) of \( D \).

\((\Rightarrow)\) Assume that \( \bar{D}_P \) is not a Prüfer domain for some \( P \in t\text{-Max}(D) \), and let \( T = \bar{D}_P \). Then there are some \( 0 \neq a, b \in T \) such that \((a, b)T \) is not invertible, and so if we let \( f = a + bX \), then \( fK[X] \cap T[X] = fct(f)^{-1}[X] \subseteq (ct(f)ct(f)^{-1})[X] \cap M[X] \) for some maximal ideal \( M \) of \( T \) (the first equality follows from \([28, Corollary 34.9]\) because \( T \) is integrally closed). Thus, \( fK[X] \cap D[X] = (fK[X] \cap T[X]) \cap D[X] \subseteq (M[X] \cap D_P[X]) \cap D[X] = P[X] \). Clearly, \( fK[X] \cap D[X] \) is an upper to zero in \( D[X] \), but \( fK[X] \cap D[X] \) is not a maximal \( t \)-ideal, a contradiction.

\((\Leftarrow)\) Assume that \( D \) is not a UMT-domain. Then there are a maximal \( t \)-ideal \( P \) of \( D \) and an upper to zero \( Q \) in \( D[X] \) such that \( Q \subseteq P[X] \) [cf. [34, Proposition 1.1]]. Since \( Q \) is an upper to zero in \( D[X] \), there is an \( f \in D[X] \) such that \( Q = fK[X] \cap D[X] \). Note that \( Q_f := fK[X] \cap \bar{D}_P \) is an upper to zero in \( \bar{D}_P[X], Q_f \cap D_P[X] = Q_{D_P}[X] \), and \( \bar{D}_P[X] \) is integral over \( D_P[X] \). Thus, there is a prime ideal \( M \) of \( D_P[X] \) such that \( Q_f \subseteq M \) and \( M \cap D_P[X] = PD_P[X] \). Clearly, \( M = (M \cap D_P)_P[X] \) because \( (M \cap D_P)[X] \cap D_P[X] = PD_P[X] \) and \( (M \cap D_P)[X] \subseteq M \). However, since \( \bar{D}_P \) is a Prüfer domain, there is a \( g \in Q_f \) such that \( \bar{D}_P = \langle g \rangle \bar{D}_P \subseteq M \cap \bar{D}_P \), a contradiction. \( \square \)

Bezout domains are Prüfer domains. Hence, if \( \bar{D}_P \) is a Bezout domain for all \( P \in t\text{-Max}(D) \), then \( D \) is a UMT-domain by Theorem 1.2. In [13, Lemma 2.2], it was shown that \( D \) is a UMT-domain if and only if the integral closure of \( D_P \) is a Bezout domain for all \( P \in t\text{-Max}(D) \). Theorem 1.2 also shows that \( D_S \) is a UMT-domain for every multiplicative set \( S \) of a UMT-domain \( D \).

**Corollary 1.3** ([34, Proposition 3.2]). \( D \) is a PrMD if and only if \( D \) is an integrally closed UMT-domain.
Proof. It is well known that \( D \) is a \( \text{Prüfer-like domain} \) if and only if \( D_P \) is a valuation domain for all \( P \in \text{Max}(D) \) [30, Theorem 5] and \( D = \bigcap_{P \in \text{Max}(D)} D_P \). Hence, the result follows directly from Theorem 1.2.

Recall that \( D \) is an \( S\)-domain if \( \text{ht}(PD[X]) = 1 \) for every prime ideal \( P \) of \( D \) with \( \text{ht} P = 1 \) [37, p. 26]. It is easy to see that a UMT-domain is an \( S\)-domain; and if \( \text{t-dim}(D) = 1 \) (e.g., \( \text{dim}(D) = 1 \)), then \( D \) is an \( S\)-domain if and only if \( D \) is a UMT-domain (cf. [43, Theorem 8]). However, \( S\)-domains need not be UMT-domains. For example, if \( D = \mathbb{R} + (X,Y)\mathbb{C}[X,Y] \), where \( \mathbb{C}[X,Y] \) is the power series ring over the field \( \mathbb{C} \) of complex numbers and \( \mathbb{R} \) is the field of real numbers, then \( D \) is a 2-dimensional Noetherian domain [12, Theorem 4 and Corollary 9] whose maximal ideal is a \( t\)-ideal. Hence, \( D \) is an \( S\)-domain [37, Theorem 148] but not a UMT-domain [34, Theorem 3.7].

We next introduce the notion of graded UMT-domains.

**Definition 1.4.** Let \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \), and assume that \( R_H \) is a UFD.

1. A nonzero prime ideal \( Q \) of \( R \) is an upper to zero in \( R \) if \( Q = fR_H \cap R \) for some \( f \in R_H \). (In this case, \( f \) is a nonzero prime element of \( R_H \) and \( Q \) is a height-one prime \( t\)-ideal of \( R \).)

2. \( R \) is a graded UMT-domain if every upper to zero in \( R \) is a maximal \( t\)-ideal of \( R \).

Recall that if \( Q \) is a maximal \( t\)-ideal of \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \) with \( Q \cap H \neq \emptyset \), then \( Q \) is homogeneous [8, Lemma 2.1]. We use this result without further citation.

**Lemma 1.5.** Let \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \) be a graded UMT-domain and \( Q \) be a nonzero prime ideal of \( R \). Then \( Q \) is a maximal \( t\)-ideal of \( R \) if and only if either \( Q \) is an upper to zero in \( R \) or \( Q \) is a homogeneous maximal \( t\)-ideal.

**Proof.** Let \( Q \) be a maximal \( t\)-ideal of \( R \). If \( Q \cap H \neq \emptyset \), then \( Q \) is homogeneous. Next, assume that \( Q \cap H = \emptyset \). Then \( Q = Q_H \cap R \), and hence \( Q \) contains an upper to zero in \( R \). Thus, \( Q \) must be an upper to zero in \( R \) because \( R \) is a graded UMT-domain. The converse is clear.

We say that \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \) is a gr-valuation ring if \( x \in R \) or \( \frac{1}{x} \in R \) for all nonzero homogeneous elements \( x \in R_H \). It is known that if \( R \) is a gr-valuation ring, then there is a valuation overring \( V \) of \( R \) such that \( V \cap R_H = R \) [35, Theorem 2.3]. Hence, a gr-valuation ring is integrally closed.

**Lemma 1.6.** Let \( \bar{R} \) be the integral closure of \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \). Then \( \bar{R} \) is a homogeneous overring of \( R \).

**Proof.** Let \( \{V_\lambda\} \) be the set of all homogeneous gr-valuation overrings of \( R \). Then \( \bar{R} = \bigcap_\lambda V_\lambda \) [35, Theorem 2.10], and since each \( V_\lambda \) is a homogeneous overring of \( R \), \( \bar{R} \) is also a homogeneous overring of \( R \).

We next show that a UMT-domain is a graded UMT-domain, while a graded UMT-domain need not be a UMT-domain (see Example 4.3).
Proposition 1.7. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a UMT-domain. Then $R$ is a graded UMT-domain.

Proof. Let $Q'$ be a prime $t$-ideal of $R$ such that $Q' \cap H = \emptyset$. Then $Q'_H$ is a $t$-ideal of $R_H$ [26, Proposition 1.4], and hence $\text{ht}Q' = \text{ht}(Q'_H) = 1$ because $R_H$ is a UFD.

Let $U_f = fR_H \cap R$ be an upper to zero in $R$. If $U_f$ is not a maximal $t$-ideal of $R$, there is a maximal $t$-ideal $Q$ of $R$ such that $U_f \subseteq Q$. By the above paragraph, $Q \cap H \neq \emptyset$, and thus $Q$ is homogeneous. Note that $U = fR_H \cap R$ is a prime ideal of $R$ and $U \cap R = U_f$; so there is a prime ideal $M$ of $R$ such that $U \subseteq M$ and $M \cap R = Q$. However, note that $R$ is a graded integral domain by Lemma 1.6; so $M^*$ is a prime ideal of $\bar{R}$ and $M^* \cap R = Q$. Hence, $M^* = M$, and since $U = fC_R(f)^{-1}$ [9, Lemma 1.2(4)], $C_R(f)C_R(f)^{-1} \subseteq M$. By Theorem 1.2, $R_M = (R_Q)_{M_0}$ is a valuation domain, and hence $R_M = (C_R(f)_{M})C_R(f)^{-1}_{M} = (C_R(f)_{M})(C_R(f)^{-1}_{M}) \subseteq M_M$, a contradiction. Thus, $U_f$ is a maximal $t$-ideal of $R$. \hfill \square

Let $D[X]$ be the polynomial ring over an integral domain $D$, and let $Q$ be an upper to zero in $D[X]$. It is known that $Q$ is a maximal $t$-ideal if and only if $c(Q)_t = D$, if and only if $Q$ is $t$-invertible [34, Theorem 1.4] (see [27, Theorem 3.3] for the case of arbitrary sets of indeterminates). This was extended to graded integral domains $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ in [8, Corollary 2.2(2)] as follows: If $Q$ is an upper to zero in $R$, then $C(Q)_t = R$ if and only if $Q$ is $t$-invertible, if and only if $Q$ is a maximal $t$-ideal. We next generalize [8, Corollary 2.2(2)] to prime $t$-ideals $Q$ of $R$ with $Q \cap H = \emptyset$.

Proposition 1.8. Let $Q$ be a prime $t$-ideal of $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ such that $Q \cap H = \emptyset$. Then the following statements are equivalent.

1. $C(Q)_t = R$.
2. $Q$ is $t$-invertible.
3. $Q$ is a maximal $t$-ideal.

In this case, $\text{ht}Q = 1$, and hence $Q$ is an upper to zero in $R$.

Proof. (1) $\Rightarrow$ (2) Since $C(Q)_t = R$, there are some $f_1, \ldots, f_k \in Q$ such that $(C(f_1) + \cdots + C(f_k))_v = R$. Assume that $\text{ht}Q \geq 2$. Since $R_H$ is a UFD, there is a $g \in Q$ such that $gR_H$ is a prime ideal and $f_1 \notin gR_H$. Clearly, $((f_1, \ldots, f_k, g)R_H)_v = R_H$, and hence if $u \in (f_1, \ldots, f_k, g)^{-1}$, then $u \in R_H$. Also, since $(C(f_1) + \cdots + C(f_k))_v = R, u \in R$. Thus, $R = (f_1, \ldots, f_k, g)^{-1} = (f_1, \ldots, f_k, g)_v \subseteq Q_t = Q \subseteq R$, a contradiction. Hence, $\text{ht}Q = 1$, and so $Q$ is an upper to zero in $R$. Thus, $Q$ is $t$-invertible [8, Corollary 2.2(2)].

(2) $\Rightarrow$ (3) [34, Theorem 1.4].

(3) $\Rightarrow$ (1) Note that $Q \subseteq C(Q)_t \subseteq R$ and $C(Q)_t$ is a $t$-ideal. Hence, if $Q$ is a maximal $t$-ideal, then $C(Q)_t = R$. \hfill \square

Corollary 1.9. Each homogeneous prime $t$-ideal of $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ has height-one if and only if $t\text{-dim}(R) = 1$. In this case, $R$ is a graded UMT-domain.
The following statements are equivalent for Theorem 1.11.

It is known that $S$ where $dD$ if for each 0

Assume that each homogeneous prime t-ideal of $R$ has height-one. Thus, $htQ = 1$ by Proposition 1.8. The converse is clear.

The “In this case” part follows because $t$-dim$(R) = 1$ implies that each prime $t$-ideal of $R$ is a maximal $t$-ideal.

Let $A \subseteq B$ be an extension of integral domains. As in [23], we say that $B$ is $t$-linked over $A$ if $I^{-1} = A$ for a nonzero finitely generated ideal $I$ of $A$ implies $(IB)^{-1} = B$. It is easy to see that $B$ is $t$-linked over $A$ if and only if $B = \bigcap_{P \in \text{Max}(A)} BP$ [14, Lemma 3.2], if and only if either $Q \cap A = (0)$ or $Q \cap A \neq (0)$ and $(Q \cap A)_t \subseteq A$ for all $Q \in t$-Max$(B)$ [4, Propositions 2.1].

**Corollary 1.10.** Let $T$ be a homogeneous overing of $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$, and assume that $T$ is $t$-linked over $R$ (e.g., $T = R_S$ for some multiplicative set $S \subseteq H$). If $R$ is a graded UMT-domain, then $T$ is a graded UMT-domain.

**Proof.** Let $U$ be an upper to zero in $T$. If $U$ is not a maximal $t$-ideal, then $C_T(U)_t \subsetneq T$ by Proposition 1.8. Hence, there is a homogeneous maximal $t$-ideal $Q$ of $T$ such that $U \subsetneq Q$. Note that $U \cap R$ is an upper to zero in $R$, $Q \cap R$ is homogeneous, $(Q \cap R)_t \subsetneq R$ because $T$ is $t$-linked over $R$, and $U \cap R \subseteq Q \cap R$. Thus, $U \cap R \subseteq (Q \cap R)_t$, a contradiction because $U \cap R$ is a maximal $t$-ideal by assumption. Hence, $U$ is a maximal $t$-ideal of $T$.

Following [3], we say that a multiplicative subset $S$ of $D$ is a $t$-splitting set if for each $0 \neq d \in D$, $dD = (AB)_t$ for some integral ideals $A$ and $B$ of $D$, where $A_t \cap sD = sA_t$ (equivalently, $(A, s)_t = D$) for all $s \in S$ and $B_t \cap S \neq \emptyset$. It is known that $S$ is a $t$-splitting set of $D$ if and only if $dd_S \cap D$ is $t$-invertible for all $0 \neq d \in D$ [3, Corollary 2.3]. Also, $D$ is a UMT-domain if and only if $D - \{0\}$ is a $t$-splitting set in $D[X]$ [16, Corollary 2.9].

**Theorem 1.11.** The following statements are equivalent for $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$.

1. $R$ is a graded UMT-domain.
2. Let $Q$ be a nonzero prime ideal of $R$ such that $C(Q)_t \subsetneq R$. Then $Q$ is homogeneous.
3. Let $Q$ be a nonzero prime ideal of $R$ such that $Q \subsetneq M$ for some homogeneous maximal $t$-ideal $M$ of $R$. Then $Q$ is homogeneous.
4. $C(Q)_t = R$ for every upper to zero $Q$ in $R$.
5. If $I = fR_H \cap R$ for $0 \neq f \in R$, then $C(I)_t = R$.
6. $H$ is a $t$-splitting set of $R$.
7. Every prime $t$-ideal of $R$ disjoint from $H$ is $t$-invertible.
8. Every prime $t$-ideal of $R$ disjoint from $H$ is a maximal $t$-ideal.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $Q$ is not homogeneous. Clearly, there is an $f \in Q \setminus H$ such that $C(f) \subsetneq Q$. Let $P$ be a prime ideal of $R$ such that $P$ is minimal over $fR$ and $P \subseteq Q$. If $P \cap H \neq \emptyset$, then $PR_{H \setminus P}$ must be a homogeneous
maximal t-ideal of $R_H \cdot P$ (cf. [8, Lemma 2.1]); so $P$ is homogeneous. Hence, $C(f) \subseteq P \subseteq Q$, a contradiction. Thus, $P \cap H = \emptyset$ and $PR_H$ is a prime t-ideal because $PR_H$ is minimal over $fR_H$, whence $P$ is an upper to zero in $R$. Thus, $P = Q$ by (1), and so $C(Q) = R$ by Proposition 1.8, a contradiction. Thus, $Q$ is homogeneous.

(2) $\iff$ (3) Clear.

(2) $\implies$ (4) Let $Q$ be an upper to zero in $R$. Then $Q$ is not homogeneous and $Q \subseteq C(Q)$. However, if $C(Q) \subseteq R$, then $Q$ is homogeneous by (2), a contradiction. Thus, $C(Q) = R$.

(4) $\implies$ (1) Proposition 1.8.

(1) $\implies$ (5) Let $f = f_1^{e_1} \cdots f_n^{e_n}$ be the prime factorization of $f$ in $R_H$, where $f_i \in R_H$ is a prime element. Then

$$I = (f_1^{e_1} \cdots f_n^{e_n})R_H \cap R,$$

$$= (f_1^{e_1}R_H \cap R) \cdots (f_n^{e_n}R_H \cap R)$$

$$= (f_1^{e_1}R_H \cap R) \cap \cdots \cap (f_n^{e_n}R_H \cap R)$$

$$= ((f_1R_H \cap R)^{e_1}) \cap \cdots \cap ((f_nR_H \cap R)^{e_n}).$$

(For the last equality, note that each $f_iR_H \cap R$ is a maximal t-ideal by (1) and $\sqrt{(f_iR_H \cap R)^{e_i}} = f_iR_H \cap R = f_iR_H \cap R = \sqrt{((f_iR_H \cap R)^{e_i})}$; so $((f_iR_H \cap R)^{e_i})$ is primary. Clearly, $((f_iR_H \cap R)^{e_i}) \cap R_H = f_i^{e_i}R_H$, and thus $((f_iR_H \cap R)^{e_i}) \cap R_H = f_i^{e_i}R_H \cap R_H$.) If $C(I) \subseteq R$, then $I \subseteq C(I) \subseteq M$ for some homogeneous maximal t-ideal $M$ of $R$. Since $M$ is a prime ideal, $f_iR_H \cap R \subseteq M$ for some $i$, and hence $R = C(f_iR_H \cap R) \subseteq C(M) = M$ by the equivalence of (1) and (4) above, a contradiction. Thus, $C(I) = R$.

(5) $\implies$ (1) Let $Q$ be an upper to zero in $R$. Then $Q = fR_H \cap R$ for some $f \in R$, and hence $C(Q) = R$ by (5). Thus, $Q$ is a maximal t-ideal by Proposition 1.8.

(1) $\implies$ (6) Let $Q$ be a prime t-ideal of $R$ such that $Q \cap H = \emptyset$. Then $Q_H$ is a prime ideal of $R_H$, and hence $fR_H \subseteq Q_H$ for some nonzero prime element $f$ of $R_H$. Hence, $fR_H \cap R \subseteq Q$, and since $fR_H \cap R$ is a maximal t-ideal of $R$ by (1), $Q = fR_H \cap R$ and $C(Q) = R$. Thus, $H$ is a t-splitting set [8, Theorem 2.1].

(6) $\implies$ (4) Let $Q$ be an upper to zero in $R$. Then $Q$ is a prime t-ideal of $R$ with $Q \cap H = \emptyset$, and thus $C(Q) = R$ [8, Theorem 2.1].

(6) $\iff$ (7) [8, Corollary 2.2].

(7) $\iff$ (8) Proposition 1.8.

Let $D[X]$ be the polynomial ring over an integral domain $D$ and $f \in D[X]$ be such that $c(f)_v = D$. If $A$ is an ideal of $D[X]$ with $f \in A$, then $A$ is $t$-invertible [34, Proposition 4.1] and $fD[X] = (Q_1^{e_1} \cdots Q_n^{e_n})_v$, for some uppers to zero $Q_i$ in $D[X]$ and integers $e_i \geq 1$ [29, p. 144]. We end this section with an extension of these results to graded integral domains.

**Proposition 1.12.** Let $A$ be a nonzero ideal of $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ such that $C(A)_v = R$. If $A$ contains a nonzero $f \in R$ with $C(f)_v = R$ (e.g., $R$ satisfies
property (#)), then \( A_t = (Q_1^{e_1} \cdots Q_n^{e_n})_t \) for some \( t \)-invertible uppers to zero \( Q_i \) in \( R \) and integers \( e_i \geq 1 \). In particular, \( A \) is \( t \)-invertible.

**Proof.** If \( A_t = R \), then \( A \) is \( t \)-invertible; so assume that \( A_t \subseteq R \). Let \( Q \) be a maximal \( t \)-ideal of \( R \) with \( A \subseteq Q \); then \( f \in Q \). If \( Q \cap H \neq \emptyset \), then \( Q \) is homogeneous, and hence \( R = C(A)_t \subseteq Q_t = Q \), a contradiction. Hence, \( Q \cap H = \emptyset \), and so \( Q \) contains an upper to zero \( U \) in \( R \) containing \( f \). Clearly, \( C(U)_t = R \); so by Proposition 1.8, \( U \) is a maximal \( t \)-ideal, and thus \( Q = U \), i.e., \( Q \) is an upper to zero in \( R \) that is \( t \)-invertible. Hence, each prime \( t \)-ideal of \( R \) containing \( A \) is an upper to zero in \( R \) that is also \( t \)-invertible. Thus, \( A_t = (Q_1^{e_1} \cdots Q_n^{e_n})_t \) for some uppers to zero \( Q_i \) in \( R \) and integers \( e_i \geq 1 \) (cf. the proof of [29, Theorem 1.3]) and \( A \) is \( t \)-invertible. \( \square \)

**Corollary 1.13.** Let \( f \in R = \bigoplus_{a \in I} R_a \) be nonzero. If \( C(f)_v = R \), then \( fR = (Q_1^{e_1} \cdots Q_n^{e_n})_t \), for some uppers to zero \( Q_i \) in \( R \) and integers \( e_i \geq 1 \).

**Proof.** Clearly, \( C(f)_R \) is \( R \) and \( f \in fR \). Thus, the result is an immediate consequence of Proposition 1.12. \( \square \)

A careful reading of the proof of Proposition 1.12 also shows:

**Corollary 1.14.** Let \( A \) be a nonzero ideal of a graded UMT-domain \( R = \bigoplus_{a \in I} R_a \) such that \( C(A)_t = R \). Then \( A_t = (Q_1^{e_1} \cdots Q_n^{e_n})_t \), for some uppers to zero \( Q_i \) in \( R \) and integers \( e_i \geq 1 \), and \( A \) is \( t \)-invertible.

Let \( D \) be an integral domain, \( S \) be a \( t \)-splitting set of \( D \), \( \mathfrak{S} = \{ A_1 \cdots A_n \} \) \( A_i = d_iD_S \cap D \) for some \( 0 \neq d_i \in D \), and \( D_{\mathfrak{S}} = \{ x \in K \mid xA \subseteq D \text{ for some } A \in \mathfrak{S} \} \). Then \( D_{\mathfrak{S}} = \cap \{ D_P \mid P \in \text{t-}\text{Max}(D) \text{ and } P \cap S \neq \emptyset \} \) [3, Lemma 4.2 and Theorem 4.3]. The \( S \) is said to be \( t \)-lcm if \( sD \cap dD \) is \( t \)-invertible for all \( s \in S \) and \( 0 \neq d \in D \); and \( S \) is called a \( t \)-complemented \( t \)-splitting set if \( D_{\mathfrak{S}} = D_T \) for some multiplicative set \( T \) of \( D \) and the saturation of \( T \) is the \( t \)-complement of \( S \).

**Corollary 1.15** (cf. [16, Proposition 3.7]). Let \( R = \bigoplus_{a \in I} R_a \) and \( N(H) = \{ f \in R \mid C(f)_v = R \} \). Then \( N(H) \) is a \( t \)-lcm \( t \)-complemented \( t \)-splitting set of \( R \).

**Proof.** Let \( 0 \neq f \in R \) and \( A = fR_{N(H)} \cap R \). For the \( t \)-splitting set property of \( N(H) \), it suffices to show that \( A \) is \( t \)-invertible [3, Corollary 2.3]. Let \( Q \) be a maximal \( t \)-ideal of \( R \). If \( Q \cap N(H) \neq \emptyset \), then \( A_Q = fR_Q \). Next, assume that \( Q \cap N(H) = \emptyset \). Then \( C(Q)_t = R \), and hence \( Q \) is an upper to zero in \( R \) and \( R_Q \) is a rank-one DVR by Proposition 1.8. Now, note that if \( Q' \) is an upper to zero in \( R \) containing \( A \), then \( f \in Q'_R \) and \( Q'_R \) is a height-one prime ideal of \( R_H \); so there are only finitely many uppers to zero in \( R \) containing \( A \), say \( Q_1, \ldots, Q_n \). Hence, if \( S = R \setminus \bigcup_{i=1}^n Q_i \), then \( R_S \) is a principal ideal domain, and thus \( AR_S = gR_S \) for some \( g \in A \). Let \( I = (f,g)_v \). Then \( IR_Q = fR_Q \) when \( Q \cap N(H) = \emptyset \), and \( IR_Q = gR_Q \) when \( Q \cap N(H) \neq \emptyset \). Thus, \( I = A \) [36, Proposition 2.8(3)]; so \( A \) is \( t \)-invertible [36, Corollary 2.7].
Next, note that every $t$-ideal of $R$ intersecting $N(H)$ is $t$-invertible by Proposition 1.12. Thus, $N(H)$ is a $t$-lcm $t$-splitting set [16, Theorem 3.4]. Also, if $\mathcal{S} = \{A_1 \cdots A_n | A_i = d_iR_{N(H)} \cap R \text{ for some } 0 \neq d_i \in R\}$, then $R_{H} \leq R_{\mathcal{S}}$ because $aR_{N(H)} \cap R = aR$ for all $a \in H$. Hence, $R_{\mathcal{S}}$ is $t$-linked over $R_{H}$ [4, Proposition 2.3], and since $R_{H}$ is a UFD, $R_{\mathcal{S}} = (R_{H})T$ for some saturated multiplicative set $T$ of $R_{H}$ [24, Theorem 1.3]. Thus, if $N = T \cap R$, then $R_{\mathcal{S}} = R_N$. □

An integral domain is called a Mori domain if it satisfies the ascending chain condition on its (integral) $v$-ideals. Clearly, Krull domains are Mori domains.

**Corollary 1.16.** Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ and $N(H) = \{f \in R \mid C(f)_v = R\}$. Then $R$ is a Mori domain (resp., UMT-domain) if and only if $R_{N(H)}$ is a Mori domain (resp., UMT-domain).

**Proof.** By Corollary 1.15, $N(H)$ is a $t$-lcm $t$-complemented $t$-splitting set of $R$. Let $N$ be the $t$-complement of $N(H)$; then $R_{H} \subseteq R_{N}$, and hence $R_{N}$ is a UFD and $R = R_{N(H)} \cap R_{N}$. Thus, $R_{N(H)}$ is a Mori domain if and only if $R$ is a Mori domain [40, Theorem 1]. The UMT-domain property follows directly from [16, Corollary 3.6] and Corollary 1.15. □

### 2. Graded integral domains with property (#)

Let $\Gamma$ be a nonzero torsionless grading monoid, $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a nontrivial $\Gamma$-graded integral domain, $H$ be the set of nonzero homogeneous elements of $R$, and $N(H) = \{f \in R \mid C(f)_v = R\}$. Let $\Omega$ be the set of all homogeneous maximal $t$-ideals of $R$, i.e., $\Omega = \{Q \in t\text{-Max}(R) \mid Q \cap H \neq \emptyset\}$, and recall that $R$ satisfies property (#) if and only if $\text{Max}(R_{N(H)}) = \{Q_{N(H)} \mid Q \in \Omega\}$ [9, Proposition 1.4].

**Lemma 2.1.** Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain with property (#), and let $Q$ be an upper to zero in $R$.

1. $Q$ is a maximal $t$-ideal if and only if $C(g)_v = R$ for some $g \in Q$.
2. If $Q$ is a maximal $t$-ideal of $R$, then $Q = (f,g)_v$ for some $f,g \in R$.

**Proof.** (1) $Q$ is a maximal $t$-ideal if and only if $C(Q)_t = R$ by Proposition 1.8, if and only if $Q \cap N(H) \neq \emptyset$ by property (#).

(2) Since $Q$ is an upper to zero in $R$, there is an $f \in R$ such that $Q = fR_{H} \cap R$. Also, there is a $g \in Q$ with $C(g)_v = R$ by (1). Clearly, $(f,g)_v \subseteq Q$. For the reverse containment, let $h \in Q$. Then $ah \in fR$ for some $a \in H$, and thus $h(\alpha) \subseteq (f,g)$. Hence, $h(\alpha)g \subseteq (f,g)_v \subseteq Q$. If $\xi \in (\alpha,g)^{-1}$, then $\alpha \in H$ implies $\xi \in R_{H}$, and since $C(g)_v = R$, $\xi g \in R$ implies $\xi \in R$. Hence, $(\alpha,g)^{-1} = R$, and thus $hR = h(\alpha)g \subseteq (f,g)_v$. Thus, $Q \subseteq (f,g)_v$. □

We next give a characterization of graded UMT-domains $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ with property (#).
Theorem 2.2. Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain with property (\#). Then the following statements are equivalent.

1. $R$ is a graded UMT-domain.
2. If $Q$ is an upper to zero in $R$, then there is an $f \in Q$ such that $C(f)_{\alpha} = R$.
3. Every prime ideal of $R_{N(H)}$ is extended from a homogeneous ideal of $R$.
4. $N(H)$ is a $t$-lcm $t$-complemented $t$-splitting set of $R$ with $t$-complement $H$.

Proof. (1) $\Leftrightarrow$ (2) This follows directly from Lemma 2.1.

(1) $\Rightarrow$ (3) Let $Q'$ be a nonzero prime ideal of $R_{N(H)}$. Then $Q' = Q_{N(H)}$ for some nonzero prime ideal $Q$ of $R$. Note that $Q \subseteq M$ for some homogeneous maximal $t$-ideal $M$ of $R$ because $R$ satisfies property (\#). Thus, $Q$ is homogeneous by Theorem 1.11.

(3) $\Rightarrow$ (1) Let $Q$ be an upper to zero in $R$, and assume that $Q$ is not a maximal $t$-ideal of $R$. Then $Q \cap N(H) = \emptyset$ by Lemma 2.1(1), and so $Q_{N(H)}$ is a proper ideal of $R_{N(H)}$. Hence, by (3), there is a homogeneous ideal $P$ of $R$ such that $Q_{N(H)} = PR_{N(H)}$. Thus, $P \subseteq PR_{N(H)} \cap R = Q_{N(H)} \cap R = Q$, and so $Q_H = R_H$, a contradiction. Thus, $Q$ is a maximal $t$-ideal of $R$.

(1) $\Rightarrow$ (4) By Corollary 1.15, $N(H)$ is a $t$-lcm $t$-complemented $t$-splitting set of $R$. Also, note that $\{Q \in t\text{-Max}(R) \mid Q \cap N(H) \neq \emptyset\}$ is the set of uppers to zero in $R$ by property (\#) and assumption; so $R_H = R_{\emptyset} = \emptyset$, where $\emptyset = \{A_1, \ldots, A_n \mid A_i = d_i R_{N(H)} \cap R$ for some $0 \neq d_i \in R\}$. Thus, $H$ is the $t$-complement of $N(H)$.

(4) $\Rightarrow$ (1) Let $Q$ be an upper to zero in $R$. Then $Q \cap H = \emptyset$, and hence $Q \cap N(H) \neq \emptyset$ [3, Theorem 4.3] because $H$ is the $t$-complement of $N(H)$. Thus, $Q$ is a maximal $t$-ideal of $R$ by Proposition 1.8.

The next result is an immediate consequence of Corollary 1.9, but we use Theorem 2.2 to give another proof.

Corollary 2.3. Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain with property (\#). Then $t\text{-dim}(R) = 1$ if and only if $\dim(R_{N(H)}) = 1$. In this case, $R$ is a graded UMT-domain.

Proof. Assume $t\text{-dim}(R) = 1$, and note that $\text{Max}(R_{N(H)}) = \{Q_{N(H)} \mid Q \in \Omega\}$. Thus, $\dim(R_{N(H)}) = 1$. Conversely, suppose $\dim(R_{N(H)}) = 1$, and let $Q$ be a maximal $t$-ideal of $R$. If $Q \cap H \neq \emptyset$, then $Q$ is homogeneous, and thus $\text{ht}Q = \text{ht}(Q_{N(H)}) = 1$. Next, if $Q \cap H = \emptyset$, then $Q_H \subseteq R_H$, and hence $Q$ contains an upper to zero $Q_0$ in $R$. However, note that since $R$ satisfies property (\#), $\dim(R_{N(H)}) = 1$ implies $(Q_0)_{N(H)} = R_{N(H)}$. Thus, $Q_0 \cap N(H) \neq \emptyset$, and so $Q_0$ is a maximal $t$-ideal of $R$ by Lemma 2.1. Hence, $Q = Q_0$ and $\text{ht}Q = 1$.

For “In this case”, note that $\dim(R_{N(H)}) = 1$ implies that every prime ideal of $R_{N(H)}$ is extended from a homogeneous ideal of $R$. Thus, $R$ is a graded UMT-domain by Theorem 2.2.
An integral domain $D$ is called an almost Dedekind domain (resp., $t$-almost Dedekind domain) if $D_P$ is a rank-one DVR for all maximal ideals (resp., maximal $t$-ideals) $P$ of $D$. Clearly, Dedekind domains are almost Dedekind domains; Krull domains are $t$-almost Dedekind domains; and if $D$ is an almost (resp., a $t$-almost) Dedekind domain, then $\dim(D) = 1$ (resp., $t$-$\dim(D) = 1$).

**Corollary 2.4** (cf. [20, Corollary 9]). Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with property $(\#)$. Then $R$ is a $t$-almost Dedekind domain if and only if $R_{N(H)}$ is an almost Dedekind domain.

**Proof.** $(\Rightarrow)$ By Corollary 2.3, $\dim(R_{N(H)}) = 1$. Note that $\Max(R_{N(H)}) = \{Q_{N(H)} \mid Q \in \Omega\}$ and $R_Q$ is a rank-one DVR for all $Q \in \Omega$. Thus, $R_{N(H)}$ is an almost Dedekind domain.

$(\Leftarrow)$ If $R_{N(H)}$ is an almost Dedekind domain, then $\dim(R_{N(H)}) = 1$, and thus $t$-$\dim(R) = 1$ by Corollary 2.3. Let $Q$ be a maximal $t$-ideal of $R$. If $Q \cap H = \emptyset$, then $ht(Q_H) = htQ = 1$, and since $R_H$ is a UFD, $R_Q$ is a rank-one DVR. Next, if $Q \cap H \neq \emptyset$, then $Q$ is homogeneous, and hence $Q_{N(H)} \subsetneq R_{N(H)}$. Thus, $R_Q$ is a rank-one DVR by assumption.\qed

An integral domain $D$ is called a weakly Krull domain if (i) $D = \bigcap_{P \in X^1(D)} D_P$, where $X^1(D)$ is the set of height-one prime ideals of $D$, and (ii) the intersection $D = \bigcap_{P \in X^1(D)} D_P$ is locally finite. It is easy to see that if $D$ is a weakly Krull domain, then $t$-$\dim(D) = 1$, i.e., $X^1(D) = t$-$\Max(D)$, and $D_S$ is a weakly Krull domain for a multiplicative set $S$ of $D$. Also, $D$ is a Krull domain if and only if $D$ is a weakly Krull domain and $D_P$ is a rank-one DVR for all $P \in X^1(D)$.

**Corollary 2.5.** The following statements are equivalent for $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$.

1. $R$ is a weakly Krull domain.
2. $R$ is a graded UMT-domain and $R_{N(H)}$ is a weakly Krull domain.
3. $R_{N(H)}$ is a weakly Krull domain.
4. $R_{N(H)}$ is an one-dimensional weakly Krull domain.

**Proof.** Note that $R_{N(H)}$ is a weakly Krull domain in this corollary. Also, $Q_{N(H)}$ is a prime $t$-ideal of $R_{N(H)}$ for all $Q \in \Omega$ [9, Proposition 1.3]. Hence, the intersection $\bigcap_{Q \in \Omega} R_Q$ is locally finite, and thus $R$ satisfies property $(\#)$ [9, Lemma 2.2].

(1) $\Rightarrow$ (2) If $R$ is a weakly Krull domain, then $t$-$\dim(R) = 1$, and hence $R$ is a graded UMT-domain by Corollary 2.3. Also, since $N(H)$ is a multiplicative subset of $R$, $R_{N(H)}$ is a weakly Krull domain.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (4) If $R_{N(H)}$ is a weakly Krull domain, then $ht(Q_{N(H)}) = 1$ for all $Q \in \Omega$. Thus, $t$-$\dim(R_{N(H)}) = 1$ because $R$ satisfies property $(\#)$.

(4) $\Rightarrow$ (1) By Corollary 2.3, $t$-$\dim(R) = 1$, and thus $R = \bigcap_{Q \in X^1(R)} R_Q$. Next, let $f \in R$ be a nonzero nonunit. Since $R_{N(H)}$ is a weakly Krull domain, $f$ is contained in only finitely many homogeneous maximal $t$-ideals of $R$. Also,
since \( R_H \) is a UFD, \( f \) is contained in only finitely many uppers to zero in \( R \). Therefore, \( R \) is a weakly Krull domain.

It is clear that \( D \) is a Krull domain if and only if \( D \) is a \( t \)-almost Dedekind weakly Krull domain and that a Krull domain \( D \) is a Dedekind domain if and only if \( \dim(D) = 1 \). Hence, by Corollaries 2.4 and 2.5, we have:

**Corollary 2.6** ([9, Corollary 2.4]). Let \( R = \bigoplus_{\alpha \in \Gamma} R_{\alpha} \). Then \( R \) is a Krull domain if and only if \( R_{N(H)} \) is a Dedekind domain.

An integral domain \( D \) is a weakly factorial domain if each nonzero nonunit of \( D \) can be written as a finite product of primary elements of \( D \). (A nonzero element \( x \in D \) is said to be primary if \( xD \) is a primary ideal.) Since a prime ideal is a primary ideal, prime elements are primary, and thus UFDs are weakly factorial domains. It is known that \( D \) is a weakly factorial domain if and only if \( D \) is a weakly Krull domain and \( Cl(D) = \{0\} \) [6, Theorem]. Note that \( X \) is a prime element of the polynomial ring \( D[X] \); so \( D[X] \) is a weakly factorial domain if and only if \( D[X, X^{-1}] \) is a weakly factorial domain. Thus, the next result is a generalization of [5, Theorem 17] that \( D \) is a weakly factorial GCD-domain if and only if \( D[X] \) is a weakly factorial domain.

**Corollary 2.7.** Let \( R = \bigoplus_{\alpha \in \Gamma} R_{\alpha} \) be a graded integral domain with a unit of nonzero degree. Then the following statements are equivalent.

1. \( R \) is a weakly factorial domain.
2. \( R \) is a weakly factorial GCD-domain.
3. \( R \) is a weakly factorial \( PrMD \).

**Proof.** (1) \( \Rightarrow \) (2) If \( R \) is a weakly factorial domain, then \( R \) is a weakly Krull domain and \( Cl(R) = \{0\} \). Hence, each upper to zero \( Q \) in \( R \) is \( t \)-invertible by Corollary 2.5 and Proposition 1.8, and so \( Q \) is principal. Thus, every upper to zero in \( R \) contains a (nonzero) prime element, and hence \( R \) is a GCD-domain [19, Theorem 2.2].

(2) \( \Rightarrow \) (3) \( \Rightarrow \) (1) Clear.

### 3. Graded integral domains with a unit of nonzero degree

Let \( R = \bigoplus_{\alpha \in \Gamma} R_{\alpha} \) be an integral domain graded by a nonzero torsionless grading monoid \( \Gamma \), \( H \) be the set of nonzero homogeneous elements of \( R \), \( N(H) = \{ f \in R \mid C(f)_{v} = R \} \), and \( \bar{R} \) be the integral closure of \( R \). Note that \( \bar{R} \) is a graded integral domain by Lemma 1.6 such that \( R \subseteq \bar{R} \subseteq R_H = R_H \). In this section, we study a graded UMT-domain property of \( R \) with a unit of nonzero degree.

**Lemma 3.1.** Let \( R = \bigoplus_{\alpha \in \Gamma} R_{\alpha} \) be a graded integral domain with a unit of nonzero degree, and let \( Q \) be a nonzero homogeneous prime ideal of \( R \). If \( Q \) is not a \( t \)-ideal, then there is an upper to zero \( U \) in \( R \) such that \( U \subseteq Q \).
Since $Q$ is not a $t$-ideal, there are some $a_0, a_1, \ldots, a_n \in Q \cap H$ such that $(a_0, a_1, \ldots, a_n)_o \not\subseteq Q$. Let
\[
f = a_0 + a_1x^{k_1} + \cdots + a_nx^{k_n},
\]
where $x \in R$ is a unit of nonzero degree and $k_i \geq 1$ is an integer such that $C(f) = (a_0, a_1, \ldots, a_n)$, and let $U \subseteq Q$ be a prime ideal of $R$ minimal over $fR$. Then $U$ is a $t$-ideal. We claim that $U$ is an upper to zero in $R$.

Let $S = H \setminus Q$. Then $Q_S$ is a unique homogeneous maximal ideal of $R_S$, and so $(C(f)R_S)_t = R_S$ because $(C(f)R_S)_t = (C(f)_tR_S)_t \not\subseteq Q_S$. Also, note that $U_S$ is a $t$-ideal of $R_S$; hence if $a \in U \cap H(\neq \emptyset)$, then $R_S = ((a, f)R_S)_t \subseteq (U_S)_t = U_S$, a contradiction. Thus, $U \cap H = \emptyset$, and so $U_H$ is a prime $t$-ideal because $U_H$ is minimal over $fR_H$. Since $R_H$ is a UFD, $U_H = gR_H$ for some $g \in R$. Thus, $U = U_H \cap R = gR_H \cap R$ is an upper to zero in $R$. $\square$

**Proposition 3.2.** Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded UMT-domain with a unit of nonzero degree, $T$ be a homogeneous overring of $R$, and $Q$ be a homogeneous prime $t$-ideal of $R$. If $M$ is a homogeneous prime ideal of $T$ such that $M \cap R = Q$, then $M$ is a $t$-ideal of $T$.

**Proof.** If $M$ is not a $t$-ideal of $T$, then there is an upper to zero $U$ in $T$ such that $U \subseteq M$ by Lemma 3.1. Clearly, $U \cap R \subseteq M \cap R = Q$. Thus, $U \cap R$ is not a maximal $t$-ideal of $R$, a contradiction. $\square$

**Corollary 3.3.** Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded UMT-domain with a unit of nonzero degree. If $Q$ is a homogeneous prime $t$-ideal of $R$, then $R_{H \setminus Q}$ is a graded UMT-domain with a unique homogeneous maximal ideal that is a $t$-ideal.

**Proof.** Clearly, $R_{H \setminus Q}$ is a homogeneous $t$-linked overring of $R$, and hence $R_{H \setminus Q}$ is a graded UMT-domain by Corollary 1.10. Also, $Q_{H \setminus Q}$ is a unique homogeneous maximal ideal of $R_{H \setminus Q}$, and by Proposition 3.2, $Q_{H \setminus Q}$ is a $t$-ideal. $\square$

**Lemma 3.4.** Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then $R$ is a graded-Prüfer domain if and only if $R_Q$ is a valuation domain for all homogeneous maximal ideals $Q$ of $R$.

**Proof.** This follows from the following two observations: (i) $R$ is a (graded) PeMD if and only if $R_Q$ is a valuation domain for all homogeneous maximal $t$-ideals $Q$ of $R$ [18, Lemma 2.7] and (ii) $R$ is a graded-Prüfer domain if and only if $R$ is a graded PeMD whose homogeneous maximal ideals are $t$-ideals. $\square$

We next give the main result of this section which provides characterizations of graded UMT-domains with a unit of nonzero degree.

**Theorem 3.5.** Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then the following statements are equivalent.

1. $R$ is a graded UMT-domain.
(2) If \( Q \) is an upper to zero in \( R \), then there is an \( f \in Q \) such that \( C(f)_e = R \).

(3) Every prime ideal of \( R_{N(I)} \) is extended from a homogeneous ideal of \( R \).

(4) \( R_{H \setminus Q} \) is a graded-Pr"ufer domain for all homogeneous maximal \( t \)-ideals \( Q \) of \( R \).

(5) \( R \) is a UMT-domain.

(6) \( R_{N(I)} \) is a Pr"ufer domain.

(7) \( R_{N(I)} \) is a UMT-domain.

(8) \( R_{N(I)} \) is a quasi-Pr"ufer domain.

Proof. (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3) Since \( R \) has a unit of nonzero degree, \( R \) satisfies property (\#). Thus, the results follow directly from Theorem 2.2.

(1) \( \Rightarrow \) (4) Let \( Q \) be a homogeneous maximal \( t \)-ideal of \( R \). Replacing \( R \) and \( Q \) with \( R_{H \setminus Q} \) and \( Q_{H \setminus Q} \) respectively, by Corollary 3.3, we may assume that \( R \) has a unique homogeneous maximal ideal \( Q \) and \( Q \) is a \( t \)-ideal.

Assume to the contrary that \( \bar{R} \) is not a graded-Pr"ufer domain. Then there are some \( a_0, a_1, \ldots, a_k \in H \) such that \( I = (a_0, a_1, \ldots, a_k) \bar{R} \) is not invertible. Let \( f = a_0 + a_1x^{m_1} + \cdots + a_kx^{m_k} \), where \( x \in R \) is a unit of nonzero degree and \( m_i \geq 1 \) is an integer such that \( C_R(f) = I \). Then \( fR_H \cap \bar{R} = fC_R(f)^{-1} \) [9, Lemma 1.2(4)], and since \( I \) is not invertible and \( C_R(f)C_R(f)^{-1} \) is homogeneous, we have \( U = fC_R(f)^{-1} \subseteq C_R(f)C_R(f)^{-1} \subseteq M \) for some homogeneous maximal ideal \( M \) of \( \bar{R} \). Note that \( R_H \) is a UFD; so \( f = f_1^{\epsilon_1} \cdots f_n^{\epsilon_n} \) for some prime elements \( f_i \in R_H \) and integers \( \epsilon_i \geq 1 \). Thus,

\[
fR_H \cap \bar{R} = ((f_1R_H)^{\epsilon_1} \cdots (f_nR_H)^{\epsilon_n}) \cap \bar{R} \\
= ((f_1R_H)^{\epsilon_1} \cap \bar{R}) \cdots (f_nR_H)^{\epsilon_n} \cap \bar{R} \\
= (f_1R_H)^{\epsilon_1} \cap \bar{R} \cdots (f_nR_H)^{\epsilon_n} \cap \bar{R} \\
\supseteq (f_iR_H \cap \bar{R})^{\epsilon_i} \cdots (f_nR_H \cap \bar{R})^{\epsilon_n}.
\]

Thus, \( M \supseteq f_iR_H \cap \bar{R} \) for some \( i \), and so

\[ Q = M \cap R \supseteq (f_iR_H \cap \bar{R}) \cap R = f_iR_H \cap R, \]

which is contrary to the fact that \( Q \) is a \( t \)-ideal. Therefore, \( \bar{R} \) is a graded-Pr"ufer domain.

(4) \( \Rightarrow \) (1) Assume that \( R \) is not a graded UMT-domain, and let \( Q_f = fR_H \cap R \) be an upper to zero in \( R \) such that \( Q_f \subseteq Q \) for some homogeneous maximal \( t \)-ideal \( Q \) of \( R \) (cf. Theorem 1.11). Let \( T = R_{H \setminus Q} \). Then by (4), \( T \) is a graded-Pr"ufer domain, and hence \( U_f = fR_H \cap T = fC_T(f)^{-1} \subseteq M_0 \) for all homogeneous maximal ideals \( M_0 \) of \( T \). Note that \( U_f \cap R_{H \setminus Q} = (Q_f)_{H \setminus Q}, \) \( (Q_f)_{H \setminus Q} \subseteq Q_{H \setminus Q} \), and \( T \) is integral over \( R_{H \setminus Q} \). Thus, there is a prime ideal \( M \) of \( T \) such that \( U_f \subseteq M \) and \( M \cap R_{H \setminus Q} = Q_{H \setminus Q} \). Since \( Q \) is homogeneous, \( M' \cap R_{H \setminus Q} = Q_{H \setminus Q} \). Thus, \( M = M' \) is homogeneous, a contradiction.

(1) \( \Rightarrow \) (5) Let \( Q \) be a maximal \( t \)-ideal of \( R \). If \( Q \cap H \neq \emptyset \), then \( Q \) is homogeneous, and thus \( R_{H \setminus Q} \) is a graded-Pr"ufer domain by the equivalence of
(1) and (4). Note that if \( M \) is a prime ideal of \( \bar{R}_{H \backslash Q} \) such that \( M \cap R_{H \backslash Q} = Q_{H \backslash Q} \), then \( M \) is homogeneous because \( Q \) is homogeneous; hence \( (\bar{R}_{H \backslash Q})_M \) is a valuation domain by Lemma 3.4. Clearly, \( R_{R \backslash Q} = (\bar{R}_{H \backslash Q})_R \). Thus, \( R_{R \backslash Q} \) is a Prüfer domain. Next, assume \( Q \cap H = \emptyset \). Then \( Q = Q_H \cap R \), and so if \( hQ \geq 2 \), then there is an \( 0 \neq f \in R \) such that \( fR_H \subseteq Q_H \) is a prime ideal of \( R_H \). Hence, \( fR_H \cap R \subsetneq Q_H \cap R = Q \), a contradiction. Thus, \( hQ = 1 \) and so \( R_Q = (R_H)_{Q_H} \) is a rank-one DVR. Therefore, by Theorem 1.2, \( R \) is a UMT-domain.

(5) \( \Rightarrow \) (6) Let \( M \) be a prime ideal of \( \bar{R} \) such that \( M_{N(H)} \) is a maximal ideal of \( \bar{R}_{N(H)} \). Then \( (M \cap R) \cap N(H) = \emptyset \), and hence \( M \cap R \) is a homogeneous maximal \( t \)-ideal of \( R \). Since \( R \) is a UMT-domain, \( \bar{R}_{M \cap R} \) is a Prüfer domain by Theorem 1.2. Note that \( \bar{R}_{M \cap R} = (\bar{R}_{N(H)})_{M_{N(H)}} \); so \( (\bar{R}_{N(H)})_{M_{N(H)}} \) is a valuation domain. Thus, \( \bar{R}_{N(H)} \) is a Prüfer domain.

(6) \( \Rightarrow \) (4) Let \( M \) be a homogeneous prime ideal of \( \bar{R} \) such that \( M_{H \backslash Q} \) is a homogeneous maximal ideal of \( \bar{R}_{H \backslash Q} \). Then \( M \cap R \subseteq Q \), and so \( M \cap N(H) = \emptyset \). Thus, \( M_{N(H)} \) is a proper prime ideal of \( \bar{R}_{N(H)} \), and so \( \bar{R}_M = (\bar{R}_{N(H)})_{M_{N(H)}} \) is a valuation domain. Thus, by Lemma 3.4, \( \bar{R}_{H \backslash Q} \) is a graded-Prüfer domain.

(6) \( \Leftrightarrow \) (8) [25, Corollary 6.5.14].

(7) \( \Leftrightarrow \) (8) This follows because each maximal ideal of \( R_{N(H)} \) is a \( t \)-ideal [9, Propositions 1.3 and 1.4].

Corollary 3.6 ([19, Theorem 2.5]). Let \( R = \bigoplus_{\alpha \in \Gamma} R_{\alpha} \) be a graded integral domain with a unit of nonzero degree. Then \( R \) is an integrally closed graded UMT-domain if and only if \( R \) is a PeMD.

Proof. \( R \) is an integrally closed graded UMT-domain if and only if \( R \) is an integrally closed UMT-domain (by Theorem 3.5), if and only if \( R \) is a PeMD (by Corollary 1.3).

Corollary 3.7. Let \( R = \bigoplus_{\alpha \in \Gamma} R_{\alpha} \) be a graded integral domain with a unit of nonzero degree. Then \( R \) is a graded-Prüfer domain if and only if \( R \) is a graded UMT-domain whose homogeneous maximal ideals are \( t \)-ideals.

Proof. (\( \Rightarrow \)) Clearly, \( \bar{R}_{H \backslash Q} \) is a graded-Prüfer domain for all homogeneous maximal \( t \)-ideals \( Q \) of \( R \). Thus, by Theorem 3.5, \( R \) is a graded UMT-domain. Next, let \( f \in R \) be nonzero such that \( fR_H \) is a prime ideal. Note that \( fR_H \cap R = fC_R(f)^{-1} \) \([9, \text{Lemma 1.2}(4)]\); so if \( h \in R_H \) with \( C_R(h) = C_R(f)^{-1} \) (such \( h \) exists because \( R \) has a unit of nonzero degree), then \( fh \in fC_R(f)^{-1} \) and \( C_R(fh) = R \). Thus, \( C(fC_R(f)^{-1}) = \bar{R} \). Note also that \( fR_H \cap R = fC_R(f)^{-1} \cap R \) and \( \bar{R} \) is integral over \( R \). Hence, \( C(fR_H \cap R) = R \). Thus, by Lemma 3.1, each homogeneous maximal ideal of \( R \) is a \( t \)-ideal.

(\( \Leftarrow \)) Let \( M \) be a homogeneous maximal ideal of \( \bar{R} \). Then \( M \cap R \) is a homogeneous ideal of \( R \); so \( (M \cap R) \cap N(H) = \emptyset \) by assumption. Hence, \( M \cap N(H) = \emptyset \), and thus \( \bar{R}_M = (\bar{R}_{N(H)})_{M_{N(H)}} \) is a valuation domain by Theorem 3.5. Thus, by Lemma 3.4, \( \bar{R} \) is a graded-Prüfer domain.
It is well known that each overring of a Prüfer domain is a Prüfer domain [28, Theorem 26.1]. The next result is the graded-Prüfer domain analog.

**Lemma 3.8** ([10, Theorem 2.5(2)]). Let $T$ be a homogeneous overring of a graded-Prüfer domain $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$. Then $T$ is a graded-Prüfer domain.

**Proof.** Let $A$ be a nonzero finitely generated homogeneous ideal of $T$. Since $R \subseteq T \subseteq \bar{R}_H$, there are an $\alpha \in H$ and a finitely generated homogeneous ideal $I$ of $R$ such that $A = \frac{1}{\alpha} IT$. Since $R$ is a graded-Prüfer domain, $I$ is invertible, and thus $A = \frac{1}{\alpha} IT$ is invertible. Hence, $T$ is a graded-Prüfer domain. □

Let $D$ be a UMT-domain, and recall that if $P$ is a nonzero prime ideal of $D$ with $P \subseteq D$, then $P$ is a $t$-ideal [26, Corollary 1.6]. We next give the graded UMT-domain analog.

**Corollary 3.9.** Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded UMT-domain with a unit of nonzero degree, and let $M$ be a homogeneous maximal $t$-ideal of $R$. If $P \subseteq M$ is a nonzero prime ideal of $R$, then $P$ is a homogeneous prime $t$-ideal.

**Proof.** Since $M$ is homogeneous, $C(P)_t \subseteq M_t = M \subseteq R$. Thus, $P$ is homogeneous by Theorem 1.11. Next, note that $\bar{R}_H \setminus M$ is a graded-Prüfer domain and $\bar{R}_H \setminus P$ is a homogeneous overring of $\bar{R}_H \setminus M$; so by Lemma 3.8, $\bar{R}_H \setminus P$ is a graded-Prüfer domain. Thus, by Corollary 3.7, $\bar{P} \setminus P$ is a prime $t$-ideal, and hence $P$ is a prime $t$-ideal of $R$. □

We next give another characterization of graded UMT-domains.

**Corollary 3.10.** Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain with a unit of nonzero degree. Then the following statements are equivalent.

1. $R$ is a graded UMT-domain.
2. Let $Q$ be a nonzero prime ideal of $R$ with $C(Q)_t \subseteq R$. Then $Q$ is a homogeneous prime $t$-ideal.
3. Let $Q$ be a nonzero prime ideal of $R$ such that $Q \subseteq M$ for some homogeneous maximal $t$-ideal $M$ of $R$. Then $Q$ is a homogeneous prime $t$-ideal.

**Proof.** (1) $\Rightarrow$ (2) Let $Q$ be a nonzero prime ideal of $R$ with $C(Q)_t \subseteq R$. Clearly, there is a homogeneous maximal $t$-ideal $M$ of $R$ such that $Q \subseteq M$. Hence, by Corollary 3.9, $Q$ is a homogeneous prime $t$-ideal.

(2) $\iff$ (3) Clear.

(3) $\Rightarrow$ (1) This follows from Theorem 1.11. □

An integral domain $D$ is called a generalized Krull domain if (i) $D = \bigcap_{P \in X^1(D)} D_P$, (ii) the intersection $D = \bigcap_{P \in X^1(D)} D_P$ is locally finite, and (iii) $D_P$ is a (rank-one) valuation domain for all $P \in X^1(D)$. Clearly, $D$ is a generalized Krull domain if and only if $D$ is a weakly Krull domain and $D_P$ is a valuation domain for all $P \in X^1(D)$, if and only if $D$ is a weakly Krull PeMD, and a generalized Krull domain $D$ is a Krull domain, if and only if $D_P$ is a DVR for all $P \in X^1(D)$. 

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Lemma 3.12. Let \( D \) be a Prüfer domain [26, Theorems 2.4 and 2.5], if and only if every prime ideal of \( D \) is a height-one prime ideal of \( R \), and since \( R \) is a UFD, \( R \) is a Prüfer domain.

Corollary 2.5 and [2, Proposition 2.6 and Corollary 3.5].

\( \square \)

Proof. (1) \( \Rightarrow \) (2) It suffices to show that \( R \) is a valuation domain for all \( Q \in X^1(R) \). Let \( Q \) be a height-one prime ideal of \( R \). If \( Q \cap H = \emptyset \), then \( Q = \emptyset \) is a valuation domain.

(2) \( \Rightarrow \) (3) \( \Rightarrow \) (1) This follows from Corollary 2.5 because \( R = R_H \cap R_N(H) \).

R \( \varnothing \) is an integrally closed weakly Krull domain.

\( \square \)

Corollary 2.6 and [2, Proposition 2.6 and Corollary 3.5]. Thus, \( R \) is a proper prime ideal of \( R \).

Let \( \bar{D} \) be the integral closure of an integral domain \( D \), \( \{X_\alpha\} \) be a nonempty set of indeterminates over \( D \), and \( N_v = \{f \in D[X_\alpha] \mid v(f) = D\} \). It is known that \( D \) is a UMT-domain if and only if \( D[X_\alpha] \) is a UMT-domain, if and only if \( D[X_\alpha]_{N_v} \) is a UMT-domain, and if and only if \( D[X_\alpha]_{N_v} \) is a Prüfer domain [26, Theorems 2.4 and 2.5], if and only if every prime ideal of \( D[X_\alpha]_{N_v} \) is extended from \( D \) (cf. [34, Theorem 3.1]). We next recover this result as a corollary of Theorem 3.5, and for this we first need a simple lemma.

Lemma 3.12. Let \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \) be a graded integral domain with a set \( \{p_\beta\} \) of non-zero homogeneous prime elements such that (i) \( \text{ht}(p_\beta R) = 1 \) for each \( \beta \) and (ii) \( \bigcap_{\beta=1}^{\infty} p_\beta R = (0) \) for any sequence \( \{p_{\beta_n}\} \) of nonassociate members of \( \{p_\beta\} \), and let \( S \) be the saturated multiplicative set of \( R \) generated by \( \{p_\beta\} \).

(1) \( R_S \) is a homogeneous overring of \( R \).

(2) \( R \) is a graded UMT-domain if and only if \( R_S \) is a graded UMT-domain.

(3) \( R \) is a UMT-domain if and only if \( R_S \) is a UMT-domain.

Proof. (1) Clear.

(2) It is clear that each upper to zero in \( R \) not comparable with \( p_\beta R \) under inclusion for all \( \beta \). Also, \( Q \) is an upper to zero in \( R \) if and only if \( Q_S \) is an upper to zero in \( R_S \). Note that \( t\text{-Max}(R_S) = \{Q_S \mid Q \in t\text{-Max}(R) \} \).

(3) Clearly, \( R_{p_\beta R} \) is a rank-one DVR for all \( \beta \). Also, if \( Q \) is a prime ideal of \( R \) with \( Q \cap S = \emptyset \), then \( Q \) is a maximal \( t \)-ideal if and only if each upper to zero in \( R_S \) is maximal \( t \)-ideal.

The result follows from Theorem 1.2 and [2, Proposition 2.6 and Corollary 3.5].

\( \square \)
Corollary 3.13. Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain with a nonzero homogeneous prime element $p$ such that $ht(pR) = 1$ and $\deg(p) \neq 0$. Then $R$ is a graded UMT-domain if and only if $R$ is a UMT-domain.

Proof. Clearly, $\{\}$ satisfies the conditions (i) and (ii) of Lemma 3.12. Also, if $S = \{up^n \mid u$ is a unit of $R$ and $n \geq 0\}$, then $R_S$ has a unit of nonzero degree. Thus, $R$ is a graded UMT-domain if and only if $R_S$ is a graded UMT-domain, and if and only if $R_S$ is a UMT-domain by Lemma 3.12 and Theorem 3.5. □

For each $\alpha$, let $\mathbb{Z}_\alpha = \mathbb{Z}$ be the additive group of integers; so if $G = \bigoplus_{\alpha \in \Gamma} \mathbb{Z}_\alpha$, then $G$ is a torsionfree abelian group and the group ring $D[G]$ of $G$ over $D$ is isomorphic to $D[[X_\alpha, X_\alpha^{-1}]]$. Thus, if $R = D[[X_\alpha, X_\alpha^{-1}]]$, then $R$ has a unit of nonzero degree and $R_{\nu(H)} = D[[X_\alpha]]$. [9, Proposition 3.1] and every homogeneous ideal of $R$ has the form $IR$ for an ideal $I$ of $D$.

Corollary 3.14. Let $D$ be an integral domain, $\{X_\alpha\}$ be a nonempty set of indeterminates over $D$, and $N_\nu = \{f \in D[[X_\alpha]] \mid c(f)_\nu = D\}$. Then the following statements are equivalent.

1. $D$ is a UMT-domain.
2. $D[[X_\alpha]]$ is a UMT-domain.
3. $D[[X_\alpha]]$ is a graded UMT-domain.
4. $D[[X_\alpha, X_\alpha^{-1}]]$ is a UMT-domain.
5. $D[[X_\alpha, X_\alpha^{-1}]]$ is a graded UMT-domain.
6. $D[[X_\alpha]]_{N_\nu}$ is a Prüfer domain.
7. $D[[X_\alpha]]_{N_\nu}$ is a UMT-domain.
8. $D[[X_\alpha]]_{N_\nu}$ is a quasi-Prüfer domain.
9. Every prime ideal of $D[[X_\alpha]]_{N_\nu}$ is extended from $D$.

Proof. (1) $\iff$ (5) Let $R = D[[X_\alpha, X_\alpha^{-1}]]$. Then $R_{\nu(H)} = D[[X_\alpha]]_{N_\nu}$ and $\{PR \mid P \in t\text{-Max}(D)\}$ is the set of homogeneous maximal $t$-ideals of $R$. Note that $R_{H\smallsetminus PR} = D[[X_\alpha, X_\alpha^{-1}]]$; and $D[[X_\alpha, X_\alpha^{-1}]]$ is a graded-Prüfer domain if and only if $D_P$ is a Prüfer domain for all $P \in t\text{-Max}(D)$ (cf. [9, Example 3.6]). Thus, the result follows from Theorems 1.2 and 3.5.

(2) $\iff$ (3) This follows from Corollary 3.13 because each $X_\beta$ is a height-one homogeneous prime element of nonzero degree.

(3) $\iff$ (5) Clearly, $\{X_\alpha\}$ is a set of nonzero homogeneous prime elements of $D[[X_\alpha]]$ satisfying the two conditions of Lemma 3.12. Also, if $S$ is the multiplicative set of $D[[X_\alpha]]$ generated by $\{X_\alpha\}$, then $D[[X_\alpha]]_S = D[[X_\alpha, X_\alpha^{-1}]]$. Thus, the result is an immediate consequence of Lemma 3.12(2).

(4) $\iff$ (5) $\iff$ (6) $\iff$ (7) $\iff$ (8) $\iff$ (9) Theorem 3.5. □

4. Counterexamples via the $D + XK[X]$ construction

In this section we use the $D + XK[X]$ construction to show that a graded UMT-domain need not be a UMT-domain in general. For this, let $D$ be an
integral domain with quotient field $K$ and $D \subseteq K$, $X$ be an indeterminate over $D$, $K[X]$ be the polynomial ring over $K$, and $R = D + XK[X]$ be a subring of $K[X]$, i.e., $R = \{ f \in K[X] \mid f(0) \in D \}$; so $D[X] \subseteq R \subseteq K[X]$ and $R$ is an $\mathbb{N}_0$-graded integral domain with $\text{deg}(aX^n) = n$ for $0 \neq a \in K$ and integer $n \geq 0$ ($a \in D$ when $n = 0$). Let $H$ be the set of nonzero homogeneous elements of $R$ and $N(H) = \{ f \in R \mid C(f)_v = R \}$; then $N(H) = \{ f \in R \mid f(0) \text{ is a unit of } R \}$ [15, Lemma 6] and $R_H = K[X, X^{-1}]$.

**Lemma 4.1.** If $Q$ is an upper to zero in $R = D + XK[X]$, then $Q = fR$ for some $f \in R$ with $f(0) = 1$, and hence $Q$ is a maximal $t$-ideal of $R$.

**Proof.** Note that $R_H = K[X, X^{-1}]$; so $Q = fK[X, X^{-1}] \cap R$ for some $f \in K[X, X^{-1}]$. Since $X$ is a unit of $K[X, X^{-1}]$ and $K$ is the quotient field of $D$, we may assume that $f \in R$ with $f(0) = 1$. Hence, if $g \in K[X, X^{-1}]$ is such that $fg \in R$, then $g \in K[X]$, and since $f(0) = 1$, we have $g(0) \in D$; so $g \in R$. Thus, $Q = fR$. □

It is known that $R = D + XK[X]$ is a PrMD if and only if $D$ is a PrMD [21, Theorem 4.43]. We next give a UMT-domain analog.

**Proposition 4.2.** Let $R = D + XK[X]$.

(1) $R$ is a graded UMT-domain.

(2) $R$ is a UMT-domain if and only if $D$ is a UMT-domain.

**Proof.** (1) Lemma 4.1.

(2) Note that $K[X]$ is a UMT-domain and $XK[X]$ is a maximal $t$-ideal of $K[X]$. Thus, $R$ is a UMT-domain if and only if $D$ is a UMT-domain [26, Proposition 3.5]. □

We end this paper with some counterexamples.

**Example 4.3.** Let $R = D + XK[X]$. Then $R$ is a graded UMT-domain.

(i) Counterexample to Proposition 1.7, Theorem 3.5, Corollary 3.6, and Corollary 3.7: Let $R$ be the field of real numbers, $\mathbb{Q}$ be the algebraic closure of the field $\mathbb{Q}$ of rational numbers in $\mathbb{R}$, $R[y]$ be the power series ring over $\mathbb{R}$, and $D = \mathbb{Q} + gR[y]$. Then $D$ is an integrally closed one-dimensional local integral domain that is not a valuation domain [11, Theorem 2.1] (hence $D$ is not a UMT-domain). Hence, $R$ satisfies property $(\#)$ [15, Corollary 9]. $R$ is an integrally closed graded UMT-domain, but $R$ is not a UMT-domain (so not a PrMD). (i) Thus, the converse of Proposition 1.7 does not hold in general. (ii) Moreover, this shows that Theorem 3.5 is not true if $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ does not contain a unit of nonzero degree. (iii) This also shows that Corollary 3.6 is not true in general. (iv) Finally, $R = D + XK[X]$ is an integrally closed domain but not a graded-Prüfer domain, while $R = D + XK[X]$ has a unique homogeneous maximal $t$-ideal (which must be a unique homogeneous maximal ideal). Thus, Corollary 3.7 does not hold in general.
(2) Let $D$ be an integral domain with a prime ideal $P$ such that $P \subseteq P_t \subseteq D$. (For example, let $D = \mathbb{R} + (X, Y, Z)\mathbb{C}[X, Y, Z]$, where $\mathbb{C}$ is the field of complex numbers and $\mathbb{C}[X, Y, Z]$ is the power series ring, and let $P = (X, Y)\mathbb{C}[X, Y, Z]$. Then $P$ is a prime ideal of $D$ such that $P \subseteq P_t \subseteq P_t$. Then $PR = P + XK[X] \subseteq P_t + XK[X] = (P + XK[X])t \subseteq R = D + XK[X]$, and hence $PR$ is a prime ideal of $R$ contained in a homogeneous maximal $t$-ideal but $PR$ is not a $t$-ideal. Thus, Corollary 3.9 does not hold if $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ does not contain a unit of nonzero degree.

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