ON COATOMIC MODULES AND LOCAL COHOMOLOGY MODULES WITH RESPECT TO A PAIR OF IDEALS

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ABSTRACT. In this paper, we show some results on the vanishing and the finiteness of local cohomology modules with respect to a pair of ideals. We also prove that \( \text{Supp}(H_{I,J}^{\dim M-1}(M)/JH_{I,J}^{\dim M-1}(M)) \) is a finite set.

1. Introduction

Throughout this paper, \( R \) is a Noetherian commutative (with non-zero identity) ring and \( I, J \) are two ideals of \( R \). It is well-known that the local cohomology theory of Grothendieck is an important tool in commutative algebra and algebraic geometry. In [8], Takahashi, Yoshino and Yoshizawa introduced the module \( H_{I,J}^i(M) \) as a generalization of the ordinary local cohomology module \( H_I^i(M) \). For an \( R \)-module \( M \), the \( (I,J) \)-torsion submodule of \( M \) is \( \Gamma_{I,J}(M) = \{ x \in M \mid I^nx \subseteq Jx \text{ for some positive integer } n \} \). They denoted by \( H_{I,J}^i \) the \( i \)-th right derived functor of the functor \( \Gamma_{I,J} \). It is clear that when \( J = 0 \), the functor \( H_{I,0}^i \) coincides with the usual local cohomology functor \( H_I^i \).

When \( M \) is a finitely generated \( R \)-module, many properties of \( H_{I,J}^i(M) \) have been studied in [2], [3], [4], [5], [6] and [8]. We now improve some results of those papers in the case \( M \) is a coatomic module. An \( R \)-module \( M \) is called coatomic if every proper submodule of \( M \) is contained in a maximal submodule of \( M \). The coatomic modules were introduced and studied by H. Zöschinger in [9]. In [1] the authors studied some properties of the local cohomology modules \( H_{I,J}^i(M) \) concerning to coatomic modules. An important result on coatomic modules was shown in [9, Satz 2.4] which says that: Let \( (R,m) \) be a local ring and \( M \) an \( R \)-module. The following statements are equivalent:

(i) \( M \) is a coatomic module;
(ii) There is an integer \( t \geq 1 \) such that \( m^tM \) is finitely generated;
(iii) There is an integer \( t \geq 1 \) such that \( M/(0:_M m^t) \) is finitely generated.

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The purpose of this paper is to show some results on the vanishing and the finiteness of local cohomology modules with respect to a pair of ideals $H^i_{I,J}(M)$. Some equivalent conditions of $(I,J)$-torsion modules when $M$ is a coatomic $R$-module are shown in Theorem 2.1. Theorem 2.3 shows that if $M$ is a coatomic module of dimension $\dim M > 0$ or a minimax module of dimension $d = \dim M > 1$ over a local ring $(R, m)$, then the module $H^d_{I,J}(M)$ is artinian and $\text{Att}(H^d_{I,J}(M)) = \{ p \in \text{Supp}(M) \cap V(J) \mid cd(I, J, R/p) = d \}$. We will see in Proposition 2.4 that if $M$ is a coatomic $R$-module with dimension $\dim M > 0$ and $t$ is a non-negative integer such that $\text{Supp}(H^i_{I,J}(M)) \subseteq \{ m \}$ for all $i < t$, then $H^i_{I,J}(M)$ is artinian for all $0 < i < t$.

An important result of this paper is Theorem 2.6 which shows that $H^i_{I,J}(M)$ is finitely generated or coatomic for all $i \geq t$ if and only if $H^i_{I,J}(M) = 0$ for all $i \geq t$.

When studying the finiteness of support of local cohomology modules with respect to an ideal, M. Aghapournahr and L. Melkersson in [1], Saremi in [7] showed that $\text{Supp}(H^{\dim M - 1}_{I}(M))$ is a finite set. Now, we prove in Theorem 2.13 that in a semi-local ring, the set $\text{Supp}(H^{\dim M - 1}_{I,J}(M)/JH^{\dim M - 1}_{I,J}(M))$ is finite.

2. Main results

Let $I, J$ be two ideals of $R$. In [8], the authors denoted by $W(I, J) = \{ p \in \text{Spec}(R) \mid I^n \subseteq p + J \text{ for some } n \geq 1 \}$ and $\bar{W}(I, J) = \{ a \triangleleft R \mid I^n \subseteq a + J \text{ for some } n \geq 1 \}$. An $R$-module $M$ is called $(I,J)$-torsion if $M$ is $(I,J)$-torsion. When $M$ is a finitely generated $R$-module, it follows from [8, 1.9] that $M/JM$ is $(I,J)$-torsion. We have the first result on the equivalent conditions of $(I,J)$-torsion modules when $M$ is a coatomic $R$-module.

**Theorem 2.1.** Let $(R, m)$ be a local ring and $M$ a coatomic $R$-module. The following statements are equivalent:

(i) $M$ is $(I,J)$-torsion;
(ii) $M/JM$ is $I$-torsion;
(iii) $H^i_{I,J}(M) = 0$ for all $i > 0$.

**Proof.** (i) $\Rightarrow$ (ii). Trivial.
(ii) $\Rightarrow$ (iii). It follows from [8, 1.13(1)].
(iii) $\Rightarrow$ (i). Assume that $M/JM$ is an $I$-torsion $R$-module. Since $M$ is coatomic, by [9, Satz 2.4] there exists a positive integer $t$ such that $M/(0 :_M m^t)$ is a finitely generated $R$-module. Let $N = 0 :_M m^t$. It is clear that $N$ is $m$-torsion and then $N$ is $(I,J)$-torsion. We see that

$$
\frac{M/N}{J(M/N)} \cong \frac{M}{JM + N} \cong \frac{M/JM}{JM + N/JM}.
$$

By the assumption, we can conclude that $(M/N)/J(M/N)$ is $I$-torsion. Since $M/N$ is finitely generated, we have by [8, 1.9] that $M/N$ is $(I,J)$-torsion. Now,
combining the short exact sequence
\[ 0 \to N \to M \to M/N \to 0 \]
with [8, 1.8 (2)] we get that \( M \) is \((I, J)\)-torsion.

(iii) \(\Rightarrow\) (i). By [9, Satz 2.4], there is an integer \( t \) such that \( M/(0 :_M m^t) \) is finitely generated. If \( M = 0 :_M m^t \), then \( M \) is an \( m \)-torsion \( R \)-module. Since \((R, m)\) is a local ring, we see that \( M \) is \((I, J)\)-torsion. Now assume that \( M/(0 :_M m^t) \neq 0 \). This implies that \( m \in \text{Supp}(M/(0 :_M m^t)) \). From the short exact sequence
\[ 0 \to 0 :_M m^t \to M \to M/(0 :_M m^t) \to 0 \]
we have
\[ \text{Supp}(M) = \text{Supp}(M/(0 :_M m^t)) \cup \text{Supp}(0 :_M m^t) = \text{Supp}(M/(0 :_M m^t)) \]
and
\[ H^i_{I,J}(M) \cong H^i_{I,J}(M/(0 :_M m^t)) \]
for all \( i > 0 \). By the assumption, \( H^i_{I,J}(M/(0 :_M m^t)) = 0 \) for all \( i > 0 \). Therefore, we can conclude that
\[ \text{Supp}(M/(0 :_M m^t)) \subseteq W(I, J) \]
by [8, 4.2] and the proof is complete. \( \square \)

Now, if \( M \) is a minimax \( R \)-module, then we have a similar result. We recall that an \( R \)-module \( M \) is minimax if there is a finitely generated submodule \( N \) of \( M \) such that \( M/N \) is artinian. Minimax modules were first introduced and studies by H. Zöschinger in [10].

**Proposition 2.2.** Let \((R, m)\) be a local ring and \( M \) a minimax \( R \)-module. The following statements are equivalent:

(i) \( M \) is \((I, J)\)-torsion.

(ii) \( M/JM \) is \( I \)-torsion.

**Proof.** (i) \(\Rightarrow\) (ii). Trivial.

(ii) \(\Rightarrow\) (i). Since \( M \) is a minimax \( R \)-module, there exists a finitely generated submodule \( N \) of \( M \) such that \( M/N \) is artinian. From the short exact sequence
\[ 0 \to N \to M \to M/N \to 0 \]
we have the following exact sequence
\[ \cdots \to \text{Tor}^R_1(R/J, M/N) \to N/JN \to M/JM \to (M/N)/J(M/N) \to 0. \]
Since \( M/N \) is an artinian \( R \)-module, we have \( \text{Supp}(M/N) \subseteq \{m\} \) and then \( M/N \) is \( I \)-torsion and \((I, J)\)-torsion. Therefore \( \text{Tor}^R_1(R/J, M/N) \) is an \( I \)-torsion \( R \)-module. It follows from the hypothesis that \( N/JN \) is \( I \)-torsion. Since \( N \) is finitely generated, we have by [8, 1.9] that \( N \) is an \((I, J)\)-torsion \( R \)-module. By [8, 1.8(2)], we imply that \( M \) is \((I, J)\)-torsion. \( \square \)
In [3, 2.1], if $M$ is a finitely generated $R$-module over a local ring $(R, \mathfrak{m})$ with $\dim M = d$, then $H^d_{I,J}(M)$ is artinian. We now prove that these properties hold for the larger class of coatomic modules of minimax modules instead of the class of finitely generated modules.

**Theorem 2.3.** Let $(R, \mathfrak{m})$ be a local ring and $M$ a coatomic $R$-module with $d = \dim M > 0$ or a minimax $R$-module with $d = \dim M > 1$. Then $H^d_{I,J}(M)$ is artinian and

$$\text{Att}(H^d_{I,J}(M)) = \{ p \in \text{Supp}(M) \cap V(J) \mid \text{cd}(I, J, R/p) = d \},$$

where $\text{cd}(I, J, M) = \sup \{ n \mid H^n_{I,J}(M) \neq 0 \}$.

**Proof.** At first we assume that $M$ is a coatomic $R$-module. Then there is an integer $k \geq 1$ such that $M/(0 :_M m^k)$ is finitely generated by [9, Satz 2.4] and $H^i_{I,J}(M) \cong H^i_{I,J}(M/(0 :_M m^k))$ for all $i > 0$. Since $\dim M > 0$, we can conclude that $\text{Supp}(M) = \text{Supp}(M/(0 :_M m^k))$ and $\dim M = \dim M/(0 :_M m^k)$.

From [3, 2.1], $H^d_{I,J}(M/(0 :_M m^k))$ is artinian and then $H^d_{I,J}(M)$ is also artinian. Now we have by [2, 2.1],

$$\text{Att}(H^d_{I,J}(M)) = \text{Att}(H^d_{I,J}(M/(0 :_M m^k)))$$

$$= \{ p \in \text{Supp}(M/(0 :_M m^k)) \cap V(J) \mid \text{cd}(I, J, R/p) = d \}$$

$$= \{ p \in \text{Supp}(M) \cap V(J) \mid \text{cd}(I, J, R/p) = d \}.$$

In the case $M$ is a minimax $R$-module. There exists a short exact sequence

$$0 \to N \to M \to A \to 0,$$

where $N$ is finitely generated and $A$ is artinian. Since $\dim M > 0$ and $A$ is artinian, we have $\text{Supp}(M) = \text{Supp}(N)$ and $\dim M = \dim N$. By applying the functor $\Gamma_{I,J}(-)$ to the above exact sequence, we obtain an exact sequence

$$0 \to H^0_{I,J}(N) \to H^0_{I,J}(M) \to H^0_{I,J}(A) \to H^1_{I,J}(N) \to H^1_{I,J}(M) \to 0$$

and

$$H^i_{I,J}(N) \cong H^i_{I,J}(M)$$

for all $i \geq 2$. Since $N$ is a finitely generated $R$-module, we have by [3, 2.1] that $H^d_{I,J}(N)$ is artinian and then so is $H^d_{I,J}(M)$. By using [2, 2.1] again, we have

$$\text{Att}(H^d_{I,J}(M)) = \text{Att}(H^d_{I,J}(N))$$

$$= \{ p \in \text{Supp}(N) \cap V(J) \mid \text{cd}(I, J, R/p) = d \}$$

$$= \{ p \in \text{Supp}(M) \cap V(J) \mid \text{cd}(I, J, R/p) = d \}$$

and the proof is complete. \qed
Note that, if $M$ is a minimax $R$-module with $\dim M = 1$, then we see that 
\[ \text{Att}(H^i_{I,J}(M)) \subseteq \{ p \in \text{Supp}(M) \cap V(J) \mid \text{cd}(I, J, R/p) = 1 \}. \]

It should be mentioned that the above result is not true when $\dim M = 0$. The example is similar to [1, 3.5]. On the other hand, if $R$ is not a local ring and $\dim M = 0$, then $H^0_{I,J}(M)$ is not artinian. Let $R = \mathbb{Z}, M = (\mathbb{Z}_2)^N$ and $I = 2\mathbb{Z}, J = 4\mathbb{Z}$. We see that $\dim M = 0$ and $H^0_{I,J}(M) = M$ is not artinian.

We see in [6, 2.8] that $H^i_{I,J}(M)$ is artinian for all $i < t$ if $M$ is a minimax module such that $\text{Supp}(H^i_{I,J}(M)) \subseteq \{ m \}$ for all $i < t$. Now, we consider in the case $M$ is a coatomic module.

**Proposition 2.4.** Let $(R, m)$ be a local ring, $M$ a coatomic $R$-module with $\dim M > 0$. Assume that $t$ is a non-negative integer such that $\text{Supp}(H^i_{I,J}(M)) \subseteq \{ m \}$ for all $i < t$. Then $H^i_{I,J}(M)$ is artinian for all $0 < i < t$.

**Proof.** It follows from the proof of Theorem 2.3, there is an integer $k \geq 1$ such that $M/(0 :_M m^k)$ is finitely generated and $\text{Supp}(M) = \text{Supp}(M/(0 :_M m^k))$. By the hypothesis we see that $\text{Supp}(H^i_{I,J}(M/(0 :_M m^k))) \subseteq \{ m \}$ for all $i < t$. Since finitely generated modules are minimax modules, we have by [6, 2.8] that $H^i_{I,J}(M/(0 :_M m^k))$ is artinian for all $i < t$. Note that $H^i_{I,J}(M/(0 :_M m^k)) \cong H^i_{I,J}(M)$ for all $i > 0$ and which completes the proof. \(\square\)

When $M$ is a finitely generated module, in [8] we see that $H^i_{I,J}(M) = 0$ for all $i > \dim M/JM$. Now, we give an extension of this property in the case $M$ is a coatomic $R$-module.

**Proposition 2.5.** Let $(R, m)$ be a local ring and $M$ a coatomic $R$-module. The following statements hold:

(i) If $J \neq R$, then $H^i_{I,J}(M) = 0$ for all $i > \dim M/JM$.

(ii) Suppose that $\sqrt{I + J} = m$. Then $\sup\{ n \mid H^n_{I,J}(M) \neq 0 \} = \dim M/JM$.

**Proof.** (i) If $\dim M/JM = -1$, then $M = JM$. Since $M$ is coatomic, there is an integer $t \geq 1$ such that $m^t M$ is finitely generated by [9, Satz 2.4]. This implies that $M$ is finitely generated since $M = J^t M \subseteq m^t M$. Therefore $M = 0$ by Nakayama’s Lemma.

Now suppose that $\dim M/JM \geq 0$. By the assumption on $M$, there exists an integer $t \geq 1$ such that $M/(0 :_M m^t)$ is finitely generated. Let $N = 0 :_M m^t$, now the short exact sequence 
\[ 0 \to N \to M \to M/N \to 0 \]
gives rise a long exact sequence 
\[ \cdots \to N/JN \to M/JM \to (M/N)/J(M/N) \to 0. \]

Note that 
\[ \text{Supp}(\text{Im} \alpha) \subseteq \text{Supp}(N/JN) \subseteq \text{Supp}(N) \subseteq \{ m \}. \]
This implies that \( \dim(\text{Im} \, \alpha) \leq 0 \). If \( M = N \), then we can easily check the claim. So in the remainder of the proof, we may and do assume that \( N \subset M \). Now from the short exact sequence

\[ 0 \to \text{Im} \, \alpha \to M/JM \to (M/N)/J(M/N) \to 0 \]

we get

\[ \dim M/JM = \dim (M/N)/J(M/N). \]

Since \( M/N \) is a finitely generated \( R \)-module, we have \( H^i_{I,J}(M/N) = 0 \) for all \( i > \dim M/JM \) by [8, 4.3]. Now the conclusion follows from the isomorphism \( H^i_{I,J}(M) \cong H^i_{I,J}(M/N) \) for all \( i > 0 \).

(ii) Combining [8, 4.5] with the isomorphism \( H^i_{I,J}(M) \cong H^i_{I,J}(M/N) \) for all \( i > 0 \), we get the assertion. \( \square \)

We are going to state and prove one of main results of this paper. The following theorem is a generalization of [1, 3.9] which shows a relationship on the vanishing, the finiteness and the coatomicness of \( H^i_{I,J}(M) \).

**Theorem 2.6.** Let \((R, m)\) be a local ring, \( M \) a finitely generated \( R \)-module and \( t \) a positive integer. The following statements are equivalent:

(i) \( H^i_{I,J}(M) = 0 \) for all \( i \geq t \);

(ii) \( H^i_{I,J}(M) \) is finitely generated for all \( i \geq t \);

(iii) \( H^i_{I,J}(M) \) is coatomic for all \( i \geq t \).

**Proof.** (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii). Trivial.

(iii) \( \Rightarrow \) (i). The proof is by induction on \( \dim M \). Let \( n = \dim M \). If \( n = 0 \), then \( H^i_{I,J}(M) = 0 \) for all \( i > 0 \).

Let \( n > 0 \), it follows from [8, 1.13] that

\[ H^i_{I,J}(M) \cong H^i_{I,J}(M/\Gamma_{I,J}(M)) \]

for all \( i > 0 \). Denote by \( \overline{M} = M/\Gamma_{I,J}(M) \), it is clear that \( \overline{M} \) is \((I,J)\)-torsion-free. This implies that \( \overline{M} \) is \( a \)-torsion-free for all \( a \in \overline{W}(I,J) \). In particular, \( \overline{M} \) is \( m \)-torsion-free and there is an element \( x \in m \) which is regular on \( \overline{M} \). Now, the short exact sequence

\[ 0 \to \overline{M} \xrightarrow{x} \overline{M} \to \overline{M}/x \overline{M} \to 0 \]

induces a long exact sequence

\[ \cdots \to H^i_{I,J}(\overline{M}) \xrightarrow{x} H^i_{I,J}(\overline{M}) \to H^i_{I,J}(\overline{M}/x \overline{M}) \to \cdots . \]

By the assumption, \( H^i_{I,J}(\overline{M}/x \overline{M}) \) is coatomic for all \( i \geq t \). Since \( \dim(\overline{M}/x \overline{M}) < \dim(M) \), \( \overline{M} \) is a finitely generated \( R \)-module, it follows from the inductive hypothesis that \( H^i_{I,J}(\overline{M}/x \overline{M}) = 0 \) for all \( i \geq t \). Now the long exact sequence yields

\[ H^i_{I,J}(\overline{M}) = xH^i_{I,J}(\overline{M}) \]

for all \( i \geq t \). Note that coatomic modules satisfy Nakayama’s Lemma. Thus \( H^i_{I,J}(\overline{M}) = 0 \) for all \( i \geq t \), and the proof is complete. \( \square \)
We may improve these results as follows.

**Corollary 2.7.** Let \((R, m)\) be a local ring, \(M\) a coatomic \(R\)-module and \(t\) a positive integer. The following statements are equivalent:

(i) \(H^i_{I,J}(M) = 0\) for all \(i \geq t\);
(ii) \(H^i_{I,J}(M)\) is finitely generated for all \(i \geq t\);
(iii) \(H^i_{I,J}(M)\) is coatomic for all \(i \geq t\).

**Proof.** Since \(M\) is a coatomic \(R\)-module, there is an integer \(k \geq 1\) such that \(M/(0 :_M m^k)\) is finitely generated by [9, Satz 2.4]. Therefore, we have the isomorphisms

\[ H^i_{I,J}(M) \cong H^i_{I,J}(M/(0 :_M m^k)) \]

for all \(i > 0\). The assertion follows immediate from 2.6. \(\square\)

**Corollary 2.8.** Let \((R, m)\) be a local ring, \(M\) a minimax \(R\)-module and \(t > 1\) a positive integer. The following statements are equivalent:

(i) \(H^i_{I,J}(M) = 0\) for all \(i \geq t\);
(ii) \(H^i_{I,J}(M)\) is finitely generated for all \(i \geq t\);
(iii) \(H^i_{I,J}(M)\) is coatomic for all \(i \geq t\).

**Proof.** (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii). Trivial. We now prove (iii) \(\Rightarrow\) (i). Since \(M\) is a minimax \(R\)-module, there is a short exact sequence

\[ 0 \to N \to M \to A \to 0, \]

where \(N\) is finitely generated and \(A\) is artinian. By applying the functor \(\Gamma_{I,J}(\cdot)\) to the above exact sequence, we get a long exact sequence

\[ 0 \to H^0_{I,J}(N) \to H^0_{I,J}(M) \to H^0_{I,J}(A) \to H^1_{I,J}(N) \to H^1_{I,J}(M) \to 0 \]

and

\[ H^i_{I,J}(N) \cong H^i_{I,J}(M) \]

for all \(i \geq 2\). By the hypothesis, \(H^i_{I,J}(N)\) is coatomic for all \(i \geq t\). It follows from 2.6 that \(H^i_{I,J}(N) = 0\) for all \(i \geq t\) and which completes the proof. \(\square\)

**Corollary 2.9.** Let \((R, m)\) be a local ring and \(M\) a finitely generated \(R\)-module with \(\text{cd}(I,J,M) > 0\). Then \(H^\text{cd}(I,J,M)_{I,J}(M)\) is not finitely generated.

Combining [8, 4.5] with 2.9, we have an immediate consequence.

**Corollary 2.10.** Let \((R, m)\) be a local ring, \(M\) a finitely generated \(R\)-module with \(\text{dim}(M/JM) > 0\) and \(\sqrt{I + J} = m\). Then \(H^\text{dim}(M/JM)_{I,J}(M)\) is not finitely generated.

In [5, Theorem 2], if \(M\) is a finitely generated with finite dimension and \(t\) is a positive integer such that \(H^i_{I,J}(M) = 0\) for all \(i > t\), then \(H^i_{I,J}(M)/aH^i_{I,J}(M) = 0\) for all \(a \in \tilde{W}(I, J)\). This property will be extended in the case \(M\) is a coatomic module.
Proposition 2.11. Let \((R, \mathfrak{m})\) be a local ring and \(M\) a coatomic \(R\)-module. Suppose that \(t\) is a positive integer such that \(H^i_{I,J}(M) = 0\) for all \(i > t\). Then \(H^i_{I,J}(M)/aH^i_{I,J}(M) = 0\) for all \(a \in \widetilde{W}(I, J)\).

Proof. Since \(M\) is a coatomic \(R\)-module, there is an integer \(k \geq 1\) such that \(M/(0 :_M \mathfrak{m}^k)\) is finitely generated. The proof above gives \(H^t_{I,J}(M) \cong H^t_{I,J}(M/(0 :_M \mathfrak{m}^k))\).

Hence, the assertion follows from [5, Theorem 2]. \(\square\)

Corollary 2.12. Let \((R, \mathfrak{m})\) be a local ring and \(M\) a coatomic \(R\)-module. Assume that \(\text{cd}(I, J, M) > 0\) and \(K\) is a proper submodule of \(H^{\text{cd}(I, J, M)}_{I,J}(M)\). Then \(H^{\text{cd}(I, J, M)}_{I,J}(M)/K\) is not a coatomic \(R\)-module.

Proof. Suppose that the conclusion is false. It follows from the definition of coatomic modules, there exists a submodule \(L\) of \(H^{\text{cd}(I, J, M)}_{I,J}(M)\) such that we have a short exact sequence

\[0 \rightarrow L/K \rightarrow H^{\text{cd}(I, J, M)}_{I,J}(M) + K \rightarrow R/\mathfrak{m} \rightarrow 0.\]

Let \(a \in \widetilde{W}(I, J)\), by applying the functor \(R/a \otimes_R -\) to the above exact sequence, there is a following exact sequence

\[\cdots \rightarrow L/aL + K \rightarrow H^{\text{cd}(I, J, M)}_{I,J}(M)/aH^{\text{cd}(I, J, M)}_{I,J}(M) + K \rightarrow R/\mathfrak{m} \rightarrow 0.\]

Note that \(H^{\text{cd}(I, J, M)}_{I,J}(M)/aH^{\text{cd}(I, J, M)}_{I,J}(M) + K\) is a homomorphic image of \(H^{\text{cd}(I, J, M)}_{I,J}(M)/aH^{\text{cd}(I, J, M)}_{I,J}(M)\).

Consequently, we can conclude that \(H^{\text{cd}(I, J, M)}_{I,J}(M)/aH^{\text{cd}(I, J, M)}_{I,J}(M) + K = 0\) by 2.11. This implies that \(R/\mathfrak{m} = 0\) which is a contradiction. \(\square\)

Next, we will consider the dimension of \(H^i_{I,J}(M)\) and the support of \(H^{d-1}_{I,J}(M)\) where \(d = \dim M\). In [1, 3.3] or [7, 2.3], when studying the local cohomology modules with respect to an ideal, the authors showed that \(\dim H^1_{I,J}(M) \leq d - i\) and \(\text{Supp}(H^{d-1}_{I,J}(M))\) is a finite set. The proof of next theorem is based on these results.

Theorem 2.13. Let \(M\) be a finitely generated \(R\)-module with \(d = \dim M < \infty\). Then

(i) \(\dim H^i_{I,J}(M) \leq d - i\).

(ii) If \(R\) is a semi-local ring, then \(\text{Supp}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M))\) is finite.

Proof. (i) Our proof starts with the observation that \(H^i_{I,J}(M) = \lim_{a \in \widetilde{W}(I, J)} H^i_a(M)\).
This implies that
\[ \text{Supp}(H^i_{I,J}(M)) \subseteq \bigcup_{a \in \tilde{W}(I,J)} \text{Supp}(H^a(M)). \]

From [7, 2.3], \( \dim(H^i_{I,J}(M)) \leq \dim M - i \) for all \( a \in \tilde{W}(I,J) \). We conclude that \( \dim(H^i_{I,J}(M)) \leq \dim M - i \).

(ii) We prove by induction on \( d = \dim M \). It is nothing to prove when \( d = 0 \).

If \( d = 1 \), we see that \( H^0_{I,J}(M) \) is finitely generated. Since \( \dim(H^0_{I,J}(M)) \leq 1 \) by (i), it follows that
\[ \text{Supp}(H^0_{I,J}(M)) \subseteq \text{Min}(H^0_{I,J}(M)) \cup \text{Max}(R). \]

Since \( H^0_{I,J}(M) \) is finitely generated, we can conclude that \( \text{Supp}(H^0_{I,J}(M)) \) is finite. Let \( d > 1 \), we now assume that the statement is true for all non-zero finitely generated modules with dimension less than \( \dim M \).

Now the short exact sequence
\[ 0 \to \Gamma_J(M) \to M \to M/\Gamma_J(M) \to 0 \]
induces a long exact sequence
\[ \cdots \to H^{d-1}_{I,J}(\Gamma_J(M)) \xrightarrow{f} H^{d-1}_{I,J}(M) \xrightarrow{g} H^{d-1}_{I,J}(M/\Gamma_J(M)) \xrightarrow{h} H^d_{I,J}(\Gamma_J(M)) \cdots . \]

It follows from [8, 2.5] that \( H^i_{I,J}(\Gamma_J(M)) \cong H^i_J(\Gamma_J(M)) \) for all \( i \geq 0 \). On the other hand \( \dim \Gamma_J(M) \leq \dim M \), so in the view of [7, 2.5] we see that \( \text{Supp}(H^{d-1}_{I,J}(\Gamma_J(M))) \) is finite. This implies that \( \text{Supp}(\text{Im } f) \) is finite. Since \( H^d_{I,J}(\Gamma_J(M)) \) is artinian, the support of \( \text{Im } h \) is finite. We now have two short exact sequences
\[ 0 \to \text{Im } f \to H^{d-1}_{I,J}(M) \to \text{Im } g \to 0 \]
and
\[ 0 \to \text{Im } g \to H^{d-1}_{I,J}(M/\Gamma_J(M)) \to \text{Im } h \to 0. \]

By applying the functor \( R/J \otimes_R - \) to above short exact sequences, we obtain the following exact sequences
\[ \cdots \to \text{Im } f/J \text{Im } f \to H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M) \to \text{Im } g/J \text{Im } g \to 0 \]
and
\[ \cdots \to \text{Tor}^1_R(R/J, \text{Im } h) \to \text{Im } g/J \text{Im } g \to H^{d-1}_{I,J}(M/\Gamma_J(M))/JH^{d-1}_{I,J}(M/\Gamma_J(M)) \to \text{Im } h/J \text{Im } h \to 0. \]

The proof is complete by showing that
\[ \text{Supp}(H^{d-1}_{I,J}(M/\Gamma_J(M))/JH^{d-1}_{I,J}(M/\Gamma_J(M))) \]
is finite. Let \( \overline{M} = M/\Gamma_J(M) \), we see that \( \overline{M} \) is \( J \)-torsion free. Then there is an element \( x \in J \) which is \( \overline{M} \)-regular. Now the short exact sequence
\[ 0 \to \overline{M} \to \overline{M} \to \overline{M}/x\overline{M} \to 0 \]
induces the following exact sequence
\[ \cdots \to H^{d-2}_{I,J}(M/xM) \to H^{d-1}_{I,J}(M) \xrightarrow{\phi} H^{d-1}_{I,J}(M) \to \cdots. \]
This gives us an exact sequence
\[ H^{d-2}_{I,J}(M/xM)/JH^{d-2}_{I,J}(M/xM) \to (0 : H^{d-1}_{I,J}(M), x)/J(0 : H^{d-1}_{I,J}(M), x) \to 0. \]
Since \( \dim(M/xM) \leq d - 1 \), we get by the inductive hypothesis that
\[ \text{Supp}(H^{d-2}_{I,J}(M/xM)/JH^{d-2}_{I,J}(M/xM)) \]
is finite and then so is \( \text{Supp}((0 : H^{d-1}_{I,J}(M), x)/J(0 : H^{d-1}_{I,J}(M), x)) \). Since \( x \in J \), it follows that the homomorphism
\[ (0 : H^{d-1}_{I,J}(M), x)/J(0 : H^{d-1}_{I,J}(M), x) \to H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M) \]
is surjective. Therefore \( \text{Supp}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M)) \) is finite, and the proof is complete.

\[ \textbf{Corollary 2.14.} \text{ Let } M \text{ be a finitely generated } R\text{-module with finite dimension } d = \dim M. \text{ Then} \]
\[ \text{Supp}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M)) \subseteq \text{Ass}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M)) \cup \text{Max}(R). \]

\[ \text{Proof.} \text{ It follows from 2.13 that } \dim(H^{d-1}_{I,J}(M)) \leq 1, \text{ we see that} \]
\[ \dim(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M)) \leq 1. \]
Therefore \( \text{Supp}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M)) \) contains minimal prime ideals of
\[ \text{Ass}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M)) \]
and maximal ideals, which completes the proof. \( \square \)

In the case \( R \) is not a semi-local ring, we will see that \( \text{Supp}(H^{\dim M-1}_{I,J}(M)) \) is not finite.

\[ \textbf{Example 2.15.} \text{ Let } R = M = \mathbb{Z} \text{ and } I = 2\mathbb{Z}, J = 4\mathbb{Z}. \text{ We see that } \dim M = 1 \text{ and } M \text{ is } (I, J)\text{-torsion. However, } \text{Supp}(H^1_{I,J}(M)) = \text{Spec}(\mathbb{Z}) \text{ is an infinite set.} \]

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