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z^J–Ideals and Strongly Prime Ideals in Posets

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Abstract. In this paper, we study the notion of z^J - ideals of posets and explore the various properties of z^J-ideals in posets. The relations between topological space on $Sspec(P)$, the set $I_Q = \{x \in P : L(x, y) \subseteq I$ for some $y \in P\backslash Q\}$ for an ideal I and a strongly prime ideal Q of P and z^J-ideals are discussed in poset P.

1. Introduction

Throughout this paper $(P, \leq)$ denotes a poset with smallest element 0 and all prime ideals are assumed to be proper. For basic terminology and notation for posets, we refer [6] and [5]. For $M \subseteq P$, let $L(M) = \{x \in P : x \leq m$ for all $m \in M\}$ denotes the lower cone of M in P and dually, let $U(M) = \{x \in P : m \leq x$ for all $m \in M\}$ be the upper cone of M in P. Let $A, B \subseteq P$, we shall write $L(A, B)$ instead of $L(A \cup B)$ and dually for the upper cones. If $M = \{x_1, x_2, ..., x_n\}$ is finite, then we use the notation $L(x_1, x_2, ..., x_n)$ instead of $L(\{x_1, x_2, ..., x_n\})$(and dually). It is clear that for any subset A of P, we have $A \subseteq L(U(A))$ and $A \subseteq U(L(A))$. If $A \subseteq B$, then $L(B) \subseteq L(A)$ and $U(B) \subseteq U(A)$. Moreover, $LUL(A) = L(A)$ and $ULU(A) = U(A)$. Following [8], a non-empty subset I of P is called semi-ideal if $b \in I$ and $a \leq b$, then $a \in I$. A subset I of P is called ideal if $a, b \in I$ implies $L(U((a, b)) \subseteq I$[6]. A proper semi-ideal(ideal) I of P is called prime if $L(a, b) \subseteq I$ implies that either $a \in I$ or $b \in I$ [5]. An ideal I of P is called semi-prime if $L(a, b) \subseteq I$ and $L(a, c) \subseteq I$ together imply $L(a, U(b, c))) \subseteq I$[6]. For any semi-ideal I of P and a subset A of P, we define $< A, I >= \{z \in P : L(a, z) \subseteq I$ for all $a \in A\} = \bigcap_{a \in A} < a, I >$[2]. If $A = \{x\}$, then we write $< \{x\}, I >=< x, I >$.

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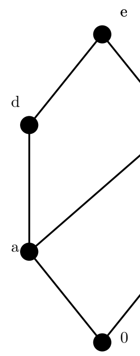
Following [2], an ideal I of P is called strongly prime if $L(A^*, B^*) \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for different proper ideals A, B of P, where $A^* = A \backslash \{0\}$. For an ideal I of P, a strongly prime ideal Q of P is said to be a minimal strongly prime ideal of I if $I \subseteq Q$ and there exists no strongly prime ideal R of P such that $I \subset R \subset Q$. The set of all strongly prime ideal of P is denoted by $Sspec(P)$ and the set of minimal strongly prime ideals of P is denoted by $Smin(P)$. For any ideal I of P, $P(I)$ denotes the intersection of all strongly prime ideals of P containing I and $P(P)$ denotes the intersection all strongly prime ideals of P. If $I = \{0\}$, then we denote $P(I) = P(P)$. From [5], the intersection of all prime semi-ideal of P containing I is I for any semi-ideal I of P. But the following example shows that the intersection of all strongly prime ideal of P containing I need not to be I for any ideal I of P.

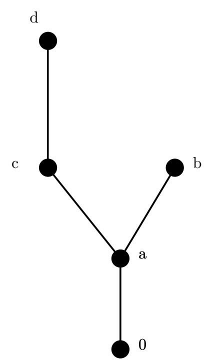
Example 1.1. Consider $P = \{0, a, b, c, d, e\}$ and define a relation $\leq$ on P as follows.



Then $(P, \leq)$ is a poset and $I_1 = \{0, a, b, c\}$ is the only strongly prime ideal of P. For $I = \{0, a\}$, we have $P(I) = \{0, a, b, c\} \neq I$. □

Following [3], a non-empty sub-set M of P is called m-system if for any $x_1, x_2 \in M$, there exists $t \in L(x_1, x_2)$ such that $t \in M$. As a generalization of m-system, we define the notion of strongly m-system as follows, a non-empty subset M of P is called strongly m-system if $A \cap M \neq \phi$ and $B \cap M \neq \phi$ implies $L(A^*, B^*) \cap M \neq \phi$ for different proper ideals A, B of P. It is clear that an ideal I of P is strongly prime if and only if $P \backslash I$ is a strongly m- system of P. Also every strongly m-system is m-system. But the converse need not be true in general. Consider the poset P depicted in Example 1.1 and for $I = \{0, a, b, c\}$, $P \backslash I$ is a m-system of P, but not a strongly m-system of P as for $A = \{0, b\}$ and $B = \{0, a, b, c\}$, we have $A \cap P \backslash I \neq \phi$ and $B \cap P \backslash I \neq \phi$, but $L(A^*, B^*) \cap P \backslash I = \phi$. For any subset X of P, we define $V(X) = \{Q \in Sspec(P) : X \subseteq Q\}$, $D(X) = Sspec(P) \backslash V(X)$, $V^{'}(X) = V(X) \cap Smin(P)$, $D^{'}(X) = D(X) \cap Smin(P)$ and $[X]$ is the smallest ideal of P containing X. It is easy to check that $D(A) = D(L(U(A)))$ and $V(A) = V(L(U(A)))$ for any subset A of P. Following [4], let τ denotes the collection of all subsets $V(I)$ of $Sspec(P)$ for any ideal I of P. Then τ is closed under finite unions and arbitrary intersections, so that there is a topology on $Sspec(P)$ for which τ is the family of closed sets. This is called the Zariski topology. Also the collection of open sets $\xi = \{D(a) : a \in P\}$ forms a basis for a topology on $Sspec(P)$. It is clear that $V(0) = Sspec(P), V(P) = \phi$ and $D(0) = \phi$. A topological space is called T_0-space if for any two distinct points there is an open set that contains one of them and not the other. A topological space is called T_2 or Hausdorff if for any two distinct points x and y there are disjoint open sets U and V such that $x \in U$ and $y \in V$. For each $a \in P$ and an ideal I of P, we define $P_a(I) = \cap\{Q \in Sspec(P) : Q \in V^{'}(I) \cap V^{'}(a)\}$. Gorden Mason studied z-ideals of commutative rings [7] and F. Azarpanah, et.al., [1] studied z^0-ideals for a commutative reduced rings. Following [1] and [7], let J be an ideal of P. An ideal I of P containing J is called z^J-ideal if for each $a \in I$, we have $P_a(J) \subseteq I$. Also if I is a z^J - ideal of P, then $P_a(J) \neq P$ for any $a \in I$. Clearly every strongly prime ideal of P is z^J- ideal. But there exists a z^J-ideal in P which is not strongly prime ideal of P shows in the following example.

Example 1.2. Consider $P = \{0, a, b, c, d\}$ and define a relation $\leq$ on P as follows.



Then $(P, \leq)$ is a poset and $I_1 = \{0, a, b\}$ and $I_2 = \{0, a, c, d\}$ are the strongly prime ideals of P. If we take $I = \{0, a\}$ and $J = \{0\}$, then $P_a(J) \subseteq I$ for all $a \in I$ which implies I is a z^J-ideal, but not a strongly prime ideal of P. □

2. Main Results

Theorem 2.1. *Let J be an ideal of P and for $a \in P \backslash J$, M be a strongly m-system of P with $< a, J > \cap M = \phi$. Then $V^{'}(a) = D^{'}(< a, J >)$ and $D^{'}(a) = V^{'}(< a, J >)$.*

Proof. Let $Q \in V^{'}(a)$ and suppose that $Q \notin D^{'}(< a, J >)$. Let $S = \{I : I$ is an ideal of P and $J \subseteq I \subset Q$ with $I \cap M = \phi\}$. Then $S \neq \phi$ as $J \in S$ and by Zorn's lemma, there exists a maximal element R in S with $R \subset Q$ and $R \cap M = \phi$. We now claim that R is a strongly prime ideal of P. Let A and B be ideals of P with $A \nsubseteq R$ and $B \nsubseteq R$.

Case 1: If $A \nsubseteq Q$ and $B \nsubseteq Q$, then $L(A^*, B^*) \nsubseteq Q$. So $L(A^*, B^*) \nsubseteq R$.

Case 2: If $A \subseteq Q$ and $B \subseteq Q$. Then $A \cap M \neq \phi$ and $B \cap M \neq \phi$. Since M is strongly m-system, we have $L(A^*, B^*) \cap M \neq \phi$ which implies $L(A^*, B^*) \nsubseteq R$.

Case 3: Let $A \subseteq Q$ and $Q \subseteq B$. Then $A \cap M \neq \phi$ and $Q \cap M \neq \phi$. Since M is strongly m-system, we have $L(A^*, Q^*) \cap M \neq \phi$ which implies $L(A^*, B^*) \nsubseteq R$. So R is a strongly prime ideal of P with $R \subset Q$, which is again a contradiction to the minimality of Q. Thus $V^{'}(a) \subseteq D^{'}(< a, J >)$. Let $Q \in D^{'}(< a, J >)$. Then there exists $t \in < a, J > \backslash Q$ with $L(L(a)^*, L(t)^*) \subseteq L(a, t) \subseteq J \subseteq Q$ which implies $a \in L(a) \subseteq Q$. Thus $Q \in V^{'}(a)$. □

Lemma 2.2. *Let I and J be ideals of P with $J \subseteq I$. If I is z^J- ideal, then I is $z^{P(J)}$-ideal. If $P(J) \subseteq I$, then I is z^J- ideal if and only if I is $z^{P(J)}$-ideal.*

Proof. It follows from $\cap\{Q \in Sspec(P) : J \subseteq Q$ and $a \in Q\} = \cap\{Q \in Sspec(P) : P(J) \subseteq Q$ and $a \in Q\}$ □

Theorem 2.3. *Let I and J be proper ideals of P with $P(J) \subseteq I$ and for $a \in P\backslash J$, M be a strongly m-system of P with $< a, J > \cap M = \phi$. Then the following are equivalent.*

(i) *I is z^J - ideal.*

(ii) *$P_a(J) = P_b(J)$ and $b \in I$ imply that $a \in I$.*

(iii) *$V^{'}(a) \cap V^{'}(J) = V^{'}(b) \cap V^{'}(J)$ and $a \in I$ imply that $b \in I$.*

(iv) *$< a, P(J) >=< b, P(J) >$ and $a \in I\backslash P(J)$ imply $b \in I$.*

(v) *$a \in I\backslash P(J)$ implies $<< a, P(J) >, P(J) >\subseteq I$.*

(vi) *I is a $z^{P(J)}$-ideal of P.*

Proof. (**i**) ⇒ (**ii**) Let I be a z^J- ideal of P and $P_a(J) = P_b(J)$ with $b \in I$. Then $P_a(J) = P_b(J) \subseteq I$. So $a \in I$.

(**ii**) ⇒ (**iii**) Let $V^{'}(a) \cap V^{'}(J) = V^{'}(b) \cap V^{'}(J)$ and $a \in I$. Then $P_a(J) = P_b(J)$, by hypothesis, $b \in I$.

(**iii**) ⇒ (**iv**) Let $< a, P(J) >=< b, P(J) >$ and $a \in I\backslash J$. Then $D^{'}(< a, P(J) >) \cap V^{'}(J) = D^{'}(< b, P(J) >) \cap V^{'}(J)$. By Theorem 2.1, we have $V^{'}(a) \cap V^{'}(J) = V^{'}(b) \cap V^{'}(J)$. So by hypothesis, $b \in I$.

(**iv**) ⇒ (**v**) For $a \in I\backslash P(J)$, let $x \in << a, P(J) >, P(J) >$. Then $L(x, t) \subseteq P(J)$ for all $t \in < a, P(J) >$ which implies $< a, P(J) >\subseteq < x, P(J) >$. Let $s \notin < a, P(J) >$. Then $<< a, P(J) >, P(J) >\nsubseteq < s, P(J) >$. Suppose $s \in < x, P(J) >$, we have $x \in < t, P(J) >$ and $<< a, P(J) >, P(J) >\subseteq < t, P(J) >$, a contradiction. So $s \notin < x, P(J) >$ and $< a, P(J) >=< x, P(J) >$. By hypothesis, $x \in I$.

(**v**) ⇒ (**vi**) For each $a \in I$, we have $V^{'}(a) \cap V^{'}(P(J)) \subseteq << a, P(J) >, P(J) >$. By hypothesis $P_a(P(J)) \subseteq I$.

(**vi**) ⇒ (**i**) It follows from Lemma 2.2. □

Lemma 2.4. *Let J be an ideal of P and $a \in P\backslash P(J)$. Then $P_a(J) \subseteq \{b \in P :< a, P(J) >\subseteq < b, P(J) >\}$.*

Proof. It is trivial. □

Theorem 2.5. *Let I and J be ideals of P with $J \subseteq I$ and S be a subset of P with $S \nsubseteq I$. If I is z^J-ideal, then $< S, I >$ is a z^J- ideal of P.*

Proof: Let $a \in < S, I >$. Then $L(a, s) \subseteq I$ for all $s \in S$. Since I is a z^J- ideal of P, we have $P_r(J) \subseteq I$ for all $r \in L(a, s)$. Let $t \in P_a(J)$. Then $L(t, s) \subseteq P_a(J) \subseteq P_r(J) \subseteq I$ which implies $t \in < s, I >$ for all $s \in S$. Hence $< S, I >$ is a z^J-ideal of P. □

Theorem 2.6. *Let I be a strongly prime ideal of P and C a subset of P. Then $< C, I >$ is a strongly prime ideal of P.*

Proof. Let A and B be strongly prime ideals of P with $L(A^*, B^*) \subseteq < C, I >$. Then $< C, I >= P$ whenever $C \subseteq I$, hence in this case $< C, I >$ is strongly prime. If $C \nsubseteq I$, then by Theorem 2.10 of [2], $< C, I >= I$, is a strongly prime ideal of P. □

Theorem 2.7. *Let I and J be ideals of P with $J \subseteq I$, and A a subset of P. Then $< A, P(J) >$ is a z^J-ideal of P.*

Proof. We have $< A, P(J) >=< A, \bigcap_{Q \in Sspec(P) \cap V(J)} Q >= \bigcap_{Q \in Sspec(P) \cap V(J)} < A, Q >$. It follows from Theorem 2.6. □

Theorem 2.8. *Let $I_1, I_2, I_3, ...$ be z^J-ideals of P. If $K = \cup I_i$ is an ideal of P, then K is a z^J-ideal of P.*

Proof. Let $a \in K$. Then $a \in I_i$ for some i. Since each I_i is a z^J- ideal of P, we have $P_a(J) \subseteq I_i \subseteq K$. □

For an ideal I and a strongly prime ideal Q of P, we define $I_Q = \{x \in P : L(x, y) \subseteq I \text{ for some } y \in P \backslash Q\} = \bigcup_{y \in P \backslash Q} < y, I >$. It is clear that I_Q is a semi-ideal of P containing I. If I is a prime ideal of P, then $I_Q = I$.

Lemma 2.9. *Let I be an ideal of P and Q be a strongly prime ideal of P containing I. Then $I \subseteq I_Q \subseteq Q$.*

Proof. Let $x \in I_Q$. Then $L(x, t) \subseteq I \subseteq Q$ for some $t \in P \backslash Q$ which implies $L(L(x)^*, L(t)^*) \subseteq Q$. Since Q is strongly prime ideal, we have $x \in Q$. □

Theorem 2.10. *Let I be an ideal of P and Q be a strongly prime ideal of P containing I. If R is a strongly prime ideal of P with $I \subseteq R \subseteq Q$, then $I_Q \subseteq R$.*

Proof. By Lemma 2.9, $I \subseteq I_Q \subseteq Q$. Let R be a strongly prime ideal of P such that $I \subseteq R \subseteq Q$ and $x \in I_Q$. Then $L(x, t) \subseteq I \subseteq R$ for some $t \in P \backslash Q$ which implies $L(L(x)^*, L(t)^*) \subseteq R$. Since R is a strongly prime ideal of P and $t \notin R$, we have $x \in R$. □

Corollary 2.11. *Let I be an ideal of P and Q be a strongly prime ideal of P containing I. Then $I_Q \subseteq \cap\{Q_i : Q_i$ is a minimal strongly prime ideal of $I\}$.*

Theorem 2.12. *Let I be a z^J-ideal of P with $J \subseteq I$ and Q be a strongly prime ideal of P. If I is semi-prime, then I_Q is a z^J -ideal of P.*

Proof. Let I be a semi-prime ideal of P. Then by Theorem 15 of [6], $< t, I >$ is an ideal of P for any $t \in P$. We now claim I_Q is an ideal of P. Let $x, y \in I_Q$. Then $L(x, t_1) \subseteq I$ and $L(y, t_2) \subseteq I$ for some $t_1, t_2 \notin Q$. Since Q is prime ideal, there exists $t \in L(t_1, t_2) \backslash Q$ with $L(x, t) \subseteq I$ and $L(y, t) \subseteq I$ which imply $x, y \in < t, I >$ and $L(U(x, y)) \subseteq < t, I >$. Since $< t, I > \subseteq I_Q$, we have $L(U(x, y)) \subseteq I_Q$. So I_Q is an ideal of P. Since I is z^J- ideal and by Theorem 2.5, we have $< s, I >$ is z^J-ideal of P. By Theorem 2.8, $I_Q = \bigcup_{s \in P \backslash Q} < s, I >$ is a z^J-ideal. □

Theorem 2.13. *Let I be an ideal of P and Q be a strongly prime ideal of P containing I. If I is strongly prime, then I_Q is a strongly prime ideal of P.*

Proof. By Theorem 2.10, it is clear that $I_Q = \bigcup_{s \in P \backslash Q} < s, I >$ is a union of strongly prime ideal and I_Q is an ideal of P. Hence I_Q is a strongly prime ideal of P. □

Theorem 2.14. *Let I and J be ideals of P with $J \subseteq I$. If I is a prime ideal and Q is a strongly prime ideal of P, then I_Q is a prime z^J- ideal of P and $< r, I >= I_Q$ for all $r \in P \backslash Q$.*

Proof. Let I be a prime ideal of P. Then $I_Q = I$ and by Theorem 2.12, I_Q is a z^J-ideal of P which implies $I_Q = I \subseteq < r, I >$ for all $r \in P$. By the definition of I_Q, $< r, I > \subseteq I_Q$ for all $r \in P \backslash Q$. Hence $< r, I >= I_Q$ for all $r \in P \backslash Q$. □

Theorem 2.15. *Let I and J be ideals of P with $J \subseteq I$, and $R \subseteq P$. If I is a semi-prime and $< R, I >$ is the maximal element among the set $\{< S, I >:< S, I > \neq P$ and $S \subseteq P\}$, then I_Q is a prime z^J- ideal of P for some strongly prime ideal Q of P and $< r, I >= I_Q$ for some $r \in P \backslash Q$.*

Proof. By Theorem 2.4 of [3], there exists $r \in R \backslash I$ such that $< r, I >=< R, I >$ is a minimal strongly prime ideal of P containing ideal I. Since we have $I \subseteq < r, I > \subseteq Q$ for some strongly prime ideal Q of P with $r \notin Q$. By Theorem 2.10, $I_Q \subseteq < r, I >$. Also by the definition of I_Q, $< r, I > \subseteq I_Q$. Hence $< r, I >= I_Q$. □

Lemma 2.16. *Let P be a poset. Then $Sspec(P)$ is a T_0- space.*

Proof. Let R and Q be two distinct points in $Sspec(P)$. Then $R \nsubseteq Q$ and $Q \nsubseteq R$. Since $R \nsubseteq Q$, we have $a \in R$ such that $a \notin Q$ which implies $Q \in D(a)$ and $R \notin D(a)$. So there is an open set $D(a)$ containing Q, but not R. Similarly there exists an open set $D(b)$ containing P, but not Q for some $b \in Q \backslash R$. □

Theorem 2.17. *Let Q be a strongly prime ideal of P. Then $Sspec(P)$ is a Hausdorff space if and only if Q is a unique strongly prime ideal containing $P(P)_Q$.*

Proof Let $Sspec(P)$ be a Hausdorff space. Let Q and S be distinct strongly prime ideals of P such that $P(P)_Q \subseteq S$. Then by Lemma 2.16, there exists disjoint sets $D(x) \supseteq Q$ and $D(y) \supseteq S$ for some $y \in Q \backslash S$ and $x \in S \backslash Q$. Since $Sspec(P)$ is Hausdorff, we have $D(x) \cap D(y) = \phi$ which implies every strongly prime ideal of P contains either x or y. Then $L(x, y) \subseteq \cap\{R : R \in Sspec(P)\}$ and $L(x, y) \subseteq P(P)$ which implies $y \in P(P)_Q \subseteq S$, a contradiction. Thus $P(P)_Q \nsubseteq S$. Let Q be the unique strongly prime ideal containing $P(P)_Q$. If $S \neq Q$ is a strongly prime ideal of P, then there exists $x \in P(P)_Q \backslash S$ which implies $L(x, y) \subseteq P(P)$ for some $y \notin Q$. Since S is strongly prime, we have $y \in S$ and $D(x)$ and $D(y)$ are the open neighborhoods of S and Q respectively. Then $D(x) \cap D(y) = D(L(x) \cap L(y)) \subseteq D(L(x, y)) = \phi$. $\square$

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