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(Σ, Δ)-Compatible Skew PBW Extension Ring

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Abstract. Ever since their introduction, skew PBW (Poincaré-Birkhoff-Witt) extensions of rings have kept growing in importance, as researchers characterized their properties (such as primeness, Krull and Goldie dimension, homological properties, etc.) in terms of intrinsic properties of the base ring, and studied their relations with other fields of mathematics, as for example quantum mechanics theory. Many rings and algebras arising in quantum mechanics can be interpreted as skew PBW extensions. Our aim in this paper is to study skew PBW extensions of Baer, quasi-Baer, principally projective and principally quasi-Baer rings, in the case when the base ring R is not assumed to be reduced. We just impose some mild compatibleness over the base ring R, and prove that these properties are stable over this kind of extensions.

1. Introduction and Preliminary Definitions

Throughout this paper, R denotes an associative ring with unity. Recall from Kaplansky [18] and Clark [9] that R is a *Baer* (resp., *quasi-Baer*) ring if the right annihilator of every non-empty subset (resp., ideal) of R is generated, as a right

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ideal, by an idempotent. Baer rings are introduced by Kaplansky to abstract various properties of von-Neumann algebras and complete -regular rings. Clark uses the quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Another generalization of Baer rings are the p.p.-rings. A ring R is called *right* (resp., *left*) *p.p.* if the right (resp., left) annihilator of each element of R is generated by an idempotent (or equivalently, rings in which each principal right (resp., left) ideal is projective). In [6], Birkenmeier et al. define a ring to be called a right (resp., left) *principally quasi-Baer* (or simply *right* (resp., *left*) *p.q.-Baer*) ring if the right annihilator of each principal right (resp., left) ideal of R is generated by an idempotent.

Pollingher and Zaks [24], showed that the class of quasi-Baer rings is closed under $n \times n$ matrix rings and under $n \times n$ upper (or lower) triangular matrix rings. It follows from this results that quasi-Baer condition is a Morita invariant property. For further works on quasi-Baer rings we direct the reader to consult the papers [4, 5, 6, 7, 8, 9, 11, 12, 13, 21, 22, 24].

There is considerable interest in studying if and how certain properties of rings are preserved under various ring-theoretic extensions. Armendariz [1] seems to be the first to consider the behavior of a polynomial rings over a Baer ring by obtaining the following result (recall that a ring is *reduced* if it has no nonzero nilpotent elements): For a reduced ring R, the polynomial ring $R[x]$ is a Baer ring if and only if R is a Baer ring [1, Theorem B]. Armendariz provided an example to show that the reduced condition was not superfluous. Note that if R is a reduced ring, then R is Baer if and only if R is quasi-Baer. A generalization of Armendariz's result for several types of polynomial extensions over Baer and quasi-Baer rings, are obtained by various authors, [4, 5, 7, 11, 13, 14, 17] . In [7] Birkenmeier et al. showed that the quasi-Baer condition is preserved by many polynomial extensions. Also, Birkenmeier et al. [4] showed that a ring R is right p.q.-Baer if and only if $R[x]$ is right p.q.-Baer.

Let σ be an endomorphism of R and δ be an σ-derivation of R (so δ is an additive map satisfying $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$), the general (left) Ore extension $R[x; \sigma, \delta]$ is the ring of polynomials over R in the variable x, with coefficients written on the left of x and with termwise addition, subject to the skew-multiplication rule $xr = \sigma(r)x + \delta(r)$ for $r \in R$. If σ is an injective endomorphism of R, then we say $R[x; \sigma, \delta]$ is an Ore extension of injective type. If σ is an identity map on R or $\delta = 0$, then we denote $R[x; \sigma, \delta]$ by $R[x; \delta]$ and $R[x; \sigma]$, respectively.

According to Krempa [19], an endomorphism σ of a ring R is called to be *rigid* if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is said to be σ-rigid if there exists a rigid endomorphism σ of R. Note that any rigid endomorphism of a ring is a monomorphism and σ-rigid rings are reduced by Hong et al. [17]. Properties of σ-rigid rings have been studied in Krempa [19], Hirano [16] and Hong et al. [17]. In [17] Hong et al. studied Ore extensions of quasi-Baer rings over σ-rigid rings.

Hashemi and Moussavi [11], used a condition of being (σ, δ)-*compatible* for an endomorphism σ and a σ-derivation δ of R. A ring R is called σ-*compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\sigma(b) = 0$. Moreover, R is said to be *δ-compatible* if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If R is both σ-compatible and δ-compatible, we say that R is *(σ, δ)-compatible*. In this case, clearly the endomorphism σ is injective. Thus the σ-compatible ring is a generalization of σ-rigid ring to the more general case where R is not assumed to be reduced. Under these conditions, they proved that a ring R is quasi-Baer (resp., p.q.-Baer) if and only if $R[x; \sigma, \delta]$ is quasi-Baer (resp., p.q.-Baer). Further work on Ore extensions over Baer and quasi-Baer rings appeared in [11, 12, 13, 19, 21, 22].

Another ring-theoretic extensions of a ring R are the Poincaré-Birkhoff-Witt (PBW for short) were defined by Bell and Goodearl [3]. The skew Poincaré-Birkhoff-Witt (skew PBW for short) extensions introduced by Gallego and Lezema [10] are a generalization of PBW extensions, which are more general than Ore extensions of injective type. These extensions include several algebras which can not be expressed as Ore extensions (universal enveloping algebras of finite Lie algebras, diffusion algebras, etc.). More exactly, it has been shown that skew PBW extensions contain various well-known groups of algebras such as some types of Auslander-Gorenstein rings, some skew Calabi-Yau algebras, quantum polynomials, some quantum universal enveloping algebras, etc. (see [10, 25]).

Since all aforementioned extensions of a ring R are particular cases of the skew PBW construction, hence any result on this construction has its counterpart in each of the mentioned classes, and these counterparts follow immediately from a single proof. This connection is a good reason to study the ring theoretical properties of the skew PBW extensions. So, it is natural to ask when the properties like Baerness, quasi-Baernes, p.q.-Baerness and principally projectivness of a ring R can be extended to the skew PBW extensions. Reyes [27], studied the behavior of skew PBW extensions over a Baer, quasi-Baer, p.p. and p.q.-Baer ring, when R is assumed to be rigid (and hence is reduced) ring.

In this paper, we study skew Poincaré-Birkhoff-Witt extensions of Baer, quasi-Baer, p.p. and p.q.-Baer rings, over a general non-reduced ring R. We just impose some compatibleness over the base ring R, and prove that these properties are stable over this kind of extensions.

For a nonempty subset U of R, $r_R(U)$ and $\ell_R(U)$ denote the right and left annihilators of U in R, respectively (if it is clear from the context, the subscript will be omitted).

2. Definitions and Basic Properties of Skew PBW Extensions

We start by recalling the definition of (skew) PBW extensions and present some key properties of these rings.

Let R and A be rings. According to Bell and Goodearl [3], we say that A is a Poincaré-Birkhoff-Witt extension (also called a *PBW extension*) of R, denoted by $A := R\langle x_1, \ldots, x_n\rangle$, if the following conditions hold:

(1) $R \subseteq A$;

(2) There exist elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the basic elements $\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n\}$.

(3) $x_i r - r x_i \in R$ for each $r \in R$ and $1 \leq i \leq n$.

(4) $x_i x_j - x_j x_i \in R + Rx_1 + \cdots + Rx_n$, for any $1 \leq i, j \leq n$.

Definition 2.1.([10, Definition 1]) Let R and A be rings. We say that A is a *skew PBW extension of* R (also called a *σ-PBW extension*) if the following conditions hold:

(1) $R \subseteq A$;

(2) There exist elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the basic elements $\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n\}$.

(3) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(4) For any elements $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$.

Under these conditions we will write $A := \sigma(R) \langle x_1, \dots, x_n \rangle$.

Clearly any PBW extension is a skew PBW extension. Observe that if σ is an injective endomorphism of the ring R and δ is a σ-derivation, then the skew polynomial ring $R[x; \sigma, \delta]$ is a trivial skew PBW extension in only one variable, $\sigma(R) \langle x \rangle$. Many important class of rings and algebras are skew PBW extensions, for example:

Example 2.2. Skew polynomial ring of derivation type $R[x; \delta]$, Ore extension of derivation type $R[x_1; \delta_1] \cdots R[x_n; \delta_n]$, Ore algebra of derivation type $R = \mathbf{k}[t_1, \dots, t_m]$, Weyl algebra $A_n(\mathbf{k})$, Extended Weyl algebra $B_n(\mathbf{k})$, Universal enveloping algebra of Lie algebra $\mathfrak{g}, \mathcal{U}(\mathfrak{g})$, Quantum plane $\mathcal{O}_q(\mathbf{k}^n)$, The algebra of q-differential operators $D_{q,h}[x, y]$, are particular examples of skew PBW extensions. A detailed list of examples of skew PBW extensions is presented in [10, 20, 25, 26].

Now we give some examples of skew PBW extensions which can not be expressed as Ore extensions (a more complete list can be found in [20, 25]).

Example 2.3.

(1) Let k be a commutative ring and $\mathfrak{g}$ a finite dimensional Lie algebra over k with basis $\{x_1, \dots, x_n\}$; the *universal enveloping algebra* of $\mathfrak{g}$, denoted by $\mathcal{U}(\mathfrak{g})$, is a PBW extension of k (see [20]). In this case, $x_i r - r x_i = 0$ and $x_i x_j - x_j x_i = [x_i, x_j] \in \mathfrak{g} = k + kx_1 + \cdots + kx_n$, for any $r \in k$ and $1 \leq i, j \leq n$.

(2) The *universal enveloping ring* $\mathcal{U}(V, R, \mathbf{k})$ introduced by Passman [23], where R is a $\mathbf{k}$-algebra, and V is a $\mathbf{k}$-vector space which is also a Lie ring containing R and $\mathbf{k}$ as Lie ideals with suitable relations. The enveloping ring $\mathcal{U}(V, R, \mathbf{k})$ is a finite skew PBW extension of R if $dim_{\mathbf{k}}(V/R)$ is finite.

(3) Diffusion algebras arise in physics as a possible way to understand a large class of 1-dimensional stochastic process. A *diffusion algebra* $\mathcal{A}$, is generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over $\mathbf{k}$ with relations

$$x_i x_j = x_j x_i, \quad x_i D_j = D_j x_i, \quad 1 \leq i, j \leq n.$$

$$c_{ij} D_i D_j - c_{ji} D_j D_i = x_j D_i - x_i D_j, \quad i < j, c_{ij}, c_{ji} \in K^*.$$

Thus, $\mathcal{A} \cong \sigma(\mathbf{k}[x_1, \ldots, x_n]) \langle D_1, \ldots, D_n \rangle$ (see [20]).

Proposition 2.4.([10, Proposition 3]) *Let A be a skew PBW extension of R. For each $1 \leq i \leq n$, there exists an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, for each $r \in R$.*

Let A be a skew PBW extension of R. According to [10, Definition 4], A is called *bijective* if σ_i is bijective for each $1 \leq i \leq n$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

Definition 2.5.([10, Definition 6]) Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \leq i \leq n$ and σ_i-derivations δ_i as in Proposition 2.4.

(1) For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n, \sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}, \delta^\alpha := \delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n}, |\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$; then $\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.

(2) For $X = x^\alpha \in \mathrm{Mon}(A), \exp(X) := \alpha$ and $\deg(X) := |\alpha|$. The symbol $\succeq$ will denote a total order defined on $\mathrm{Mon}(A)$ (a total order on $\mathbb{N}_0^n$). For an element $x^\alpha \in \mathrm{Mon}(A), \exp(x^\alpha) := \alpha \in \mathbb{N}_0^n$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$.
Every element $f \in A$ can be expressed uniquely as $f = a_0 + a_1 X_1 + \cdots + a_m X_m$, with $a_i \in R \backslash \{0\}$, and $X_m \succ \cdots \succ X_1$. With this notation, we define $lm(f) := X_m$, the *leading monomial* of $f; lc(f) := a_m$, the *leading coefficient* of f; $lt(f) := a_m X_m$, the *leading term* of f; $\exp(f) := \exp(X_m)$, the *order* of f; and $E(f) := \{\exp(X_i) \mid 1 \leq i \leq t\}$. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$. Finally, if $f = 0$, then $lm(0) := 0, lc(0) := 0, lt(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$.

Remark 2.6.([10, Remark 2])

(1) Since that $\mathrm{Mon}(A)$ is a R-basis for A, the elements $c_{i,r}$ and $c_{i,j}$ in the Definition 2.1 are unique.

(2) If $r = 0$, then $c_{i,0} = 0$. Moreover, in Definition 2.1(4), $c_{i,i} = 1$.

(3) Let $i < j$, there exist $c_{j,i}, c_{i,j} \in R$ such that $x_ix_j - c_{j,i}x_jx_i \in R + Rx_1 + \cdots + Rx_n$ and $x_jx_i - c_{i,j}x_ix_j \in R + Rx_1 + \cdots + Rx_n$, but since $\mathrm{Mon}(A)$ is a R-basis then $1 = c_{j,i}c_{i,j}$, i.e., for every $1 \leq i < j \leq n$, $c_{i,j}$ has a left inverse and $c_{j,i}$ has a right inverse.

(4) Each element $f \in A \setminus \{0\}$ has a unique representation in the form $f = a_0 + a_1X_1 + \cdots + a_mX_m$, with $a_i \in R \setminus \{0\}$ and $X_i \in \mathrm{Mon}(A)$, $1 \leq i \leq m$.

Skew PBW extensions can be characterized in the following way.

Theorem 2.7.([10, Theorem 7]) *Let A be a polynomial ring over R with respect to $\{x_1, \ldots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions are satisfied:*

(1) *For each $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}, p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α.*

(2) *For each $x^\alpha, x^\beta \in \mathrm{Mon}(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = c_{\alpha,\beta}x^{\alpha+\beta} + p_{\alpha,\beta}$, where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.*

We remember also the following facts from [10, Remark 8].

Remark 2.8.

(1) A left inverse of $c_{\alpha,\beta}$ will be denoted by $c'_{\alpha,\beta}$. We observe that if $\alpha = 0$ or $\beta = 0$, then $c_{\alpha,\beta} = 1$ and hence $c'_{\alpha,\beta} = 1$.

(2) From Theorem 2.8, we get also that if A is a bijective skew PBW extension, then $c_{\alpha,\beta}$ is invertible for any $\alpha, \beta \in \mathbb{N}_0^n$.

In the next Remark we will look more closely at the form of the polynomials $p_{\alpha,r}$ and $p_{\alpha,\beta}$ which appear in Theorem 2.7.

Remark 2.9.([27, Remark 2.10])

(1) Let x_n be a variable and α_n an element of $\mathbb{N}_0$. Then we have

$$x_n^{\alpha_n} r = \sigma_n^{\alpha_n}(r)x_n^{\alpha_n} + \sum_{j=1}^{\alpha_n} x_n^{\alpha_n - j}\delta_n(\sigma_n^{j-1}(r))x_n^{j-1}, \quad \sigma_n^0 := id_R \tag{2.1}$$

and so

$$\begin{aligned} x_n^{\alpha_n} r = {} & \sigma_n^{\alpha_n}(r)x_n^{\alpha_n} + x_n^{\alpha_n-1}\delta_n(r) \\ & + x_n^{\alpha_n-2}\delta_n(\sigma_n(r))x_n + x_n^{\alpha_n-3}\delta_n(\sigma_n^2(r))x_n^2 \\ & + \cdots + x_n\delta_n(\sigma_n^{\alpha_n-2}(r))x_n^{\alpha_n-2} + \delta_n(\sigma_n^{\alpha_n-1}(r))x_n^{\alpha_n-1}, \quad \sigma_n^0 := id_R. \end{aligned}$$

Note that

$$\begin{aligned}p_{\alpha_n,r} &= x_n^{\alpha_n-1}\delta_n(r)\\ &\quad + x_n^{\alpha_n-2}\delta_n(\sigma_n(r))x_n\\ &\quad + x_n^{\alpha_n-3}\delta_n(\sigma_n^2(r))x_n^2\\ &\quad + \cdots + x_n\delta_n(\sigma_n^{\alpha_n-2}(r))x_n^{\alpha_n-2} + \delta_n(\sigma_n^{\alpha_n-1}(r))x_n^{\alpha_n-1},\end{aligned}$$

where $p_{\alpha_n,r}=0$ or $\deg(p_{\alpha_n,r})<|\alpha_n|$ if $p_{\alpha_n,r}\neq 0$. It is clear that $\exp(p_{\alpha_n,r})\prec\alpha_n$. Again, using (2.1) in every term of the product $x_n^{\alpha_n}r$ above, we obtain

$$\begin{aligned}x_n^{\alpha_n}r &= \sigma_n^{\alpha_n}(r)x_n^{\alpha_n}\\ &\quad + \sigma_n^{\alpha_n-1}(\delta_n(r))x_n^{\alpha_n-1} + \sum_{j=1}^{\alpha_n-1} x_n^{\alpha_n-1-j}\delta_n(\sigma_n^{j-1}(\delta_n(r)))x_n^{j-1}\\ &\quad + \left[\sigma_n^{\alpha_n-2}(\delta_n(\sigma_n(r)))x_n^{\alpha_n-2} + \sum_{j=1}^{\alpha_n-2} x_n^{\alpha_n-2-j}\delta_n(\sigma_n^{j-1}(\delta_n(\sigma_n(r))))x_n^{j-1}\right]x_n\\ &\quad + \left[\sigma_n^{\alpha_n-3}(\delta_n(\sigma_n^2(r)))x_n^{\alpha_n-3} + \sum_{j=1}^{\alpha_n-3} x_n^{\alpha_n-3-j}\delta_n(\sigma_n^{j-1}(\delta_n(\sigma_n^2(r))))x_n^{j-1}\right]x_n^2\\ &\quad + \cdots + \left[\sigma_n(\delta_n(\sigma_n^{\alpha_n-2}(r)))x_n + \delta_n(\delta_n(\sigma_n^{\alpha_n-2}(r)))\right]x_n^{\alpha_n-2} + \delta_n(\sigma_n^{\alpha_n-1}(r))x_n^{\alpha_n-1},\end{aligned}$$

which shows that

$$lc(p_{\alpha_n,r}) = \sum_{p=1}^{\alpha_n}\sigma_n^{\alpha_n-p}(\delta_n(\sigma_n^{p-1}(r))).$$

In this way, we can see that $lc(p_{\alpha_n,r})$ involves elements obtained evaluating σ_n and δ_n in the element r of R.

(2) Let $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}_0^n$, $r\in R$ and $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. Then

$$\begin{aligned}x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}r &= \sigma_1^{\alpha_1}(\cdots(\sigma_n^{\alpha_n}(r)))x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}\\ &\quad + p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))}x_2^{\alpha_2}\cdots x_n^{\alpha_n}\\ &\quad + x_1^{\alpha_1}p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))}x_3^{\alpha_3}\cdots x_n^{\alpha_n}\\ &\quad + x_1^{\alpha_1}x_2^{\alpha_2}p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))}x_4^{\alpha_4}\cdots x_n^{\alpha_n}\\ &\quad + \cdots + x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-2}^{\alpha_{n-2}}p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)}x_n^{\alpha_n}\\ &\quad + x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}}p_{\alpha_n,r}.\end{aligned}$$

Considering the leading coefficients of $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}r$ we can write this term as

$$
\begin{aligned}
&= \sigma_1^{\alpha_1}(\cdots(\sigma_n^{\alpha_n}(r)))x_1^{\alpha_1}\cdots x_n^{\alpha_n}\\
&+\left[\sum_{p=1}^{\alpha_1}\sigma_1^{\alpha_1-p}(\delta_1(\sigma_1^{p-1}(\sigma_2^{\alpha_2}(\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))))))\right]x_1^{\deg(p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))})}x_2^{\alpha_2}\cdots x_n^{\alpha_n}\\
&+\left[\sum_{p=1}^{\alpha_2}\sigma_1^{\alpha_1}(\sigma_2^{\alpha_2-p}(\delta_2(\sigma_2^{p-1}(\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))))))\right]x_1^{\alpha_1}x_2^{\deg(p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))})}x_3^{\alpha_3}\cdots x_n^{\alpha_n}\\
&+\left[\sum_{p=1}^{\alpha_3}\sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\sigma_3^{\alpha_3-p}(\delta_3(\sigma_3^{p-1}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r))))))))\right]x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\deg(p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))})}\\
&\quad x_4^{\alpha_4}\cdots x_n^{\alpha_n}+\cdots\\
&+\left[\sum_{p=1}^{\alpha_{n-1}}\sigma_1^{\alpha_1}(\cdots(\sigma_{n-2}^{\alpha_{n-2}}(\sigma_{n-1}^{\alpha_{n-1}-p}(\delta_{n-1}(\sigma_{n-1}^{p-1}(\sigma_n^{\alpha_n}(r)))))))\right]x_1^{\alpha_1}\cdots x_{n-2}^{\alpha_{n-2}}\\
&\quad x_{n-1}^{\deg(p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)})}x_n^{\alpha_n}\\
&+\left[\sum_{p=1}^{\alpha_n}\sigma_1^{\alpha_1}(\cdots(\sigma_{n-1}^{\alpha_{n-1}}(\sigma_n^{\alpha_n-p}(\delta_n(\sigma_n^{p-1}(r))))))\right]x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\deg(p_{\alpha_n},r)}\\
&+\text{other terms of degree less than }\deg(p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))})+\alpha_2+\cdots+\alpha_n\\
&+\text{other terms of degree less than }\alpha_1+\deg(p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))})+\alpha_3+\cdots+\alpha_n\\
&+\text{other terms of degree less than }\alpha_1+\alpha_2+\deg(p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))})+\alpha_4\\
&+\cdots+\alpha_n\\
&\ \vdots\\
&+\text{other terms of degree less than }\alpha_1+\cdots+\alpha_{n-2}+\deg(p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)})+\alpha_n\\
&+\text{other terms of degree less than }\alpha_1+\cdots+\alpha_{n-1}+\deg(p_{\alpha_n,r}).
\end{aligned}
$$

Therefore we can see that the polynomials $p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))}, p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))}$, $p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))}, \ldots, p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)}$, and $p_{\alpha_n,r}$ in the expression above for the term $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}r$, involve elements obtained evaluating σ's and δ's in the element r of R.

(3) Let $X_i := x_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}$, $Y_j := x_1^{\beta_{j1}}\cdots x_n^{\beta_{jn}}$ and $a_i, b_j \in R$. Then

$$
\begin{aligned}
a_iX_ib_jY_j &= a_i\sigma^{\alpha_i}(b_j)x^{\alpha_i}x^{\beta_j} + a_ip_{\alpha_{i1},\sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_2^{\alpha_{i2}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
&+ a_ix_1^{\alpha_{i1}}p_{\alpha_{i2},\sigma_3^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_3^{\alpha_{i3}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
&+ a_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}p_{\alpha_{i3},\sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_4^{\alpha_{i4}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
&+\cdots+a_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}\cdots x_{i(n-2)}^{\alpha_{i(n-2)}}p_{\alpha_{i(n-1)},\sigma_{in}^{\alpha_{in}}(b_j)}x_n^{\alpha_{in}}x^{\beta_j}\\
&+ a_ix_1^{\alpha_{i1}}\cdots x_{i(n-1)}^{\alpha_{i(n-1)}}p_{\alpha_{in},b_j}x^{\beta_j}
\end{aligned}
$$

As we saw above, the polynomials $p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))}$, $p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))}$, $p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))}, \ldots, p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)}$, and $p_{\alpha_n,r}$, involve elements of R obtained evaluating σ_j and δ_j in the element r of R. So, when we compute every summand of $a_iX_ib_jY_j$ we obtain products of the coefficient a_i with several evaluations of b_j in σ's and δ's depending of the coordinates of α_i.

3. (Σ, Δ)-Compatible Skew PBW Extension Rings

Throughout this section, let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of a ring R. Let $\Delta := \{\delta_1, \ldots, \delta_n\}$ be the $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ derivations as mentioned in the proposition 2.4. According to Reyes [27], Σ is called a *rigid endomorphisms family* if $a\sigma^\alpha(a) = 0$ implies $a = 0$ for each $a \in R$ and $\alpha \in \mathbb{N}_0^n$, where σ^α is as mentioned in the Definition 2.5. A ring R is called Σ-*rigid* if there exists a rigid endomorphisms family Σ of R. Since Ore extensions of injective type are particular examples of skew PBW extensions, the concepts of Baer, quasi-Baer, p.p. and p.q.-Baer are interesting for the ring theoretical study of skew PBW extensions. Hence, in this section we generalize the results presented in [27], with the purpose of establishing necessary and sufficient conditions to guarantee that these concepts are stable under skew PBW extensions.

Definition 3.1. Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of a ring R. Let $\Delta := \{\delta_1, \ldots, \delta_n\}$ be the $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ derivations as mentioned in the proposition 2.4. We say that R is Σ-*compatible* if for each $a, b \in R$ and $\alpha \in \mathbb{N}_0^n$, $ab = 0 \Leftrightarrow a\sigma^\alpha(b) = 0$. Moreover, R is said to be Δ-*compatible* if for each $a, b \in R$ and $\alpha \in \mathbb{N}_0^n$, $ab = 0 \Rightarrow a\delta^\alpha(b) = 0$, where σ^α and δ^α are as mentioned in the Definition 2.5. If R is both Σ-*compatible* and Δ-*compatible*, we say that R is (Σ, Δ)-*compatible*.

The definition is quite natural, in the light of its similarity with the notion of Σ-rigid rings, where in Lemma 3.5, we will show that R is Σ-rigid if and only if R is Σ-compatible and reduced. Thus the Σ-compatible ring is a generalization of Σ-rigid ring to the more general case where R is not assumed to be reduced.

In the following, we give some examples of skew PBW extension $A = \sigma(R)$ $\langle x_1, \ldots, x_n\rangle$ which R satisfies the (Σ, Δ)-compatible conditions.

Example 3.2.

(1) ([12, Example 1.2]) Let δ be an σ-derivation of R and R be an σ-rigid ring. Let

$$R_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in R \right\}$$

be a subring of $T_3(R)$. The endomorphism σ of R is extended to the endomorphism $\overline{\sigma} : R_3 \to R_3$ defined by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ and the σ-derivation δ of R is also extended to $\overline{\delta} : R_3 \to R_3$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$ which $\overline{\delta}$

is an $\overline{\delta}$-derivation of R_3. Then R_3 is a $(\overline{\sigma}, \overline{\delta})$-compatible ring. Note that any skew polynomial ring $R_3[x; \overline{\sigma}, \overline{\delta}]$, with $\overline{\sigma}$ injective, is a skew PBW extension. In this case we have $R_3[x; \overline{\sigma}, \overline{\delta}] = \overline{\sigma}(R_3)\langle x\rangle$.

(2) Let R be a domain and σ be the automorphism on the polynomial ring $R[x, y]$ in two indeterminates x, y, given by $\sigma(x) = y$ and $\sigma(y) = x$. Then $R[x, y]$ is a σ-rigid ring. Hence by Lemma 3.5, $R[x, y]$ is a σ-compatible ring. Also any skew polynomial ring $R[x, y][z; \sigma]$, with σ bijective, is a bijective skew PBW extension. In this case we have $R[x, y][z; \sigma] = \sigma(R[x, y])\langle z\rangle$.

(3) Let $\mathbf{k}$ be a field, the $\mathbf{k}$-algebra $A_n(q_1, \ldots, q_n)$ is generated by $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ and subject to the relations:

$$\begin{aligned} &x_j x_i = x_i x_j, \quad y_j y_i = y_i y_j, \quad 1 \le i, j \le n, \\ &y_i x_j = x_j y_i, \quad i \ne j, \\ &y_i x_i = q_i x_i y_i + 1, \quad 1 \le i \le n, \end{aligned}$$

where $q_i \in \mathbf{k} - \{0\}$. We observe that $A_n(q_1, \ldots, q_n)$ is isomorphic to the iterated skew polynomial ring $\mathbf{k}[x_1, \ldots, x_n][y_1; \sigma_1, \delta_1] \ldots [y_n; \sigma_n, \delta_n]$ over the commutative polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$:

$$\begin{aligned} &\sigma_j(y_i) := y_i, \quad \delta_j(y_i) := 0, \quad 1 \le i, j \le n, \\ &\sigma_i(x_j) := x_j, \quad \delta_i(x_j) := 0, \quad 1 \le i, j \le n, \\ &\sigma_i(x_i) := q_i x_i, \quad \delta_i(x_i) := 1, \quad 1 \le i \le n. \end{aligned}$$

Thus

$$A_n(q_1, \ldots, q_n) \cong \sigma(\mathbf{k}[x_1, \ldots, x_n])\langle y_1, \ldots, y_n\rangle .$$

Also it is easy to see that $\mathbf{k}[x_1, \ldots, x_n]$ is a (Σ, Δ)-compatible ring.

In the following, we mention some properties of (Σ, Δ)-compatible rings.

Lemma 3.3. *Let R be a (Σ, Δ)-compatible ring. Then we have the following:*

(1) *If $ab = 0$ then $a\sigma^{\alpha}(b) = \sigma^{\alpha}(a)b = 0$ for each $\alpha \in \mathbb{N}_0^n$.*

(2) *If $ab = 0$ then $a\delta^{\beta}(b) = \delta^{\beta}(a)b = 0$ for each $\beta \in \mathbb{N}_0^n$.*

(3) *If $ab = 0$ then $a\sigma^{\alpha}(\delta^{\beta}(b)) = a\delta^{\beta}(\sigma^{\alpha}(b)) = 0$ for each $\alpha, \beta \in \mathbb{N}_0^n$.*

(4) *If $a\sigma^{\theta}(b) = \sigma^{\theta}(a)b = 0$ for some $\theta \in \mathbb{N}_0^n$, then $ab = 0$.*

Proof. (1) It is sufficient to prove that if $ab = 0$, then $a\sigma_t(b) = \sigma_t(a)b = 0$ for every $1 \le t \le n$. If $ab = 0$, then $\sigma_t(a)\sigma_t(b) = 0$ and hence by Σ-compatibility of R, $\sigma_t(a)(b) = 0$ for every $1 \le t \le n$.

(2) Similar to above, it is sufficient to prove that if $ab = 0$, then $a\delta_t(b) = \delta_t(a)b = 0$ for every $1 \le t \le n$. If $ab = 0$, then by (1) and Δ-compatibility of R, $\sigma_t(a)\delta_t(b) = 0$. Hence $\delta_t(a)b = \delta_t(ab) - \sigma_t(a)\delta_t(b) = 0$.

(3) It follows from (1) and (2).

(4) Suppose that $a\sigma^{\theta}(b) = 0$ for some $\theta \in \mathbb{N}_0^n$. Then by (1) we have $\sigma^{\theta}(ab) = \sigma^{\theta}(a)\sigma^{\theta}(b) = 0$. Since σ^{θ} is injective, $ab = 0$. Similarly, one can see that if $\sigma^{\theta}(a)b = 0$ for some $\theta \in \mathbb{N}_0^n$, then $ab = 0$. □

Corollary 3.4. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of a ring R. If R is (Σ, Δ)-compatible and $ab = 0$ for $a, b \in R$, then $ax^{\alpha}bx^{\beta} = 0$ in A for every $\alpha, \beta \in \mathbb{N}_0^n$.*

Proof. It follows from Lemma 3.3 and Remark 2.9. □

Lemma 3.5. *Let Σ be a family of endomorphisms and Δ be the Σ-derivations of a ring R. Then R is (Σ, Δ)-compatible and reduced if and only if R is Σ-rigid.*

Proof. We only need to prove that for each an endomorphism σ_t and σ_t-derivation δ_t, R is (σ_t, δ_t)-compatible and reduced if and only if R is σ_t-rigid for every $1 \leq t \leq n$. Let R be a (σ_t, δ_t)-compatible and reduced for every $1 \leq t \leq n$ and $r \in R$ such that $r\sigma_t(r) = 0$. Then we have $\sigma_t(r)\sigma_t(r) = 0$ for every $1 \leq t \leq n$ by Lemma 3.3. Since σ_t is a monomorphism for every $1 \leq t \leq n$ and R is reduced, $r = 0$. Conversely, let R be a σ_t-rigid ring for every $1 \leq t \leq n$. Since σ_t-rigid rings are reduced, $ab = 0$ if and only if $ba = 0$. Then $a\sigma_t(b)\sigma_t(a\sigma_t(b)) = a\sigma_t(ba){\sigma_t}^2(b) = 0$ and also Σ-rigidity of R yields $a\sigma_t(b) = 0$ for every $1 \leq t \leq n$. Similarly, one can see that $ba = 0$ implies that $\sigma_t(a)b = 0$ for every $1 \leq t \leq n$. Now suppose that $a\sigma_t(b) = 0$, then $ba\sigma_t(ba) = 0$ for every $1 \leq t \leq n$ and hence $ab = ba = 0$, Since R is σ_t-rigid. On the other hand, from $ab = 0$ we have $\delta_t(ba) = \delta_t(b)a + \sigma_t(b)\delta_t(a) = 0$ for every $1 \leq t \leq n$. Multiplying $\sigma_t(b)\delta_t(a)$ from right-hand side of the above, we have $(\sigma_t(b)\delta_t(a))^2 = -\delta_t(b)a\sigma_t(b)\delta_t(a) = 0$ for every $1 \leq t \leq n$. Since R is reduced, $\sigma_t(b)\delta_t(a)) = 0$, so $\delta_t(b)a = 0$ and hence $a\delta_t(b) = 0$ for every $1 \leq t \leq n$. □

Lemma 3.6. *Let R be a (Σ, Δ)-compatible ring and $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of a ring R. If $f = a_0 + a_1X_1 + \cdots + a_mX_m \in A$, $r \in R$ and $fr = 0$, then $a_ir = 0$ for each i.*

Proof. Consider $f = a_0 + a_1X_1 + \cdots + a_mX_m$, where $a_i \in R$, $1 \leq i \leq m$, $a_m \neq 0$, with $X_i = x^{\alpha_i} = x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$, and $X_m \succ X_{m-1} \succ \cdots \succ X_1$. By Theorem 2.7 (1) we have

$$\begin{aligned} fr &= (a_0 + a_1X_1 + \cdots + a_mX_m)r \\ &= \text{“lower terms”} + a_mX_mr \\ &= \cdots + a_m[\sigma^{\alpha_m}(r)x^{\alpha_m} + p_{\alpha_m,r}], \\ &= \cdots + a_mp_{\alpha_m,r} + a_m\sigma^{\alpha_m}(r)x^{\alpha_m}, \end{aligned}$$

where $p_{\alpha_m,r} = 0$ or $\deg(p_{\alpha_m,r}) < |\alpha_m|$ if $p_{\alpha_m,r} \neq 0$. From the $fr = 0$ we have $a_m\sigma^{\alpha_m}(r) = 0$ and Σ-compatibility gives $a_mr = 0$. By Remark 2.9, we can see that the polynomial $p_{\alpha_m,r}$ involve elements obtained evaluating σ's and δ's in the element r of R. Since R is (Σ, Δ)-compatible and using Lemma 3.3, we obtain $a_mp_{\alpha_m,r} = 0$. Hence $(a_0 + a_1X_1 + \cdots + a_{m-1}X_{m-1})r = 0$. By above argument and using induction on $|\alpha_m|$, we obtain $a_ir = 0$ for all $i \geq 0$. □

Definition 3.7. Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of a ring R. Let $\Delta := \{\delta_1, \ldots, \delta_n\}$ be the $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ derivations as mentioned in the proposition 2.4.

(1) We say R satisfies the $(*)$ *condition* if whenever $fAg = 0$ for elements $f = a_0 + a_1X_1 + \cdots + a_mX_m$ and $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in A$, then $a_iX_iRb_jY_j = 0$ for all i, j.

(2) We say that R is a (Σ, Δ)-*Armendariz* ring, if for elements $f = a_0 + a_1X_1 + \cdots + a_mX_m$ and $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in A$, $fg = 0$ implies $a_iX_ib_jY_j = 0$ for all i, j.

Lemma 3.8. *Let R be a reduced (Σ, Δ)-compatible ring. Then R is (Σ, Δ)-Armendariz.*

Proof. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m$ and $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ such that $fg = 0$. Since each reduced (Σ, Δ)-compatible ring is Σ-rigid and by [27, Proposition 3.6], we have $a_ib_j = 0$ for each i, j. Therefore $a_iX_ib_jY_j = 0$ for each i, j, by Corollary 3.4. □

For a ring R, put $rAnn_R(2^R) = \{r_R(U) | U \subseteq R\}$ and $\ell Ann_R(2^R) = \{\ell_R(U) | U \subseteq R\}$.

Proposition 3.9. *Let R be a (Σ, Δ)-compatible ring and $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of a ring R. Then the following statements are equivalent:*

(1) *R is (Σ, Δ)-Armendariz;*

(2) *$\psi : rAnn_R(2^R) \to rAnn_A(2^A)$; $S \to SA$ is bijective;*

(3) *$\varphi : \ell Ann_R(2^R) \to \ell Ann_A(2^A)$; $V \to AV$ is bijective.*

Proof. (1) $\Rightarrow$ (2). For a element $f = a_0 + a_1X_1 + \cdots + a_mX_m \in A$, f^* denotes the set of coefficients of f and for a subset S of A, S^* denotes the set $\bigcup_{f\in S} f^*$. Let S be a subset of A and let $f \in S$. Since R is (Σ, Δ)-compatible (Σ, Δ)-Armendariz ring, $r_A(f) = r_A(f^*) = r_R(f^*)A$. Hence $r_A(S) = \bigcap_{f\in S} r_A(f) = \bigcap_{f\in S} r_A(f^*) = r_R(f^*)A$.

(2) $\Rightarrow$ (1). Let $f = a_0 + a_1X_1 + \cdots + a_mX_m \in A$. By hypothesis, $r_A(f) = IA$ for some right ideal I of R. If $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in A$ satisfies $fg = 0$ then $g \in IA$, and hence $b_0, b_1, \ldots, b_t \in I \subseteq r_A(f)$. Then $a_ib_j = 0$ for every i, j. Therefore $a_iX_ib_jY_j = 0$ for every i, j, Since R is (Σ, Δ)-compatible.

Similarly we can prove (1) $\Leftrightarrow$ (3). □

Corollary 3.10. *Let R be a (Σ, Δ)-compatible ring and $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of a ring R. If R is (Σ, Δ)-Armendariz, then R is Baer (resp., p.p.) if and only if A is Baer (resp., p.p.).*

Proof. It follows from Proposition 3.9. □

Corollary 3.11.([27, Theorem 3.9 and 3.12]) *Let R be a Σ-rigid ring. Then R is Baer (resp., p.p.) if and only if the skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is*

Baer (resp., p.p.).

Proof. Since Σ-rigid rings are reduced and (Σ, Δ)-compatible, the proof follows from Lemma 3.8 and Corollary 3.10. □

Following [15], for a ring R, put $rAnn_R(id(R)) = \{r_R(U)|U$ is an ideal of $R\}$ and $\ell Ann_R(id(R)) = \{\ell_R(U)|U$ is an ideal of $R\}$.

Proposition 3.12. *Let R be a (Σ, Δ)-compatible ring and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of a ring R. Then the following statements are equivalent:*

(1) *R satisfies condition $(*)$;*

(2) *$\psi : rAnn_R(id(R)) \to rAnn_A(id(A))$; $S \to SA$ is bijective;*

(3) *$\varphi : \ell Ann_R(id(R)) \to \ell Ann_A(id(A))$; $V \to AV$ is bijective.*

Proof. (1) ⇒ (2). Let $S \in rAnn_R(id(R))$. Then there exists an ideal I of R such that $S = r_R(I)$. So by Lemma 3.3, $r_A(AIA) = SA$. This shows that ψ is a well defined mapping. Suppose that $V \in rAnn_A(id(A))$, then there exists an ideal J of A such that $V = r_A(J)$. We show that $r_R(J_1R) = V_1R$, where V_1 and J_1 are the set of coefficients of elements of V and J in A. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m \in J$ and $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in V = r_A(J)$. Then $fAg = 0$. Since R satisfies condition $(*)$ we have $a_iRb_j = 0$ for all a_i, b_j. Thus $(J_1R)(V_1R) = 0$, and hence $V_1 \subseteq r_R(J_1R)$. On the other hand, (Σ, Δ)-compatibility gives, $r_R(J_1R) \subseteq V_1R$. Thus $r_R(J_1R) = V_1R$, and therefore $V = r_A(J) = (V_1R)A$.

(2) ⇒ (1). Assume that $fAg = 0$, where $f = a_0 + a_1X_1 + \cdots + a_mX_m$ and $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in A$. Then for an ideal I of R, $g \in r_A(AfA) = IA$. Hence $b_0, b_1, \dots, b_t \in I$ and so $fRb_j = 0$ for $j = 0, \dots, t$. Therefore by Lemma 3.6, $a_iRb_j = 0$ for $i = 0, \dots, m$ and $j = 0, \dots, t$.

Similarly we can prove (1) ⇔ (3). □

We recall the definition of a *right s-unital* ideal from [**?**]. An ideal I of R is said to be *right s-unital* if, for each $a \in I$ there is an $x \in I$ such that $ax = a$. If an ideal I of R is right s-unital, then for any finite subset F of I, there exists an element $e \in I$ such that $xe = x$ for all $x \in F$.

Theorem 3.13. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of a ring R. If R is (Σ, Δ)-compatible ring, then the following statements are equivalent:*

(1) *$\ell_R(Ra)$ is a right s-unital ideal of R for any element $a \in R$;*

(2) *$\ell_A(Af)$ is a right s-unital ideal of A for any element $f \in A$.*
In this case R satisfies condition $(*)$.

Proof. (1) ⇒ (2). First we prove that R satisfies condition $(*)$. Consider $f = a_0 + a_1X_1 + \cdots + a_mX_m$, $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in I$, where $a_i \in R$, $1 \le i \le m$, $a_m \neq 0$, with $X_i = x^{\alpha_i} = x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$, $X_m \succ X_{m-1} \succ \cdots \succ X_1$, and $b_j \in R$,

$1 \le j \le t$, $b_t \neq 0$, with $Y_j = x^{\alpha_j} = x_1^{\alpha_{j1}} \cdots x_n^{\alpha_{jn}}$, $\quad Y_t \succ Y_{t-1} \succ \cdots \succ Y_1$. Assume that $(a_0 + a_1X_1 + \cdots + a_mX_m)A(b_0 + b_1Y_1 + \cdots + b_tY_t) = 0$, with $a_i, b_j \in R$. Then

$$(a_0 + a_1X_1 + \cdots + a_mX_m)R(b_0 + b_1Y_1 + \cdots + b_tY_t) = 0, \tag{3.1}$$

and hence

$$\text{"other terms of order less than"} + a_mX_mRb_tY_t = 0.$$

Thus by Theorem 2.7 and (3.1), we get

$$\begin{aligned} a_mX_mRb_tY_t &= a_m[\sigma^{\alpha_m}(Rb_t)x^{\alpha_m} + p_{\alpha_m,Rb_t}]x^{\beta_t} \\ &= a_m\sigma^{\alpha_m}(Rb_t)x^{\alpha_m}x^{\beta_t} + a_mp_{\alpha_m,Rb_t}x^{\beta_t} \\ &= a_m\sigma^{\alpha_m}(Rb_t)[c_{\alpha_m,\beta_t}x^{\alpha_m+\beta_t} + p_{\alpha_m,\beta_t}] + a_mp_{\alpha_m,Rb_t}x^{\beta_t} \\ &= a_m\sigma^{\alpha_m}(Rb_t)c_{\alpha_m,\beta_t}x^{\alpha_m+\beta_t} + a_m\sigma^{\alpha_m}(Rb_t)p_{\alpha_m,\beta_t} + a_mp_{\alpha_m,Rb_t}x^{\beta_t} \\ &= 0, \end{aligned}$$

where $p_{\alpha_m,Rb_t} = 0$ or $\deg(p_{\alpha_m,Rb_t}) < |\alpha_m|$ if $p_{\alpha_m,Rb_t} \neq 0$ and $p_{\alpha_m,\beta_t} = 0$ or $\deg(p_{\alpha_m,\beta_t}) < |\alpha_m + \beta_t|$ if $p_{\alpha_m,\beta_t} \neq 0$. Since A is bijective by Remark 2.8, from the equality $lc(fAg) = a_m\sigma^{\alpha_m}(Rb_t)c_{\alpha_m,\beta_t} = 0$ we obtain $a_m\sigma^{\alpha_m}(Rb_t) = 0$ and hence $a_mRb_t = 0$, since R is (Σ, Δ)-compatible. So that $a_m \in \ell_R(Rb_t)$. Also by Remark 2.9, we can see that the polynomial $p_{\alpha_m,r}$ involve elements obtained evaluating σ's and δ's in the element r of R. Thus (Σ, Δ)-compatibility gives

$$a_mp_{\alpha_m,Rb_t} = 0. \tag{3.2}$$

Since $\ell_R(Rb_t)$ is right s-unital, there exists $e_t \in \ell_R(Rb_t)$ such that $a_me_t = a_m$. If we replace R by e_tR in (3.1) and using (3.2), then we get $a_m\sigma^{\alpha_m}(e_tRb_{t-1}) = 0$. Thus

$$a_me_tRb_{t-1} = a_mRb_{t-1} = a_mp_{\alpha_m,Rb_{t-1}} = 0, \tag{3.3}$$

since R is (Σ, Δ)-compatible. Therefore $a_m \in \ell_R(Rb_t) \cap \ell_R(Rb_{t-1})$. Since $\ell_R(Rb_{t-1})$ is right s-unital, there exists $h \in \ell_R(Rb_{t-1})$ such that $a_mh = a_m$. Hence $a_m\delta(h) = 0$ and $a_m\sigma^s(h) = a_m$ for all $s \ge 0$ by Lemma 3.3. If we take $e_{t-1} = e_th$, then we have $a_mh = a_m$ and $e_{t-1} \in \ell_R(Rb_t) \cap \ell_R(Rb_{t-1})$. Similar above, replacing R by $e_{t-1}R$ in (3.1), and using (3.2), (3.3) and (Σ, Δ)-compatibility of R, we obtain $a_mRb_{t-2} = 0$ and hence $a_m \in \ell_R(Rb_t) \cap \ell_R(Rb_{t-1}) \cap \ell_R(Rb_{t-2})$. Continuing in this way, we obtain $a_mRb_k = 0$ for $k = 0, \ldots, t$. Hence we get $(a_0 + a_1X_1 + \cdots + a_mX_m)A(b_0 + b_1Y_1 + \cdots + b_{t-1}Y_{t-1}) = 0$, since R is (Σ, Δ)-compatible. Using induction on $|\alpha_m + \beta_t|$ we obtain $a_iRb_j = 0$ for all i, j. Hence $a_iX_iRb_jY_j = 0$ for all i, j, by (Σ, Δ)-compatibility of R. Therefore R satisfies condition $(*)$. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m$ and $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in \ell_A(Af)$. Then $b_jRa_i = 0$ for all i, j. Since $\ell_R(a_i)$ is right s-unital, there exists $e_i \in \ell_R(a_i)$ such that $b_j = b_je_i$ for $j = 0, \ldots, t$. Put $e = e_0e_1 \ldots e_t$, then $b_j = b_je$ for $j = 0, \ldots, t$. Hence $b_j\sigma^s(e) = b_j$ and $b_j\delta(e) = 0$

for $s \geq 0$ and $j = 0, \dots, t$, by Lemma 3.3. Hence $e \in \ell_A(Af)$ and also $ge = g$. Therefore $\ell_A(Af)$ is right s-unital.

(2) $\Rightarrow$ (1). Let a be an element of R. Since R is (Σ,Δ)-compatible, $\ell_R(Ra) \subseteq \ell_A(Aa)$. Hence for any $b \in \ell_R(Ra)$, there exists a element $h \in A$ such that $bh = b$. Let a_0 be the constant term of h. Then $ba_0 = b$ and by (Σ,Δ)-compatibility of R, $a_0 \in \ell_R(Ra)$. This implies that $\ell_R(Ra)$ is right s-unital. □

Since quasi-Baer (left p.q.-Baer) rings satisfy the hypothesis of Theorem 3.13, hence we have the following.

Corollary 3.14. *Let R be a (Σ,Δ)-compatible ring and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of a ring R. Then R is quasi-Baer (resp., left p.q.-Baer) if and only if A is quasi-Baer (resp., left p.q.-Baer). In this case R satisfies condition $(*)$.*

The next examples show that without (Σ,Δ)-compatibility conditions, Corollary 3.14 is not true in general.

Example 3.15.([22, Example 3.1]) Let $\mathbb{Z}$ and $\mathbb{Q}$ be the ring of integers and rational numbers, respectively. Suppose $S = \prod_{i\in\mathbb{Z}} \mathbb{Q}_i$ with $\mathbb{Q}_i = \mathbb{Q}$ for each $i \in \mathbb{Z}$ and R be the ring generated by $\bigoplus_{i\in\mathbb{Z}} \mathbb{Q}_i$ and 1_S, where $\mathbb{Q}_i = \mathbb{Q}$ for each $i \in \mathbb{Z}$. Then R is a reduced p.q.-Baer ring. Let $\sigma : R \to R$ be the map given by $\sigma((a_i)_{i\in\mathbb{Z}}) = (a_{i+1})_{i\in\mathbb{Z}}$. Then σ is an automorphism of R. Since $(0,1,0,\dots,0,1)(1,0,1,\dots,1,0) = 0$ but $(0,1,0,\dots,0,1)\sigma(1,0,1,\dots,1,0) = (0,1,\dots,0,1) \neq 0$, the ring R is not σ-compatible and also the skew PBW extension $R[x;\sigma]$ is neither right nor left p.q.-Baer ring.

Example 3.16.([2, Example 11]) There is a ring R and a derivation δ of R such that $R[x;\delta]$ is a Baer (hence quasi-Baer) ring, but R is not quasi-Baer. In fact, let $R = \mathbb{Z}_2[t]/(t^2)$ with the derivation δ such that $\delta(\bar{t}) = 1$ where $\bar{t} = t + (t^2)$ in R and $\mathbb{Z}_2[t]$ is the polynomial ring over a field $\mathbb{Z}_2$ of two elements. Consider skew PBW extension of derivation type $R[x;\delta]$. Note that since $\bar{t}^2 = 0$ but $\bar{t}\delta(\bar{t}) \neq 0$, the δ-compatibility condition fails here. If we set $e_{11} = \bar{t}x, e_{12} = \bar{t}, e_{21} = \bar{t}x^2 + x$ and $e_{22} = 1 + \bar{t}x$ in $R[x;\delta]$, then they from a system of matrix units in $R[x;\delta]$. Now the centralizer of these matrix units in $R[x;\delta]$ is $\mathbb{Z}_2[x^2]$. Therefore $R[x;\delta] \cong M_2(\mathbb{Z}_2[x^2]) \cong M_2(\mathbb{Z}_2)[y]$, where $M_2(\mathbb{Z}_2)[y]$ is the polynomial ring over $M_2(\mathbb{Z}_2)$. So $R[x;\delta]$ is a Baer ring but R is not quasi-Baer.

Corollary 3.17.([12, Corollary 2.8]) *Let R be a (σ,δ)-compatible ring and σ be an automorphism of a ring R . Then R is quasi-Baer (resp., left p.q.-Baer) if and only if $R[x;\sigma,\delta]$ is quasi-Baer (resp., left p.q.-Baer).*

Corollary 3.18.([27, Theorems 3.10 and 3.13]) *Let R be a Σ-rigid ring. Then R is quasi-Baer (resp., left p.q.-Baer) if and only if bijective skew PBW extension $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is quasi-Baer (resp., left p.q.-Baer).*

Proof. Since Σ-rigid rings are (Σ,Δ)-compatible and R is a Σ-rigid ring, this follows from Corollary 3.14. □

We conclude by noting that, the class of (Σ, Δ)-compatible rings which satisfies condition $(*)$ are independent of the class of quasi-Baer rings. In fact, there exists a non quasi-Baer (Σ, Δ)-compatible ring R which satisfies condition $(*)$ (see [12, Example 2.12]).

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References

[1] E. P. Armendariz, *A note on extensions of Baer and p.p.-rings*, J. Austral. Math. Soc., **18**(1974), 470–473.

[2] E. P. Armendariz, H. K. Koo and J. K. Park, *Isomorphic Ore extensions*, Comm. Algebra, **15**(1987), 2633–2652.

[3] A. Bell and K. R. Goodearl, *Uniform rank over differential operator rings and Poincaré-Birkhoff-Witt extensons*, Pacific J. Math., **131**(1988), 13–37.

[4] G. F. Birkenmeier, Y. Kim and J. K. Park, *On quasi-Baer rings*, Contemp. Math., **259**(2000), 67–92.

[5] G. F. Birkenmeier, J. Y. Kim and J. K. Park, *On polynomial extensions of principally quasi-Baer rings*, Kyungpook Math. J., **40**(2000), 247–253.

[6] G. F. Birkenmeier, J. Y. Kim and J. K. Park, *Principally quasi-Baer rings*, Comm. Algebra, **29**(2001), 639–660.

[7] G. F. Birkenmeier, J. Y. Kim and J. K. Park, *Polynomial extensions of Baer and quasi-Baer rings*, J. Pure Appl. Algebra, **159**(2001), 25–42.

[8] G. F. Birkenmeier, H. E. Heatherly, Y. Kim and J. K. Park, *Triangular matrix representations*, J. Algebra, **230**(2000), 558–595.

[9] W. E. Clark, *Twisted matrix units semigroup algebras*, Duke Math. J., **34**(1967), 417–424.

[10] C. Gallego and O. Lezama, *Gröbner bases for ideals of σ-PBW extensions*, Comm. Algebra, **39**(2011), 50–75.

[11] E. Hashemi and A. Moussavi, *Polynomial extensions of quasi-Baer rings*, Acta Math. Hungar., **107** (2005), 207–224.

[12] E. Hashemi, *A characterization of δ-quasi-Baer rings*, Math. J. Okayama Univ., **49**(2007), 197–200.

[13] J. Han, Y. Hirano and H. Kim, *Semiprime Ore extensions*, Comm. Algebra, **28**(2000), 3795–3801.

[14] J. Han, Y. Hirano and H. Kim, *Some results on skew polynomial rings over a reduced ring*, in International Symposium on Ring Theory (Kyongju, (1999)), Trends Math., Birkhauser Boston, Boston, MA (2001), 123–129.

[15] Y. Hirano, *On annihilator ideals of a polynomial ring over a noncommutative ring*, J. Pure Appl. Algebra, **168**(2002), 45–52.

[16] Y. Hirano, *On the uniqueness of rings of coefficients in skew polynomial rings*, Publ. Math. Debrecen, **54**(1999), 489–495.

[17] C. Y. Hong, N. K. Kim and T. K. Kwak, *Ore extensions of Baer and p.p.-rings*, J. Pure Appl. Algebra **151**(2000), 215–226.

[18] I. Kaplansky, *Rings of operators*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[19] J. Krempa, *Some examples of reduced rings*, Algebra Colloq., **3**(1996), 289–300.

[20] O. Lezama and A. Reyes, *Some homological properties of skew PBW extensions*, Comm. Algebra, **42**(2014), 1200–1230.

[21] A. R. Nasr-Isfahani and A. Moussavi, *and quasi-Baer differential polynomial rings*, Comm. Algebra, **36**(2008), 3533–3542.

[22] A. R. Nasr-Isfahani and A. Moussavi, *On Ore extensions of quasi-Baer rings*, J. Algebra Appl., **2** (2008), 211–224.

[23] D. S. Passman, *Prime ideals in enveloping rings*, Trans. Amer. Math. Soc., **302**(1987), 535–560.

[24] P. Pollingher and A. Zaks, *Baer and quasi-Baer rings*, Duke Math. J., **37**(1970), 127–138.

[25] A. Reyes, *and module theoretical properties of skew PBW extensions*, Thesis (Ph.D.), Universidad Nacional de Colomia, Bogota, 2013, 142 pp.

[26] A. Reyes, *Jacobson's conjecture and skew PBW extension*, Rev. Integr. Temas Mat., **32**(2014), 139–152.

[27] A. Reyes, *Skew PBW Extensions of Baer, quasi-Baer, p.p. and p.q.-rings*, Rev. Integr. Temas Mat., **33**(2015), 173–189.