KYUNGPOOK Math. J. 57(2017), 457-471
https://doi.org/10.5666/KMJ.2017.57.3.457
pISSN 1225-6951 eISSN 0454-8124

On Normal Products of Selfadjoint Operators

Il Bong Jung*
Department of Mathematics, Kyungpook National University, Daegu 41566, Korea
e-mail : ibjung@knu.ac.kr

Mohammed Hichem Mortad
Department of Mathematics, University of Oran 1 (Ahmed Ben Bella), B.P. 1524, El Menouar, Oran 31000, Algeria
e-mail : mhmortad@gmail.com, mortad@univ-oran.dz

Jan Stochel
Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, PL-30348 Kraków, Poland
e-mail : Jan.Stochel@im.uj.edu.pl

Abstract. A necessary and sufficient condition for the product AB of a selfadjoint operator A and a bounded selfadjoint operator B to be normal is given. Various properties of the factors of the unitary polar decompositions of A and B are obtained in the case when the product AB is normal. A block operator model for pairs (A, B) of selfadjoint operators such that B is bounded and AB is normal is established. The case when both operators A and B are bounded is discussed. In addition, the example due to Rehder is reexamined from this point of view.

1. Introduction

The question of when the product AB of two bounded selfadjoint operators A and B is selfadjoint has simple answer, namely, AB is selfadjoint if and only if A and B commute. If the product AB is in a wider class of operators, for example,

* Corresponding Author.
Received July 7, 2017; accepted 26, August.
2010 Mathematics Subject Classification: Primary 47B15, 47B20, 47B25.
Key words and phrases: Selfadjoint operator, normal operator, unitary polar decomposition, block operator model, normal product.
The research of the first author was supported by Kyungpook National University. The research of the third author was supported by the NCN (National Science Center), decision No. DEC-2013/11/B/ST1/03613.

in the class of normal operators, then the operators A and B may not commute. A simple counterexample can be given even in the case of 2×2 matrices (see Example 5.3). It was Rehder who noticed and proved that if at least one of the factors of the product AB is positive (still under the assumption that A and B are bounded and selfadjoint), then AB is normal if and only if AB is selfadjoint (see [11, Theorem]). The main idea of his proof relies on applying the Fuglede-Putnam theorem ([10, Lemma]). There were many attempts to generalize Rehder's theorem to the more general context of Banach algebras as well as to the context of unbounded operators (e.g., see [1, 6, 8, 9]). We mention two recent results of this type. The first states that if A and B are selfadjoint operators such that A is positive, B is bounded and AB is normal, then AB is selfadjoint (see [6, Theorem 1.5]). The second result states that if A and B are self-adjoint operators such that B is bounded and positive, BA is hyponormal and closed, and the spectrum of BA is not the whole complex plane, then both BA and AB are self-adjoint (see [4, Theorem 3.6]).

In this paper we are searching for necessary and sufficient conditions for the product AB of a selfadjoint operator A and a bounded selfadjoint operator B to be normal. It is worth pointing out that there is an asymmetry when considering the products AB and BA. Namely, if A and B are selfadjoint (or even less, closed) and B is bounded, then the product AB is closed, while BA may not be (cf. Lemma 3.1). In other words, the assumption that BA is normal is stronger than that on normality of AB. Indeed, if BA is normal, then $AB = (BA)^*$ is normal. This is the reason why we concentrate on studying the normality of products of the form AB, where B is bounded. Our idea is to use unitary polar decompositions instead of commonly used polar decompositions. Unitary polar decompositions are investigated in Section 2. Theorem 3.2, which is the main result of Section 3, characterizes the normality of the product AB of a selfadjoint operator A and a bounded selfadjoint operator B. We also derive many of the properties of the factors of unitary polar decompositions of A and B. In particular, it is shown that the modulus of the product AB equals the product of the moduli of A and B whenever A is selfadjoint, B is bounded and selfadjoint and AB is normal. As a consequence, some recent results related to this issue are immediately deduced (see Corollaries 3.3 and 3.4). In Section 4, using Theorem 3.2, we construct a block operator model for pairs (A, B) of selfadjoint operators A and B such that B is bounded and AB is normal (see Theorem 4.4). In Section 5, we obtain the block operator model for pairs (A, B) of bounded operators. We conclude the paper by reexamining the Rehder's example (see Example 5.3).

2. Unitary Polar Decompositions

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces. A linear mapping $A\colon \mathcal{H} \supseteq \mathscr{D}(A) \to \mathcal{K}$ defined on a vector subspace $\mathscr{D}(A)$ of $\mathcal{H}$ is called an *operator* from $\mathcal{H}$ to $\mathcal{K}$. The domain, the kernel and the range of A are denote by $\mathscr{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively. We write A^* and $|A|$ for the adjoint and the modulus of A (provided they exist). We denote by $\boldsymbol{B}(\mathcal{H}, \mathcal{K})$ the Banach space of all bounded operators from

$\mathcal{H}$ to $\mathcal{K}$ with the domain $\mathcal{H}$. We abbreviate $\boldsymbol{B}(\mathcal{H},\mathcal{H})$ to $\boldsymbol{B}(\mathcal{H})$ and write $I_{\mathcal{H}}$, or simply I when no confusion arises, for the identity operator on $\mathcal{H}$. If $\mathcal{A} \subseteq \boldsymbol{B}(\mathcal{H})$, then we denote by $\mathcal{A}'$ the *commutant* of $\mathcal{A}$, i.e., the set of all $T \in \boldsymbol{B}(\mathcal{H})$ such that $TA = AT$ for all $A \in \mathcal{A}$, and write $\mathcal{A}'' = (\mathcal{A}')'$.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be closed vector subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. We say that an operator A in $\mathcal{H}$ has the block matrix form $\left[\begin{smallmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{smallmatrix}\right]$ with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and write $A = \left[\begin{smallmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{smallmatrix}\right]$, if $A_{i,j}$ is an operator from $\mathcal{H}_j$ to $\mathcal{H}_i$, $\mathscr{D}(A) = \mathcal{E} \oplus \mathcal{F}$, where $\mathcal{E} = \mathscr{D}(A_{1,1}) \cap \mathscr{D}(A_{2,1})$ and $\mathcal{F} = \mathscr{D}(A_{1,2}) \cap \mathscr{D}(A_{2,2})$ and

$$A(f_1 \oplus f_2) = (A_{1,1}f_1 + A_{1,2}f_2) \oplus (A_{2,1}f_1 + A_{2,2}f_2)$$

for all $f_1 \in \mathcal{E}$ and $f_2 \in \mathcal{F}$. In particular, the following basic identity holds:

$$A_{1,1} \oplus A_{2,2} = \left[\begin{smallmatrix} A_{1,1} & 0 \\ 0 & A_{2,2} \end{smallmatrix}\right].$$

Recall that if A is a closed densely defined operator in $\mathcal{H}$, then there exists a unique partial isometry $V \in \boldsymbol{B}(\mathcal{H})$ such that $A = V|A|$ and $\mathcal{N}(A) = \mathcal{N}(V)$, where $|A|$ is the square root of A^*A (see e.g., [2, Theorem 8.1.2]). Such decomposition is called the *polar decomposition* of A. In what follows, we will often use the fact that

$$\mathcal{N}(A) = \mathcal{N}(|A|) \text{ whenever } A \text{ is closed and densely defined,} \tag{2.1}$$

which is a direct consequence of the polar decomposition of A.

Definition 2.1. We say that a closed densely defined operator A in $\mathcal{H}$ has the *unitary polar decomposition* if there exists a unitary operator $U \in \boldsymbol{B}(\mathcal{H})$ such that $A = U|A|$ and $U|_{\mathcal{N}(A)} = I_{\mathcal{N}(A)}$.

Clearly, the unitary polar decomposition of A is unique, namely, if $A = U_1|A|$ and $A = U_2|A|$ are unitary polar decompositions of A, then $U_1 = U_2$. We refer the reader to the monograph [5] for more information on polar decompositions of variety of classes of bounded operators in which the partial isometry factor is replaced by an isometric or a unitary operator.

The following proposition gives necessary and sufficient condition for an operator to have the unitary polar decomposition.

Proposition 2.2. *A closed densely defined operator A in $\mathcal{H}$ has the unitary polar decomposition if and only if $\mathcal{N}(A) = \mathcal{N}(A^*)$. If $A = V|A|$ and $A = U|A|$ are the polar and the unitary polar decompositions of A respectively, then $U = V + P$, where $P \in \boldsymbol{B}(\mathcal{H})$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(A)$.*

Proof. If $V \in \boldsymbol{B}(\mathcal{H})$ is such that $A = V|A|$, then

$$\|V(|A|h)\| = \|Ah\| = \||A|h\|, \quad h \in \mathscr{D}(A) = \mathscr{D}(|A|), \tag{2.2}$$

which implies that the operator $V|_{\overline{\mathcal{R}(|A|)}}\colon \overline{\mathcal{R}(|A|)} \to \overline{\mathcal{R}(A)}$ is unitary. This combined with the polar decomposition completes the proof. □

Now we state some basic properties of the unitary polar decomposition of a normal operator.

Proposition 2.3. *Suppose A is a normal operator in $\mathcal{H}$. Then A has the unitary polar decomposition. Moreover, if $A = U|A|$ is the unitary polar decomposition of A, then the following assertions hold*:

(i) $\overline{\mathcal{R}(|A|)}$ *reduces* $|A|$, U *and* A,

(ii) $A = 0_{\mathcal{N}(A)} \oplus A_1$ *with* $A_1 = A|_{\overline{\mathcal{R}(|A|)}}$,

(iii) $U = I_{\mathcal{N}(A)} \oplus U_1$ *with* $U_1 = U|_{\overline{\mathcal{R}(|A|)}}$,

(iv) $A_1 = U_1|A_1|$ *is the polar decomposition of* A_1 *with* $|A_1| = |A||_{\overline{\mathcal{R}(|A|)}}$,

(v) *if A is selfadjoint, then $A = |A|U$ and U is a fundamental symmetry*[1], *i.e.*, $U = U^*$ *and* $U^2 = I$.

Proof. In view of Proposition 2.2, only the "moreover" part requires the proof. Assume that $A = U|A|$ is the unitary polar decomposition of A. Note that $\mathcal{N}(|A|)$ reduces $|A|$. By (2.1) and Definition 2.1, $U(\mathcal{N}(|A|)) = \mathcal{N}(|A|)$, which implies that $\mathcal{N}(|A|)$ reduces U. Using the fact that a closed vector subspace $\mathcal{M}$ of $\mathcal{H}$ reduces an operator T in $\mathcal{H}$ if and only if $PT \subseteq TP$, where $P \in \boldsymbol{B}(\mathcal{H})$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$, we conclude that $\mathcal{N}(|A|)$ reduces A as well. It is a routine matter to verify the assertions (ii)-(iv). To prove (v), assume that $A = A^*$. Since, by (iv), $A_1^* = U_1^*|A_1^*|$ is the polar decomposition of A_1^* and A_1 is selfadjoint, we see that U_1 is selfadjoint and unitary. As a consequence, $U = U^*$, $U^2 = I$ and $A = |A|U$. This completes the proof. □

Corollary 2.4. *Let A be a closed densely defined operator in $\mathcal{H}$ and $U \in \boldsymbol{B}(\mathcal{H})$. Then the following conditions are equivalent*:

(i) *A is selfadjoint and $A = U|A|$ is the unitary polar decomposition of A,*

(ii) $U = U^*$, $A = U|A| = |A|U$ *and* $U|_{\mathcal{N}(A)} = I_{\mathcal{N}(A)}$.

Proof. (i)⇒(ii) Apply Proposition 2.3(v).

(ii)⇒(i) Clearly $A^* = |A|U = A$, which together with (2.1) yields $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(|A|)}$. This combined with the equalities $A = U|A|$ and (2.2) implies that the mapping

$$U|_{\overline{\mathcal{R}(|A|)}}\colon \overline{\mathcal{R}(|A|)} \to \overline{\mathcal{R}(|A|)}$$

[1] The notion of a fundamental symmetry comes from the theory of Krein spaces (see [3, Chap. II, Sec. 11]). Obviously, fundamental symmetries are unitary.

is a unitary operator. Since $U|_{\mathcal{N}(A)} = I_{\mathcal{N}(A)}$ and $\mathcal{N}(A) = \mathcal{N}(|A|)$, we see that U is a unitary operator. □

3. A Characterization of Normal Products

Before stating the main result of this section we formulate some important properties of products of strongly commuting selfadjoint operators.

Lemma 3.1. *Let A and B be selfadjoint operators in $\mathcal{H}$. Suppose $B \in \boldsymbol{B}(\mathcal{H})$. Then the following assertions hold*:

(i) *AB is a selfadjoint operator if and only if $BA \subseteq AB$; if this is the case, then $AB = \overline{BA}$,*

(ii) *if A and B are positive and $BA \subseteq AB$, then AB is positive.*

Proof. (i) If AB is selfadjoint, then

$$BA = B^*A^* \subseteq (AB)^* = AB.$$

To prove the converse, assume that $BA \subseteq AB$. By [2, Theorem 6.3.2], the spectral measures E_A and E_B of A and B commute. Let E be the joint spectral measure of the pair (A, B) (see [2, Theorem 6.5.1]). Then, by [2, Theorem 5.4.7], we have

$$\overline{BA} = \overline{AB} = \int_{\mathbb{R}\times\mathbb{R}} x \cdot t \, dE(x,t), \tag{3.1}$$

where $\mathbb{R}$ is the set of all real numbers. Since the operator AB is closed and the spectral integral appearing on right-hand side of (3.1) is selfadjoint (see [2, Theorem 5.4.5]), we conclude that AB is selfadjoint and thus, by (3.1), $AB = \overline{BA}$.

(ii) If A and B are positive and $BA \subseteq AB$, then we can replace $\mathbb{R} \times \mathbb{R}$ by $\mathbb{R}_+ \times \mathbb{R}_+$ in (3.1), where $\mathbb{R}_+ = [0, \infty)$. As a consequence, AB is positive. □

It is worth mentioning that if A and B are selfadjoint operators in $\mathcal{H}$ and B is bounded, then the product AB is closed, however the product BA may not be closed even if $BA \subseteq AB$. This is the reason why we are interested in answering the question of when the product AB, not BA, is normal.

Now we characterize normality of the product of two selfadjoint operators with bounded second factor in terms of their unitary polar decompositions.

Theorem 3.2. *Let A and B be selfadjoint operators in $\mathcal{H}$. Suppose $B \in \boldsymbol{B}(\mathcal{H})$. Let $A = U|A|$ and $B = V|B|$ be the unitary polar decompositions of A and B, respectively. Then the following conditions are equivalent*:

(i) *AB is normal,*

(ii) *$|B|A \subseteq A|B|$ and*[2] *$V|A| \subseteq |A|V$.*

[2] In fact, $V|A| = |A|V$; see Lemma 4.1.

Moreover, if (i) *holds, then the following assertions are valid*:

(a) $V(AB)^* = ABV$ *and* $VAB = (AB)^*V$,

(b) $V|AB| = |AB|V$,

(c) $|AB| = |A||B|$,

(d) $|A|B$ *and* $A|B|$ *are selfadjoint,*

(e) $U|AB| \subseteq |AB|U$,

(f) $AB = UV|AB|$,

(g) $(AB)^* = VU|AB|$,

(h) $|B||A| \subseteq |A||B|$ *and* $B|A| \subseteq |A|B$.

Proof. (i)⇒(ii) Assume AB is normal. First we show that

$$|B|A \subseteq A|B| \text{ and } |B||A| \subseteq |A||B|. \tag{3.2}$$

Indeed, since $B(AB) \subseteq (AB)^*B$, the Putnam-Fuglede theorem (see [10, Lemma]) implies that $B(AB)^* \subseteq (AB)B$, and thus

$$|B|^2A = B^2A \subseteq B(AB)^* \subseteq AB^2 = A|B|^2.$$

This and [2, Theorem 6.3.2] imply that $|B|^2E_A = E_A|B|^2$, where E_A is the spectral measure of A. By the square root theorem, $|B|E_A = E_A|B|$. Applying [2, Theorem 6.3.2] again, we get the first inclusion in (3.2) (this inclusion can also be deduced from [7, Lemma 2.1] applied to $\phi(x) = \sqrt{x}$). The second inclusion in (3.2) is a consequence of the first one. Indeed, the first inclusion implies that $|B|A^2 \subseteq A^2|B|$. Hence, by the square root theorem (see [2, Theorems 5.4.8 and 6.1.4]), we see that $|B||A| \subseteq |A||B|$. This is also the first inclusion in (h).

Now we prove (a) and (b). It follows from Proposition 2.3(v) that $|B|V = B$, $V = V^*$ and $V^2 = I$, and thus $VB = |B|$. This yields

$$VBAV = |B|AV \overset{(3.2)}{\subseteq} A|B|V = AB,$$

which in turn implies that

$$(AB)^* \subseteq V(BA)^*V = VABV.$$

By maximality of normal operators, we get $(AB)^* = VABV$, so (a) holds. Since

$$|AB|^2 = (AB)^*VVAB \overset{(a)}{=} VAB(AB)^*V = V|AB|^2V, \tag{3.3}$$

we get $|AB| = V|AB|V$, which implies (b).

Our next aim is to prove (c). First we show that

$$\text{if } C \in \{A, |A|\}, \text{ then } (C|B|)^2 = A^2|B|^2. \tag{3.4}$$

Indeed, by (3.2) and Lemma 3.1, $C|B|$ and $A^2|B|^2$ are selfadjoint and $|A||B|$ is positive. Since squares of selfadjoint operators are selfadjoint and, by (3.2), $(C|B|)^2 \subseteq A^2|B|^2$, maximality of selfadjoint operators implies that $(C|B|)^2 = A^2|B|^2$. This proves the assertion (3.4). Since the operator $A|B|$ is selfadjoint, we infer from Proposition 2.3(v) that

$$|AB|^2 = (A|B|V)^*A|B|V = V(A|B|)^2V \overset{(3.4)}{=} VA^2|B|^2V \overset{(3.4)}{=} V(|A||B|)^2V.$$

This together with (3.3) implies that $|AB|^2 = (|A||B|)^2$. Taking square roots gives (c); recall that the product $|A||B|$ is positive and selfadjoint.

Now we show that

$$V \in \{B\}''. \tag{3.5}$$

Indeed, if $T \in \{B\}'$, then $T|B| = |B|T$ and so $\mathcal{N}(|B|)$ reduces T. This implies that

$$(TV)(|B|h) = TBh = BTh = V|B|Th = (VT)(|B|h), \quad h \in \mathcal{H},$$

and thus $TVh = VTh$ for all $h \in \overline{\mathcal{R}(|B|)}$. As $TVh = Th = VTh$ for all $h \in \mathcal{N}(|B|)$, we get $TV = VT$ which proves (3.5).

Next we show that (ii), (d) and (h) hold. Using Proposition 2.3(v), we obtain

$$B|A| = V|B||A| \overset{(3.2)}{\subseteq} V|A||B| \overset{(c)}{=} V|AB| \overset{(b)}{=} |AB|V \overset{(c)}{=} |A||B|V = |A|B,$$

which yields (h). Hence, by [2, Theorem 6.3.2], $E_{|A|}(\varDelta) \in \{B\}'$ for all Borel subsets $\varDelta$ of $[0, \infty)$. Therefore, by (3.5), $VE_{|A|} = E_{|A|}V$, or equivalently

$$V|A| \subseteq |A|V, \tag{3.6}$$

which together with (3.2) implies (ii). As a consequence, we have

$$B|A| = |B|V|A| \overset{(3.6)}{\subseteq} |B||A|V \overset{(3.2)}{\subseteq} |A||B|V = |A|B.$$

Thus, by Lemma 3.1 and (3.2), the operators $|A|B$ and $A|B|$ are selfadjoint which proves (d).

(ii)⇒(i) Assume (ii) holds. Since the first inclusion in (ii) implies the second inclusion in (3.2) (see the first paragraph of this proof), the condition (3.2) is valid. Since, by (3.2) and Lemma 3.1, $A|B|$ is selfadjoint, we see that $AB = (A|B|)V$ is closed and densely defined. For the same reason, $|A||B|$ is positive and selfadjoint. Hence, because U and V are selfadjoint and unitary (see Proposition 2.3(v)), we have

$$|AB|^2 = (U|A||B|V)^*U|A||B|V = V|A||B|UU|A||B|V = V(|A||B|)^2V,$$

which, by the square root theorem, yields

$$|AB| = V|A||B|V. \tag{3.7}$$

By the von Neumann theorem (see [2, Theorem 3.3.7]), the operator $(AB)^*$ is closed and densely defined, and (recall that $|A||B|$ is positive and selfadjoint)

$$|(AB)^*|^2 = U|A||B|V(U|A||B|V)^* = U|A||B|VV|A||B|U = U(|A||B|)^2U,$$

which implies that

$$|(AB)^*| = U|A||B|U. \tag{3.8}$$

As a consequence of (ii) and Proposition 2.3(v), we have

$$V|A||B| \subseteq |A|V|B| = |A||B|V, \tag{3.9}$$

and thus

$$|AB| \overset{(3.7)}{=} V|A||B|V \overset{(3.9)}{\subseteq} |A||B|V^2 = |A||B|, \tag{3.10}$$

which by maximality of selfadjoint operators implies that

$$|AB| = |A||B|. \tag{3.11}$$

Observe now that

$$|B||A|U = |B|A \overset{\text{(ii)}}{\subseteq} A|B| = U|A||B|. \tag{3.12}$$

Taking closures and noting that $\overline{|B||A|} = |A||B|$ (use (3.2) and Lemma 3.1), we get

$$|AB|U \overset{(3.11)}{=} |A||B|U \overset{(3.12)}{\subseteq} U|A||B| \overset{(3.11)}{=} U|AB|.$$

This together with $U = U^*$ and $U^2 = I$ yields (e). Next, by (3.8) and (3.11), we have

$$|(AB)^*| = U|AB|U \overset{\text{(e)}}{\subseteq} |AB|U^2 = |AB|,$$

which by maximality of selfadjoint operators implies that $|(AB)^*| = |AB|$. Therefore AB is normal, which means that (i) holds.

It remains to prove (f) and (g). First note that

$$AB = U|A||B|V \overset{\text{(c)}}{=} U|AB|V \overset{\text{(b)}}{=} UV|AB|,$$

which gives (f). This in turn implies that

$$(AB)^* = |AB|VU \overset{\text{(b)}}{=} V|AB|U \overset{\text{(e)}}{\supseteq} VU|AB|. \tag{3.13}$$

Since AB is normal, we obtain

$$\mathscr{D}((AB)^*) = \mathscr{D}(AB) = \mathscr{D}(|AB|) = \mathscr{D}(VU|AB|).$$

This combined with (3.13) gives (g). This completes the proof. □

The following is a direct consequence of Theorem 3.2(d) and Lemma 3.1.

Corollary 3.3. *Assume that A and B are selfadjoint operators in $\mathcal{H}$ such that $B \in \boldsymbol{B}(\mathcal{H})$ and AB is normal. If at least one of the operators A and B is positive, then AB is selfadjoint, $BA \subseteq AB$ and $AB = \overline{BA}$.*

Note that the case when A is positive has been recently proved in [6, Theorem 1.5]. The case when B is positive has been considered in [8, Corollary 3]. The next corollary can be easily deduced from Corollary 3.3 by observing that $(BA)^* = AB$.

Corollary 3.4. ([6, Theorem 1.1]) *Assume that A and B are selfadjoint operators in $\mathcal{H}$ such that A is positive, $B \in \boldsymbol{B}(\mathcal{H})$ and BA is normal. Then AB is selfadjoint and $BA = AB$.*

4. A Block Operator Model

Our goal in this section is to give a block operator model for two selfadjoint operators A and B such that $B \in \boldsymbol{B}(\mathcal{H})$ and AB is normal. We begin by proving two necessary lemmata.

Lemma 4.1. *Suppose A and B are selfadjoint operators in $\mathcal{H}$ such that $B \in \boldsymbol{B}(\mathcal{H})$ and AB is normal. Let $B = V|B|$ be the unitary polar decomposition of B. Then there exist closed vector subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of $\mathcal{H}$ such that*

(i) $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$,

(ii) *$\mathcal{H}_+$ and $\mathcal{H}_-$ reduce $|A|$, $|B|$, B and V,*

(iii) *$|A| = A_+ \oplus A_-$, $|B| = B_+ \oplus B_-$, $B = B_+ \oplus (-B_-)$ and $V = I_{\mathcal{H}_+} \oplus (-I_{\mathcal{H}_-})$, where $A_\pm$ (resp., $B_\pm$) is the restriction of $|A|$ (resp., $|B|$) to $\mathcal{H}_\pm$; $A_\pm$ and $B_\pm$ are positive selfadjoint operators in $\mathcal{H}_\pm$ and $B_\pm \in \boldsymbol{B}(\mathcal{H}_\pm)$,*

(iv) *$\mathcal{N}(B_-) = \{0\}$ and $B_\pm A_\pm \subseteq A_\pm B_\pm$,*

(v) *$V|A| = |A|V$.*

Proof. (i)-(iii) Denote by E_V and $E_{|A|}$ the spectral measures of the selfadjoint operators V and $|A|$, respectively. Since, by Proposition 2.3(v), $V = V^*$ and $V^2 = I$, the spectral mapping theorem and the spectral theorem imply that $\sigma(V) \subseteq \{-1, 1\}$ and $V = I_{\mathcal{H}_+} \oplus (-I_{\mathcal{H}_-})$, where $\mathcal{H}_\pm := \mathcal{R}(E_V(\{\pm 1\}))$. By Theorem 3.2(ii) and [2, Theorems 6.2.4 and 6.3.2], $E_V(\varDelta)|A| \subseteq |A|E_V(\varDelta)$ for all Borel subsets $\varDelta$ of $\mathbb{R}$. In particular, we have

$$E_V(\{\pm 1\})|A| \subseteq |A|E_V(\{\pm 1\}),$$

which implies that $\mathcal{H}_\pm$ reduces $|A|$. Clearly, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Since, by Proposition 2.3(v), $B = V|B| = |B|V$, the same argument as above shows that $\mathcal{H}_\pm$ reduces $|B|$ and consequently $\mathcal{H}_\pm$ reduces B. This implies (i), (ii) and (iii).

(iv) Since $V|_{\mathcal{N}(|B|)} = I_{\mathcal{N}(|B|)}$ and, by (iii), $\mathcal{N}(|B|) = \mathcal{N}(B_+) \oplus \mathcal{N}(B_-)$ and $V = I_{\mathcal{H}_+} \oplus (-I_{\mathcal{H}_-})$, we deduce that $\mathcal{N}(B_-) = \{0\}$. It follows from (iii) and Theorem 3.2(h) that $B_\pm A_\pm \subseteq A_\pm B_\pm$. This yields (iv).

(v) This is a direct consequence of (iii) and Theorem 3.2(ii). $\square$

Lemma 4.2. *Suppose A and B are selfadjoint operators in $\mathcal{H}$ such that $B \in \boldsymbol{B}(\mathcal{H})$ and AB is normal. Let $A = U|A|$ and $B = V|B|$ be the unitary polar decompositions of A and B, respectively, and let $\mathcal{H}_\pm$, $A_\pm$ and $B_\pm$ be as in Lemma* 4.1. *Then there exist operators $X \in \boldsymbol{B}(\mathcal{H}_+)$, $Y \in \boldsymbol{B}(\mathcal{H}_-, \mathcal{H}_+)$ and $Z \in \boldsymbol{B}(\mathcal{H}_-)$ such that*

(i) $X = X^*$ *and* $Z = Z^*$,

(ii) $X^2 + YY^* = I_{\mathcal{H}_+}$, $Y^*Y + Z^2 = I_{\mathcal{H}_-}$ *and* $XY + YZ = 0$,

(iii) $X|_{\mathcal{N}(A_+)} = I_{\mathcal{N}(A_+)}$, $Y|_{\mathcal{N}(A_-)} = 0$, $Y^*|_{\mathcal{N}(A_+)} = 0$ *and* $Z|_{\mathcal{N}(A_-)} = I_{\mathcal{N}(A_-)}$,

(iv) $XA_+ \subseteq A_+X$, $YA_- \subseteq A_+Y$ *and* $ZA_- \subseteq A_-Z$,

(v) $B_+XA_+ \subseteq XA_+B_+$, $B_+YA_- \subseteq YA_-B_-$, $B_-Y^*A_+ \subseteq Y^*A_+B_+$ *and* $B_-ZA_- \subseteq ZA_-B_-$,

(vi) $U = \left[\begin{smallmatrix} X & Y \\ Y^* & Z \end{smallmatrix}\right]$ *with respect to* $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

Proof. (i)-(iii)&(vi) Let $\left[\begin{smallmatrix} X & Y \\ \widetilde{Y} & Z \end{smallmatrix}\right]$ be the block matrix form of U with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $X \in \boldsymbol{B}(\mathcal{H}_+)$, $Y \in \boldsymbol{B}(\mathcal{H}_-, \mathcal{H}_+)$, $\widetilde{Y} \in \boldsymbol{B}(\mathcal{H}_+, \mathcal{H}_-)$ and $Z \in \boldsymbol{B}(\mathcal{H}_-)$. It follows from Proposition 2.3(v) that $U = U^*$ and $U^2 = I$. The first equality implies that $X = X^*$, $Z = Z^*$ and $\widetilde{Y} = Y^*$, which gives (i) and (vi). This and the equality $U^2 = I$ yield (ii). In turn, since $U|_{\mathcal{N}(|A|)} = I_{\mathcal{N}(|A|)}$ and, by Lemma 4.1(iii), $\mathcal{N}(|A|) = \mathcal{N}(A_+) \oplus \mathcal{N}(A_-)$, we see that (iii) is valid.

(iv) It follows from Lemma 4.1(iii) that $|A| = A_+ \oplus A_-$ and so

$$\mathscr{D}(|A|) = \mathscr{D}(A_+) \oplus \mathscr{D}(A_-). \tag{4.1}$$

Take $f_- \in \mathscr{D}(A_-)$. Then

$$0 \oplus f_- \in \mathscr{D}(|A|) = \mathscr{D}(A) = \mathscr{D}(U|A|),$$

and thus, since $U|A| \subseteq |A|U$ (see Proposition 2.3(v)), we get

$$Yf_- \oplus Zf_- \overset{\text{(vi)}}{=} U(0 \oplus f_-) \in \mathscr{D}(|A|).$$

This together with (4.1) implies that $Yf_- \in \mathscr{D}(A_+)$ and $Zf_- \in \mathscr{D}(A_-)$. Moreover, we have

$$YA_-f_- \oplus ZA_-f_- = U|A|(0 \oplus f_-) = |A|U(0 \oplus f_-) = A_+Yf_- \oplus A_-Zf_-.$$

Putting this all together yields $YA_- \subseteq A_+Y$ and $ZA_- \subseteq A_-Z$. Taking $f_+ \in \mathscr{D}(A_+)$ and arguing as above with $f_+ \oplus 0$ in place of $0 \oplus f_-$, we get $XA_+ \subseteq A_+X$. This completes the proof of (iv).

(v) It follows from Proposition 2.3(v) and Theorem 3.2(ii) that $A = U|A| = |A|U$ and $|B|A \subseteq A|B|$. This implies that $|B|U|A| \subseteq U|A||B|$. In turn, by Lemma 4.1(iii), we have $|B| = B_+ \oplus B_-$. Take $f_+ \in \mathscr{D}(A_+)$. Then, by (4.1), $f_+ \oplus 0 \in \mathscr{D}(|A|) = \mathscr{D}(|B|U|A|)$, and consequently $B_+f_+ \oplus 0 = |B|(f_+ \oplus 0) \in \mathscr{D}(|A|)$. Hence $B_+f_+ \in \mathscr{D}(A_+)$. Moreover, we have

$$\begin{aligned} B_+XA_+f_+ \oplus B_-Y^*A_+f_+ &= |B|U|A|(f_+ \oplus 0) \\ &= U|A||B|(f_+ \oplus 0) \\ &= XA_+B_+f_+ \oplus Y^*A_+B_+f_+. \end{aligned}$$

This implies that $B_+XA_+ \subseteq XA_+B_+$ and $B_-Y^*A_+ \subseteq Y^*A_+B_+$. Taking $f_- \in \mathscr{D}(A_-)$ and arguing as above with $0 \oplus f_-$ in place of $f_+ \oplus 0$, we get $B_+YA_- \subseteq YA_-B_-$ and $B_-ZA_- \subseteq ZA_-B_-$. This completes the proof. □

Before stating the main result of this section, we give a primary definition.

Definition 4.3. Let A and B be operators in $\mathcal{H}$, $\mathcal{H}_\pm$ be closed vector subspaces of $\mathcal{H}$, $A_\pm$ be operators in $\mathcal{H}_\pm$, $B_\pm \in \boldsymbol{B}(\mathcal{H}_\pm)$, $X \in \boldsymbol{B}(\mathcal{H}_+)$, $Y \in \boldsymbol{B}(\mathcal{H}_-, \mathcal{H}_+)$ and $Z \in \boldsymbol{B}(\mathcal{H}_-)$. We say that $(\mathcal{H}_\pm, A_\pm, B_\pm, X, Y, Z)$ is a *block operator model* for the pair (A, B) if the following conditions hold:

1° $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$,

2° $A_\pm$ and $B_\pm$ are positive and selfadjoint and $\mathcal{N}(B_-) = \{0\}$,

3° the conditions (i)-(v) of Lemma 4.2 hold,

4° $A = \left[\begin{smallmatrix} XA_+ & YA_- \\ Y^*A_+ & ZA_- \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} B_+ & 0 \\ 0 & -B_- \end{smallmatrix}\right]$ with respect to $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

Now, we are in a position to establish a block operator model for two selfadjoint operators whose product is normal still under the assumption that the second factor is bounded.

Theorem 4.4. *Let A and B be operators in a complex Hilbert space $\mathcal{H}$. Then the following conditions are equivalent*:

(i) *A and B are selfadjoint, $B \in \boldsymbol{B}(\mathcal{H})$ and AB is normal,*

(ii) *(A, B) has a block operator model,*

(iii) *(A, B) has a block operator model $(\mathcal{H}_\pm, A_\pm, B_\pm, X, Y, Z)$ such that $B_\pm A_\pm \subseteq A_\pm B_\pm$.*

If $(\mathcal{H}_\pm, A_\pm, B_\pm, X, Y, Z)$ is a block operator model for (A, B) and $A = U|A|$ and $B = V|B|$ are the unitary polar decompositions of A and B, respectively, then

$$\mathcal{H}_\pm = \mathcal{N}(I_{\mathcal{H}} \mp V), \tag{4.2}$$

$$U = \left[\begin{smallmatrix} X & Y \\ Y^* & Z \end{smallmatrix}\right], \quad |A| = \left[\begin{smallmatrix} A_+ & 0 \\ 0 & A_- \end{smallmatrix}\right], \quad V = \left[\begin{smallmatrix} I_{\mathcal{H}_+} & 0 \\ 0 & -I_{\mathcal{H}_-} \end{smallmatrix}\right], \quad |B| = \left[\begin{smallmatrix} B_+ & 0 \\ 0 & B_- \end{smallmatrix}\right]. \tag{4.3}$$

Furthermore, (A, B) has at most one block operator model.

Proof. (i)⇒(iii) This is a direct consequence of Proposition 2.2 and Lemmata 4.1 and 4.2.

(iii)⇒(ii) Obvious.

(ii)⇒(i) Let $(\mathcal{H}_\pm, A_\pm, B_\pm, X, Y, Z)$ be a block operator model for (A, B). Set $\widehat{V} = I_{\mathcal{H}_+} \oplus (-I_{\mathcal{H}_-})$. It follows from 2° and 4° of Definition 4.3 that $B \in \boldsymbol{B}(\mathcal{H})$, B is selfadjoint, $|B| = B_+ \oplus B_-$ and $\widehat{V}$ is a unitary operator such that $B = \widehat{V}|B| = |B|\widehat{V}$ and $\widehat{V}|_{\mathcal{N}(|B|)} = I_{\mathcal{N}(|B|)}$. Hence, by the uniqueness of the unitary factor in the unitary polar decomposition of B, we see that $V = \widehat{V}$. The operator $\widehat{U} := \left[\begin{smallmatrix} X & Y \\ Y^* & Z \end{smallmatrix}\right]$ is bounded and selfadjoint, and by the condition (ii) of Lemma 4.2, $\widehat{U}^2 = I$. This means that $\widehat{U}$ is unitary. It follows from 4° of Definition 4.3 that $A = \widehat{U}(A_+ \oplus A_-)$. As a consequence, the operator A is closed and densely defined. This yields

$$|A|^2 = A^*A = (A_+ \oplus A_-)\widehat{U}^*\widehat{U}(A_+ \oplus A_-) = (A_+ \oplus A_-)^2.$$

Since, by 2° of Definition 4.3, the operator $A_+ \oplus A_-$ is positive and selfadjoint, we deduce that $|A| = A_+ \oplus A_-$. This implies that $A = \widehat{U}|A|$. It follows from the condition (iv) of Lemma 4.2 that $YA_- \subseteq A_+Y$, which yields $Y^*A_+ \subseteq A_-Y^*$. Now, it is a routine matter to verify, by using the condition (iv) of Lemma 4.2, that $\widehat{U}|A| \subseteq |A|\widehat{U}$. Thus $|A| \subseteq \widehat{U}|A|\widehat{U}$. Since the operators $|A|$ and $\widehat{U}|A|\widehat{U}$ are selfadjoint, maximality of selfadjoint operators yields $|A| = \widehat{U}|A|\widehat{U}$. Therefore $A = \widehat{U}|A| = |A|\widehat{U}$. Applying the condition (iii) of Lemma 4.2 and using the fact that $\mathcal{N}(|A|) = \mathcal{N}(A_+) \oplus \mathcal{N}(A_-)$, we verify that $\widehat{U}|_{\mathcal{N}(|A|)} = I_{\mathcal{N}(|A|)}$. By Corollary 2.4, the operator A is selfadjoint and $A = \widehat{U}|A|$ is the unitary polar decomposition of A. By the uniqueness of the unitary factor in the unitary polar decomposition of A, we have $U = \widehat{U}$. This completes the proof of (4.3). As a consequence, (4.2) holds. Since $V = I_{\mathcal{H}_+} \oplus (-I_{\mathcal{H}_-})$ and $|A| = A_+ \oplus A_-$, we see that $V|A| = |A|V$. Using the condition (v) of Lemma 4.2 and the first equality in 4° of Definition 4.3, we verify that $|B|A \subseteq A|B|$. It follows from Theorem 3.2 that the product AB is normal.

Combining (4.2) with (4.3) and the uniqueness of the unitary factors in the unitary polar decompositions of B and A shows that (A, B) has at most one block operator model. This completes the proof. □

The following corollary shows that two pairs of operators having block operator models are unitarily equivalent if and only if their models are unitarily equivalent. It can be deduced from Theorem 4.4 via (4.2) and (4.3). The details are left to the reader.

Corollary 4.5. *Let $(\mathcal{H}_\pm, A_\pm, B_\pm, X, Y, Z)$ and $(\mathcal{H}'_\pm, A'_\pm, B'_\pm, X', Y', Z')$ be block operator models for pairs (A, B) and (A', B') of operators in complex Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, respectively. Then the following conditions are equivalent*:

(i) *there exists a unitary isomorphism $W\colon \mathcal{H} \to \mathcal{H}'$ such that*

$$WA = A'W, \quad WB = B'W,$$

(ii) *there exist unitary isomorphisms* $W_\pm\colon \mathcal{H}_\pm \to \mathcal{H}'_\pm$ *such that*

$$W_\pm A_\pm = A'_\pm W_\pm, \quad W_\pm B_\pm = B'_\pm W_\pm,$$
$$W_+ X = X' W_+, \quad W_+ Y = Y' W_-, \quad W_- Z = Z' W_-.$$

5. The Bounded Case

We begin this section by adapting the block operator model given in Section 4 to the context of bounded operators. The following theorem is a direct consequence of Theorem 4.4.

Theorem 5.1. *Let* $A, B \in \boldsymbol{B}(\mathcal{H})$, $\mathcal{H}_\pm$ *be closed vector subspaces of* $\mathcal{H}$, $A_\pm, B_\pm \in \boldsymbol{B}(\mathcal{H}_\pm)$, $X \in \boldsymbol{B}(\mathcal{H}_+)$, $Y \in \boldsymbol{B}(\mathcal{H}_-, \mathcal{H}_+)$ *and* $Z \in \boldsymbol{B}(\mathcal{H}_-)$. *Then* $(\mathcal{H}_\pm, A_\pm, B_\pm, X, Y, Z)$ *is a block operator model for* (A, B) *if and only if the following conditions hold*:

(a) $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$,

(b) $A_\pm$ *and* $B_\pm$ *are positive and selfadjoint,* $\mathcal{N}(B_-) = \{0\}$ *and* $A_\pm B_\pm = B_\pm A_\pm$,

(c) *the conditions* (i)-(iii) *of Lemma* 4.2 *hold,*

(d) $X \in \{A_+\}'$, $A_+ Y = Y A_-$ *and* $Z \in \{A_-\}'$,

(e) $X \in \{A_+ B_+\}'$, $A_+ B_+ Y = Y A_- B_-$ *and* $Z \in \{A_- B_-\}'$,

(f) $A = \begin{bmatrix} XA_+ & YA_- \\ Y^*A_+ & ZA_- \end{bmatrix}$ *and* $B = \begin{bmatrix} B_+ & 0 \\ 0 & -B_- \end{bmatrix}$ *with respect to* $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

The case when the product AB is selfadjoint makes all block operators appearing in our model diagonal.

Corollary 5.2. *Let* $(\mathcal{H}_\pm, A_\pm, B_\pm, X, Y, Z)$ *be a block operator model for* (A, B), *where* $A, B \in \boldsymbol{B}(\mathcal{H})$. *Then* $A_\pm \in \boldsymbol{B}(\mathcal{H}_\pm)$ *and the following conditions are equivalent*:

(i) AB *is selfadjoint,*

(ii) $A_+ B_+ Y = 0$,

(iii) $Y A_- B_- = 0$,

(iv) $Y = 0$.

Proof. By 2° and 4° in Definition 4.3 and the closed graph theorem, $A_\pm \in \boldsymbol{B}(\mathcal{H}_\pm)$. Since the operators X and Z are selfadjoint, we infer from the conditions (b) and (e) of Theorem 5.1 that

(5.1) the operators XA_+B_+ and ZA_-B_- are selfadjoint,

(5.2) $$(YA_-B_-)^* = (A_+B_+Y)^* = Y^*B_+A_+ = Y^*A_+B_+.$$

It follows from the condition (f) of Theorem 5.1 that

$$AB = \begin{bmatrix} XA_+B_+ & -YA_-B_- \\ Y^*A_+B_+ & -ZA_-B_- \end{bmatrix}. \tag{5.3}$$

(i)⇒(ii) If AB is selfadjoint, then by (5.2) and (5.3), we have

$$Y^*A_+B_+ = -Y^*A_+B_+,$$

hence $Y^*A_+B_+ = 0$, and thus by the condition (b) of Theorem 5.1, $A_+B_+Y = 0$.

(ii)⇒(iii) Apply the condition (e) of Theorem 5.1.

(iii)⇒(iv) Taking adjoints, we see that $B_-A_-Y^* = 0$. Since, by the condition (b) of Theorem 5.1, $\mathcal{N}(B_-) = \{0\}$, we deduce that $A_-Y^* = 0$. Hence $YA_- = 0$, which implies that $Y|_{\overline{\mathcal{R}(A_-)}} = 0$. However, by the condition (iii) of Lemma 4.2, $Y|_{\mathcal{N}(A_-)} = 0$. As a consequence, we have $Y = 0$.

(iv)⇒(i) Apply (5.1) and (5.3). □

We conclude this paper by reexamining the example established by Rehder (see [11, p. 815]).

Example 5.3. Let $\mathcal{H} = \mathbb{C}^2$ and $A, B \in \boldsymbol{B}(\mathcal{H})$ be selfadjoint operators having the following block matrix forms

$$A = \begin{bmatrix} -1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

with respect to the standard orthogonal decomposition $\mathcal{H} = \mathbb{C} \oplus \mathbb{C}$. Since

$$AB = -BA = \begin{bmatrix} i & -1 \\ 1 & -\mathrm{i} \end{bmatrix},$$

the product AB is normal but not selfadjoint. It is easily seen that the factors of the unitary polar decompositions $A = U|A|$ and $B = V|B|$ have the following block matrix forms

$$U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} \\ -\frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad |A| = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad |B| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

with respect to the standard orthogonal decomposition $\mathcal{H} = \mathbb{C} \oplus \mathbb{C}$.

Now, we describe the block operator model for (A, B). It follows from (4.2) that $\mathcal{H}_\pm$ is equal to the linear span of $u_\pm$, where $u_\pm = \frac{1}{\sqrt{2}}(1, \pm 1)$. This implies that the operators U, $|A|$, V and $|B|$ have the following block matrix forms

$$U = \begin{bmatrix} 0 & -\frac{1+\mathrm{i}}{\sqrt{2}} \\ \frac{-1+\mathrm{i}}{\sqrt{2}} & 0 \end{bmatrix}, \quad |A| = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad |B| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Hence, the remaining components of the block operator model for (A, B) take the form

$$A_\pm = \sqrt{2}, \quad B_\pm = 1, \quad X = 0, \quad Y = -\frac{1+\mathrm{i}}{\sqrt{2}}, \quad Z = 0.$$

Acknowledgement. A substantial part of this paper was written while the first author visited Jagiellonian University during the spring of 2017 and the second author visited Kyungpook National University during the summer of 2017. They wish to thank the faculties and the administrations of these units for their warm hospitality.

References

[1] E. Albrecht, P. G. Spain, *When products of self-adjoints are normal*, Proc. Amer. Math. Soc., **128**(2000), 2509-2511.

[2] M. Sh. Birman, M. Z. Solomjak, *Spectral theory of selfadjoint operators in Hilbert space*, D. Reidel Publishing Co., Dordrecht, 1987.

[3] J. Bognár, Indefinite inner product spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 78. Springer-Verlag, New York-Heidelberg, 1974.

[4] S. Dehimi, M. H. Mortad, *Right (or left) invertibility of bounded and unbounded operators and applications to the spectrum of products*, Complex Anal. Oper. Theory (2017). https://doi.org/10.1007/s11785-017-0687-z.

[5] T. Furuta, Invitation to linear operators, Taylor & Francis, Ltd., London, 2001.

[6] K. Gustafson, M. H. Mortad, *Conditions implying commutativity of unbounded self-adjoint operators and related topics*, J. Operator Theory, **76**(2016), 159-169.

[7] Z. J. Jabłoński, I. B. Jung, J. Stochel, *Unbounded quasinormal operators revisited*, Integr. Equ. Oper. Theory, **79**(2014), 135-149.

[8] M. H. Mortad, *An application of the Putnam-Fuglede theorem to normal products of selfadjoint operators*, Proc. Amer. Math. Soc., **131**(2003), 3135-3141.

[9] M. H. Mortad, *On some product of two unbounded self-adjoint operators*, Integr. Equ. Oper. Theory, **64**(2009), 399-408.

[10] C. R. Putnam, *On normal operators in Hilbert space*, Amer. J. Math., **73**(1951), 357-362.

[11] W. Rehder, *On the Product of Self-adjoint Operators*, Internat. J. Math. Math. Sci., **5**(1982), 813-816.