KYUNGPOOK Math. J. 57(2017), 503-516
https://doi.org/10.5666/KMJ.2017.57.3.503
pISSN 1225-6951 eISSN 0454-8124

Totally Umbilical Slant Lightlike Submanifolds of Indefinite Kaehler Manifolds

Rashmi Sachdeva
Department of Basic and Applied Sciences, Punjabi University, Patiala 147-002, India
e-mail : rashmi.sachdeva86@gmail.com

Rakesh Kumar*
Department of Basic and Applied Sciences, Punjabi University, Patiala 147-002, India
e-mail : dr_rk37c@yahoo.co.in

Satvinder Singh Bhatia
School of Mathematics, Thapar University, Patiala 147-001, India
e-mail : ssbhatia@thapar.edu

Abstract. In this paper, we study totally umbilical slant lightlike submanifolds of indefinite Kaehler manifolds. We prove that there do not exist totally umbilical proper slant lightlike submanifolds in indefinite Kaehler manifolds other than totally geodesic proper slant lightlike submanifolds. We also prove that there do not exist totally umbilical proper slant lightlike submanifolds of indefinite Kaehler space forms. Finally, we give a characterization theorem on minimal slant lightlike submanifolds.

1. Introduction

The notion of slant submanifolds was initiated by B. Y. Chen, as a generalization of both holomorphic and totally real submanifolds in complex geometry [5, 6]. Since then such submanifolds have been studied by many authors. In particular, N. Papaghiuc [15] introduced semi-slant submanifolds. A. Lotta [12, 13], defined and studied slant submanifolds in contact geometry. J. L. Cabrerizo et al. studied slant, semi-slant and bi-slant submanifolds in contact geometry [3, 4]. They all studied the geometry of slant submanifolds with positive definite metric. Therefore

* Corresponding Author.
Received March 4, 2015; accepted July 8, 2017.
2010 Mathematics Subject Classification: 53C15, 53C40, 53C50.
Key words and phrases: Slant lightlike submanifolds, totally umbilical lightlike submanifolds, totally geodesic lightlike submanifolds.

this geometry may not be applicable to the other branches of mathematics and physics, where the metric is not necessarily definite. Thus the geometry of slant submanifolds with indefinite metric became a topic of chief discussion and Sahin [21] played a very crucial role in this study by introducing the notion of slant lightlike submanifolds of indefinite Hermitian manifolds. R. S. Gupta et al. [11], introduced the notion of a slant lightlike submanifold of an indefinite Cosymplectic manifold and obtained necessary and sufficient conditions for the existence of a slant lightlike submanifold. Recently in [18, 19, 20], we also studied the geometry of slant and hemi-slant lightlike submanifolds of indefinite contact manifolds.

Sahin [22] proved that there do not exist totally umbilical proper slant submanifolds in Kaehler manifolds other than totally geodesic proper slant submanifolds. It is known that a proper slant submanifold of a Kaehler manifold is even dimensional, but this is not true for slant lightlike submanifold [21]. In [16] and [17], authors already proved that there do not exist totally contact umbilical proper slant lightlike submanifolds in indefinite Cosymplectic and indefinite Sasakian manifolds other than totally contact geodesic proper slant lightlike submanifolds, respectively. In this paper, we study totally umbilical slant lightlike submanifolds of indefinite Kaehler manifolds. We prove that there do not exist totally umbilical proper slant lightlike submanifolds in indefinite Kaehler manifolds other than totally geodesic proper slant lightlike submanifolds. We also prove that there do not exist totally umbilical proper slant lightlike submanifolds of indefinite Kaehler space forms. Finally, we give characterization theorems on minimal slant lightlike submanifolds.

2. Lightlike Submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $RadTM : x \in M \longrightarrow RadT_xM$, defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold (for detail see [7]) and $RadTM$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is,

$$TM = RadTM \perp S(TM), \tag{2.1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to

TM in $T\bar{M}\mid_M$ and to $RadTM$ in $S(TM^\perp)^\perp$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^\perp). \tag{2.2}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^\perp). \tag{2.3}$$

Let u be a local coordinate neighborhood of M then for quasi-orthonormal fields of frames $\{\xi_1, ..., \xi_r, W_{r+1}, ..., W_n, N_1, ..., N_r, X_{r+1}, ..., X_m\}$, we have

Theorem 2.1.([7]) *Let $(M, g, S(TM), S(TM^\perp))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^\perp)^\perp$ and a basis of $\Gamma(ltr(TM)\mid_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TM^\perp)^\perp\mid_{\mathrm{u}}$, where u is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad \text{for any} \quad i, j \in \{1, 2, .., r\}, \tag{2.4}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then according to the decomposition (2.3), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^\perp_X U, \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^\perp_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is linear a operator on M, known as shape operator.

According to (2.2), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$, respectively then Gauss and Weingarten formulas give

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.6}$$

where we put $h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)), D^l_X U = L(\nabla^\perp_X U)$, $D^s_X U = S(\nabla^\perp_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^\perp))$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad \bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.7}$$

where $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$. By using (2.2)-(2.3) and (2.6)-(2.7), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.8}$$

for any $\xi \in \Gamma(RadTM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N, N' \in \Gamma(ltr(TM))$.

Let $\bar{P}$ is a projection of TM on $S(TM)$. Now, we consider the decomposition (2.4), we can write

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \quad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X \bar{P}Y,\ A^*_\xi X\}$ and $\{h^*(X, \bar{P}Y),\ \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$ respectively. Here ∇^* and ∇^{*t}_X are linear connections on $S(TM)$ and $RadTM$ respectively. By using (2.7) and (2.9), we obtain

$$\bar{g}(h^l(X, \bar{P}Y), \xi) = g(A^*_\xi X, \bar{P}Y), \quad \bar{g}(h^*(X, \bar{P}Y), N) = \bar{g}(A_N X, \bar{P}Y). \tag{2.10}$$

In [1], Barros and Romero defined indefinite Kaehler manifolds as

Definition 2.2. Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called an indefinite Kaehler manifold if $\bar{J}$ is parallel with respect to $\bar{\nabla}$, that is, $(\bar{\nabla}_X \bar{J})Y = 0$, for any $X, Y \in \Gamma(T\bar{M})$.

3. Slant Lightlike Submanifolds

A lightlike submanifold has two distributions, namely radical distribution and screen distribution. The radical distribution is totally lightlike and it is not possible to define angle between two vector fields of radical distribution. Furthermore, the screen distribution is non-degenerate. There are some definitions for angle between two vector fields in Lorentzian setup [14], but not appropriate for our goal because a manifold with Lorentzian metric can not admit an almost Hermitian structure [10]. Therefore to introduce the notion of slant lightlike submanifolds one needs a Riemannian distribution. For such distribution Sahin [21] proved the following lemma.

Lemma 3.1. *Let M be an r-lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$ of index $2r$. Suppose that $\bar{J}RadTM$ is a distribution on M such that $RadTM \cap \bar{J}RadTM = \{0\}$. Then any complementary distribution to $\bar{J}RadTM \oplus \bar{J}ltr(TM)$ in $S(TM)$ is Riemannian.*

In the light of above lemma Sahin [21], defines slant lightlike submanifolds as

Definition 3.2. Let M be a r-lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$ of index $2r$. Then M is a slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied

(A) $Rad(TM)$ is a distribution on M such that

$$\bar{J}RadTM \cap Rad(TM) = \{0\}.$$

(B) For each non-zero vector field tangent to D at $p \in U \subset M$, the angle $\theta(X)$ between $\bar{J}X$ and the vector space D_p is constant, that is, it is independent of the choice of $p \in U \subset M$ and $X \in D_p$, where D is complementary distribution to $\bar{J}RadTM \oplus \bar{J}ltr(TM)$ in the screen distribution $S(TM)$.

This constant angle $\theta(X)$ is called slant angle of the distribution D. A slant lightlike submanifold is said to be proper if $D \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$. Since a submanifold M is invariant (respectively anti-invariant) if $\bar{J}T_pM \subset T_pM$, (respectively $\bar{J}T_pM \subset T_pM^{\perp}$), for any $p \in M$. Therefore from above definition, it is clear that M is invariant (respectively anti-invariant) if $\theta(X) = 0$, (respectively $\theta(X) = \frac{\pi}{2}$).

The tangent bundle TM of M is decomposed as

$$TM = RadTM \perp S(TM) = RadTM \perp (\bar{J}RadTM \oplus \bar{J}ltr(TM)) \perp D. \tag{3.1}$$

For any $X \in \Gamma(TM)$ we write

$$\bar{J}X = TX + FX, \tag{3.2}$$

where TX is the tangential component of $\bar{J}X$ and FX is the transversal component of $\bar{J}X$. Similarly for any $V \in \Gamma(tr(TM))$ we write

$$\bar{J}V = BV + CV, \tag{3.3}$$

where BV is the tangential component of $\bar{J}V$ and CV is the transversal component of $\bar{J}V$. Using the decomposition in (3.1), we denote by P_1, P_2, Q_1 and Q_2 the projection on the distributions $RadTM$, $\bar{J}RadTM$, $\bar{J}ltr(TM)$ and D, respectively. Then for any $X \in \Gamma(TM)$, we can write

$$X = P_1X + P_2X + Q_1X + Q_2X. \tag{3.4}$$

Applying $\bar{J}$ to (3.4) we obtain

$$\bar{J}X = \bar{J}P_1X + \bar{J}P_2X + FQ_1X + TQ_2X + FQ_2X. \tag{3.5}$$

Then using (3.2) and (3.3), we get

$$\bar{J}P_1X = TP_1X \in \Gamma(\bar{J}RadTM), \quad \bar{J}P_2X = TP_2X \in \Gamma(RadTM),$$

$$FP_1X = FP_2X = 0, \quad TQ_2X \in \Gamma(D), \quad FQ_1X \in \Gamma(ltr(TM)).$$

Lemma 3.3. *Let M be a slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then $FQ_2X \in \Gamma(S(TM^{\perp}))$, for any $X \in \Gamma(TM)$.*

Proof. Using (2.2) and (2.4) it is clear that $FQ_2X \in \Gamma(S(TM^{\perp}))$ if and only if $g(FQ_2X, \xi) = 0$, for any $\xi \in \Gamma(RadTM)$. Therefore $g(FQ_2X, \xi) = g(\bar{J}Q_2X - TQ_2X, \xi) = g(\bar{J}Q_2X, \xi) = -g(Q_2X, \bar{J}\xi) = 0$, gives the result. □

Thus from the Lemma 3.3. it follows that $F(D_p)$ is a subspace of $S(TM^{\perp})$. Therefore there exists an invariant subspace μ_p of $T_p\bar{M}$ such that

$$S(T_pM^{\perp}) = F(D_p)\perp\mu_p, \tag{3.6}$$

and

$$T_p\bar{M} = S(T_pM)\perp\{Rad(T_pM)\oplus ltr(T_pM)\}\perp\{F(D_p)\perp\mu_p\}. \tag{3.7}$$

Differentiating (3.5) and using (2.6)-(2.7), (3.2) and (3.3), for any $X, Y \in \Gamma(TM)$, we have

$$(\nabla_X T)Y = A_{FQ_1Y}X + A_{FQ_2Y}X + Bh(X,Y),$$

and

$$\begin{aligned} D^s(X, FQ_1Y) + D^l(X, FQ_2Y) &= F\nabla_X Y - h(X,TY) + Ch(X,Y) \\ &\quad -\nabla^s_X FQ_2Y - \nabla^l_X FQ_1Y. \end{aligned} \tag{3.8}$$

Corollary 3.4.([21]) *Let M be a slant lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$. Then we have*

$$g(TQ_2X, TQ_2Y) = cos^2\theta g(Q_2X, Q_2Y), \tag{3.9}$$

and

$$g(FQ_2X, FQ_2Y) = sin^2\theta g(Q_2X, Q_2Y), \tag{3.10}$$

for any $X, Y \in \Gamma(TM)$.

4. Totally Umbilical Slant Lightlike Submanifolds

Definition 4.1.([8]) A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that, for $X, Y \in \Gamma(TM)$,

$$h(X,Y) = H\bar{g}(X,Y). \tag{4.1}$$

Using (2.6)-(2.7) it is clear that M is a totally umbilical, if and only if, on each coordinate neighborhood u there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^{\perp}))$ such that

$$h^l(X,Y) = H^l g(X,Y), \quad h^s(X,Y) = H^s g(X,Y), \quad D^l(X,W) = 0, \tag{4.2}$$

for $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$. A lightlike submanifold is said to be totally geodesic if $h(X,Y) = 0$, for any $X, Y \in \Gamma(TM)$. Therefore in other words, a lightlike submanifold is totally geodesic if $H^l = 0$ and $H^s = 0$.

Theorem 4.2. *Let M be a totally umbilical slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then at least one of the following statements is true:*
(i) *M is an anti-invariant submanifold.*
(ii) *$D = \{0\}$.*
(iii) *If M is a proper slant submanifold, then $H^s \in \Gamma(\mu)$.*

Proof. Let M be a totally umbilical slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then for any $X = Q_2X \in \Gamma(D)$ using (4.1), we have

$$h(TQ_2X, TQ_2X) = g(TQ_2X, TQ_2X)H,$$

therefore from (2.5), (3.9) and above equation, we get

$$\bar{\nabla}_{TQ_2X}TQ_2X - \nabla_{TQ_2X}TQ_2X = cos^2\theta g(Q_2X, Q_2X)H.$$

Using (3.2) and the fact that $\bar{M}$ is Kaehler manifold, we obtain

$$\bar{J}\bar{\nabla}_{TQ_2X}Q_2X - \bar{\nabla}_{TQ_2X}FQ_2X - \nabla_{TQ_2X}TQ_2X = cos^2\theta g(Q_2X, Q_2X)H.$$

Then using (2.6) and (2.7), we get

$$\begin{aligned}&\bar{J}\nabla_{TQ_2X}Q_2X + \bar{J}h^l(TQ_2X, X) + \bar{J}h^s(TQ_2X, X) + A_{FQ_2X}TQ_2X\\ &-\nabla^s_{TQ_2X}FQ_2X - D^l(TQ_2X, FQ_2X) - \nabla_{TQ_2X}TQ_2X = cos^2\theta g(Q_2X, Q_2X)H.\end{aligned}$$

Thus using (3.2), (3.3) and (4.2), we have

$$\begin{aligned}&T\nabla_{TQ_2X}Q_2X + F\nabla_{TQ_2X}Q_2X + g(TQ_2X, X)\bar{J}H^l + g(TQ_2X, X)BH^s\\ &+g(TQ_2X, X)CH^s + A_{FQ_2X}TQ_2X - \nabla^s_{TQ_2X}FQ_2X - D^l(TQ_2X, FQ_2X)\\ &-\nabla_{TQ_2X}TQ_2X = cos^2\theta g(Q_2X, Q_2X)H.\end{aligned}$$

Equating the transversal components, we get

$$\begin{aligned}&F\nabla_{TQ_2X}Q_2X + g(TQ_2X, X)CH^s - \nabla^s_{TQ_2X}FQ_2X - D^l(TQ_2X, FQ_2X)\\ &= cos^2\theta g(Q_2X, Q_2X)H.\end{aligned} \tag{4.3}$$

On the other hand, (3.10) holds for any $X = Y \in \Gamma(D)$ and by taking the covariant derivative with respect to TQ_2X, we obtain

$$g(\nabla^s_{TQ_2X}FQ_2X, FQ_2X) = sin^2\theta g(\nabla_{TQ_2X}Q_2X, Q_2X). \tag{4.4}$$

Now taking the inner product in (4.3) with FQ_2X, we obtain

$$\begin{aligned}&g(F\nabla_{TQ_2X}Q_2X, FQ_2X) - g(\nabla^s_{TQ_2X}FQ_2X, FQ_2X)\\ &= cos^2\theta g(Q_2X, Q_2X)g(H^s, FQ_2X).\end{aligned}$$

Then using (3.10) and (4.4), we get

$$cos^2\theta g(Q_2X, Q_2X)g(H^s, FQ_2X) = 0. \tag{4.5}$$

Thus from (4.5), it follows that either $\theta = \pi/2$ or $Q_2X = 0$ or $H^s \in \Gamma(\mu)$. This completes the proof. □

Lemma 4.3. *Let M be a totally umbilical slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then $g(\nabla_X X, \bar{J}\xi) = 0$, for any $X \in \Gamma(D)$ and $\xi \in \Gamma(RadTM)$.*

Proof. Let $X \in \Gamma(D)$ then using (2.6) and (2.7) for a totally umbilical slant lightlike submanifold, we have

$$\begin{aligned} g(\nabla_X X, \bar{J}\xi) &= \bar{g}(\bar{\nabla}_X X, \bar{J}\xi) = -\bar{g}(\bar{\nabla}_X TQ_2X, \xi) - \bar{g}(\bar{\nabla}_X FQ_2X, \xi) \\ &= -\bar{g}(D^l(X, FQ_2X), \xi). \end{aligned} \tag{4.6}$$

Now replace W by FQ_2X and Y by ξ in (2.8) and using that M is totally umbilical slant lightlike submanifold, we obtain

$$\begin{aligned} \bar{g}(D^l(X, FQ_2X), \xi) &= -\bar{g}(h^s(X, \xi), FQ_2X) = -g(Q_2X, \xi)g(H^s, FQ_2X) \\ &= 0. \end{aligned} \tag{4.7}$$

Therefore from (4.6) and (4.7), the result follows. □

Theorem 4.4. *Every totally umbilical proper slant lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ is totally geodesic.*

Proof. Since M is a totally umbilical slant lightlike submanifold therefore we have $h(TQ_2X, TQ_2X) = g(TQ_2X, TQ_2X)H$, for any $X = Q_2X \in \Gamma(D)$. Using (3.9), we get

$$h(TQ_2X, TQ_2X) = cos^2\theta g(Q_2X, Q_2X)H. \tag{4.8}$$

For any $X \in \Gamma(D)$, using (3.8), we obtain

$$\begin{aligned} F\nabla_{TQ_2X}X &= h(TQ_2X, TQ_2X) - Ch(TQ_2X, Q_2X) + \nabla^s_{TQ_2X}FQ_2X \\ &\quad + D^l(TQ_2X, FQ_2X), \end{aligned}$$

since M is a totally umbilical slant lightlike submanifold therefore $Ch(TQ_2X, X) = g(TQ_2X, X)CH = 0$ and using (4.8), we get

$$cos^2\theta g(Q_2X, Q_2X)H = F\nabla_{TQ_2X}X - \nabla^s_{TQ_2X}FQ_2X - D^l(TQ_2X, FQ_2X). \tag{4.9}$$

Taking the scalar product of both sides of (4.9) with respect to FQ_2X, we obtain

$$\begin{aligned} cos^2\theta g(Q_2X, Q_2X)\bar{g}(H^s, FQ_2X) &= \bar{g}(F\nabla_{TQ_2X}X, FQ_2X) \\ &\quad - \bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X), \end{aligned}$$

using (3.10), we get

$$cos^2\theta g(Q_2X, Q_2X)\bar{g}(H^s, FQ_2X) = sin^2\theta\bar{g}(\nabla_{TQ_2X}X, X) - \bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X). \tag{4.10}$$

Since (3.10) holds for any $X = Y \in \Gamma(D)$ and by taking the covariant derivative with respect to $\bar{\nabla}_{TQ_2X}$, we get

$$\bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X) = sin^2\theta g(\nabla_{TQ_2X}Q_2X, Q_2X). \tag{4.11}$$

Using (4.11) in (4.10), we obtain

$$cos^2\theta g(Q_2X, Q_2X)\bar{g}(H^s, FQ_2X) = 0.$$

Since M is a proper slant lightlike submanifold and g is a Riemannian metric on D therefore we have $\bar{g}(H^s, FQ_2X) = 0$. Thus using Lemma 3.3. and (3.6), we obtain

$$H^s \in \Gamma(\mu). \tag{4.12}$$

Now, since $\bar{M}$ is an indefinite Kaehler manifold therefore for any $X, Y \in \Gamma(D)$, we have $\bar{\nabla}_X\bar{J}Y = \bar{J}\bar{\nabla}_XY$, this implies that

$$\nabla_XTQ_2Y + g(X, TQ_2Y)H - A_{FQ_2Y}X + \nabla^s_XFQ_2Y + D^l(X, FQ_2Y) = T\nabla_XY + F\nabla_XY + g(X, Y)\bar{J}H. \tag{4.13}$$

Taking the scalar product of both sides of (4.13) with respect to $\bar{J}H^s$ and then using (3.7) and (4.12), we obtain

$$\bar{g}(\nabla^s_XFQ_2X, \bar{J}H^s) = g(X, Y)g(H^s, H^s). \tag{4.14}$$

Since μ is an invariant subspace therefore using the Kaehlerian character of $\bar{M}$, we have $\bar{\nabla}_X\bar{J}H^s = \bar{J}\bar{\nabla}_XH^s$, this implies that

$$-A_{\bar{J}H^s}X + \nabla^s_X\bar{J}H^s + D^l(X, \bar{J}H^s) = -TA_{H^s}X - FA_{H^s}X + B\nabla^s_XH^s + C\nabla^s_XH^s + \bar{J}D^l(X, H^s). \tag{4.15}$$

Taking the scalar product of both sides of above equation with respect to FQ_2Y and using invariant character of μ, that is, $C\nabla^s_XH^s \in \Gamma(\mu)$, we get

$$\bar{g}(\nabla^s_X\bar{J}H^s, FQ_2Y)) = -g(FA_{H^s}X, FQ_2Y) = -sin^2\theta g(A_{H^s}X, Q_2Y). \tag{4.16}$$

Since $\bar{\nabla}$ is a metric connection therefore $(\bar{\nabla}_Xg)(FQ_2Y, \bar{J}H^s) = 0$ this further implies that $\bar{g}(\nabla^s_XFQ_2Y, \bar{J}H^s) = \bar{g}(\nabla^s_X\bar{J}H^s, FQ_2Y)$, therefore using (4.16), we obtain

$$\bar{g}(\nabla^s_XFQ_2Y, \bar{J}H^s) = -sin^2\theta g(A_{H^s}X, Q_2Y). \tag{4.17}$$

From (4.14) and (4.17), we have

$$g(X, Y)g(H^s, H^s) = -sin^2\theta g(A_{H^s}X, Q_2Y), \tag{4.18}$$

using (2.8) in (4.18), we obtain

$$g(X,Y)g(H^s,H^s) = -sin^2\theta\bar{g}(h^s(X,Q_2Y),H^s) = -sin^2\theta g(X,Y)g(H^s,H^s),$$

this implies that

$$(1+sin^2\theta)g(X,Y)g(H^s,H^s) = 0.$$

Since g is a Riemannian metric on D therefore we obtain

$$H^s = 0. \tag{4.19}$$

Furthermore, using the Kaehler character of $\bar{M}$, we have $\bar{\nabla}_X\bar{J}X = \bar{J}\bar{\nabla}_X X$ for any $X = Q_2X \in \Gamma(D)$, this implies that $\nabla_X TQ_2X + h(X,TQ_2X) - A_{FQ_2X}X + \nabla^s_X FQ_2X + D^l(X,FQ_2X) = T\nabla_X X + F\nabla_X X + Bh(X,X) + Ch(X,X)$. Since M is totally umbilical slant lightlike manifold therefore using $h(X,TQ_2X) = 0$ in above equation and then comparing the tangential components, we obtain

$$\nabla_X TQ_2X - A_{FQ_2X}X = T\nabla_X X + Bh(X,X). \tag{4.20}$$

Taking the scalar product of both sides of (4.20) with respect to $\bar{J}\xi \in \Gamma(\bar{J}Rad(TM))$ and using the Lemma 4.3., we get

$$g(A_{FQ_2X}X,\bar{J}\xi) + \bar{g}(h^l(X,X),\xi) = 0. \tag{4.21}$$

Now using (2.8), we have

$$\bar{g}(h^s(X,\bar{J}\xi),FQ_2X) + \bar{g}(\bar{J}\xi,D^l(X,FQ_2X)) = g(A_{FQ_2X}X,\bar{J}\xi),$$

using M is totally umbilical slant lightlike submanifold and (4.19), the above equation implies that

$$g(A_{FQ_2X}X,\bar{J}\xi) = 0. \tag{4.22}$$

Using (4.22) in (4.21), we obtain that $\bar{g}(h^l(X,X),\xi) = 0$, this implies that

$$g(Q_2X,Q_2X)\bar{g}(H^l,\xi) = 0.$$

Since g is a Riemannian metric on D therefore $\bar{g}(H^l,\xi) = 0$ then using (2.4), we obtain that

$$H^l = 0. \tag{4.23}$$

Thus from (4.19) and (4.23), the proof is complete. □

Theorem 4.5. *Let M be a totally umbilical proper slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then the induced connection ∇ is a metric connection on M.*

Proof. Using (4.2) and (4.23), we have $h^l = 0$ then using the Theorem 2.2 in [7], at page 159, the induced connection ∇ becomes a metric connection on M. □

Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇ respectively then by straightforward calculations ([7]), we have

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y\\ &\quad - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z)\\ &\quad + D^l(X,h^s(Y,Z)) - D^l(Y,h^s(X,Z)) + (\nabla_X h^s)(Y,Z)\\ &\quad - (\nabla_Y h^s)(X,Z) + D^s(X,h^l(Y,Z)) - D^s(Y,h^l(X,Z)),\end{aligned} \tag{4.24}$$

where

$$(\nabla_X h^s)(Y,Z) = \nabla^s_X h^s(Y,Z) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z).$$

$$(\nabla_X h^l)(Y,Z) = \nabla^l_X h^l(Y,Z) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z). \tag{4.25}$$

An indefinite complex space form is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c and denoted by $\bar{M}(c)$. Then the curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by

$$\begin{aligned}\bar{R}(X,Y)Z &= \frac{c}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX\\ &\quad - \bar{g}(JX,Z)JY + 2\bar{g}(X,JY)JZ\},\end{aligned} \tag{4.26}$$

for X, Y, Z vector fields on $\bar{M}$. □

Theorem 4.6. *There exists no totally umbilical proper slant lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, $c \neq 0$.*

Proof. Suppose M be a totally umbilical proper lightlike submanifold of $\bar{M}(c)$ such that $c \neq 0$. Then using (4.26), for any $X \in \Gamma(D)$, $Z \in \Gamma(\bar{J}ltr(TM))$ and $\xi \in \Gamma(Rad(TM))$, we obtain

$$\bar{g}(\bar{R}(X,\bar{J}X)Z,\xi) = -\frac{c}{2}g(X,X)g(\bar{J}Z,\xi). \tag{4.27}$$

On the other hand using (4.2) and (4.24), we get

$$\bar{g}(\bar{R}(X,\bar{J}X)Z,\xi) = \bar{g}((\nabla_X h^l)(\bar{J}X,Z),\xi) - \bar{g}((\nabla_{\bar{J}X} h^l)(X,Z),\xi). \tag{4.28}$$

Using (4.2) and (4.25), we have

$$(\nabla_X h^l)(\bar{J}X,Z) = -g(\nabla_X TQ_2X, Z)H^l - g(TQ_2X, \nabla_X Z)H^l. \tag{4.29}$$

Similarly

$$(\nabla_{\bar{J}X} h^l)(X,Z) = -g(\nabla_{\bar{J}X} X, Z)H^l - g(X, \nabla_{\bar{J}X} Z)H^l. \tag{4.30}$$

Using (4.29) and (4.30) in (4.28), we obtain

$$\begin{aligned}\bar{g}(\bar{R}(X,\bar{J}X)Z,\xi) &= -g(\nabla_X TQ_2X,Z)\bar{g}(H^l,\xi) - g(TQ_2X,\nabla_X Z)\bar{g}(H^l,\xi)\\ &\quad + g(\nabla_{\bar{J}X}X,Z)\bar{g}(H^l,\xi) + g(X,\nabla_{\bar{J}X}Z)\bar{g}(H^l,\xi).\end{aligned} \tag{4.31}$$

Now using (4.2), we have

$$g(TX,\nabla_X Z) = -\bar{g}(\bar{\nabla}_X TQ_2X,Z) = -g(\nabla_X TQ_2X,Z), \tag{4.32}$$

and

$$g(X,\nabla_{\bar{J}X}Z) = -\bar{g}(\bar{\nabla}_{\bar{J}X}X,Z) = -g(\nabla_{\bar{J}X}X,Z). \tag{4.33}$$

Using (4.32) and (4.33) in (4.31), we obtain

$$\bar{g}(\bar{R}(X,\bar{J}X)Z,\xi) = 0. \tag{4.34}$$

Thus using (4.34) in (4.27), we have

$$\frac{c}{2}g(X,X)g(\bar{J}Z,\xi) = 0.$$

Since g is a Riemannian metric on D and (2.4) implies that $g(\bar{J}Z,\xi) \neq 0$, therefore $c = 0$. This contradiction completes the proof. □

In [7], a minimal lightlike submanifold M is defined when M is a hypersurface of a 4-dimensional Minkowski space. Then in [2], a general notion of minimal lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ is introduced as follows:

Definition 4.7. A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if

(i) $h^s = 0$ on $Rad(TM)$ and

(ii) $trace\ h = 0$, where trace is written with respect to g restricted to $S(TM)$.

We use the quasi orthonormal basis of M given by

$$\{\xi_1, \ldots\ldots, \xi_r, \bar{J}\xi_1, \ldots\ldots, \bar{J}\xi_r, e_1, \ldots\ldots, e_q, \bar{J}N_1, \ldots\ldots, \bar{J}N_r\},$$

such that $\{\xi_1, \ldots\ldots, \xi_r\}$, $\{\bar{J}\xi_1, \ldots\ldots, \bar{J}\xi_r\}$, $\{e_1, \ldots\ldots, e_q\}$ and $\{\bar{J}N_1, \ldots\ldots, \bar{J}N_r\}$ form a basis of $Rad(TM)$, $\bar{J}(Rad(TM))$, D and $\bar{J}(ltr(TM))$ respectively.

Definition 4.8.([9]) A lightlike submanifold is called irrotational if and only if $\bar{\nabla}_X\xi \in \Gamma(TM)$ for all $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$.

Theorem 4.9. *Let M be an irrotational slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is minimal if and only if*

$$traceA_{W_k}|_{S(TM)} = 0, traceA^*_{\xi_i}|_{S(TM)} = 0,$$

where $\{W_k\}_{k=1}^{l}$ *is a basis of* $S(TM^{\perp})$ *and* $\{\xi_i\}_{i=1}^{r}$ *is a basis of* $Rad(TM)$.

Proof. Since M is an irrotational so this implies that $h^s(X,\xi)=0$ for $X\in\Gamma(TM)$ and $\xi\in\Gamma(Rad(TM))$. Thus h^s vanishes on $Rad(TM)$. Hence M is minimal if and only if trace $h=0$ on $S(TM)$, that is, M is minimal if and only if

$$\sum_{i=1}^{r}h(\bar{J}\xi_i,\bar{J}\xi_i)+\sum_{i=1}^{r}h(\bar{J}N_i,\bar{J}N_i)+\sum_{j=1}^{q}h(e_j,e_j)=0.$$

Using (2.8) and (2.10) we obtain

$$\sum_{i=1}^{r}h(\bar{J}\xi_i,\bar{J}\xi_i) = \sum_{i=1}^{r}\{\frac{1}{r}\sum_{a=1}^{r}g(A^*_{\xi_a}\bar{J}\xi_i,\bar{J}\xi_i)N_a + \frac{1}{l}\sum_{k=1}^{l}g(A_{W_k}\bar{J}\xi_i,\bar{J}\xi_i)W_k\}. \tag{4.35}$$

Similarly, we have

$$\sum_{i=1}^{r}h(\bar{J}N_i,\bar{J}N_i) = \sum_{i=1}^{r}\{\frac{1}{r}\sum_{a=1}^{r}g(A^*_{\xi_a}\bar{J}N_i,\bar{J}N_i)N_a + \frac{1}{l}\sum_{k=1}^{l}g(A_{W_k}e_j,e_j)W_k\}, \tag{4.36}$$

and

$$\sum_{j=1}^{q}h(e_j,e_j)=\sum_{j=1}^{q}\{\frac{1}{r}\sum_{i=1}^{r}g(A^*_{\xi_i}e_j,e_j)N_i+\frac{1}{l}\sum_{k=1}^{l}g(A_{W_k}e_j,e_j)W_k\}. \tag{4.37}$$

Thus our assertion follows from (4.35)-(4.37). □

References

[1] M. Barros and A. Romero, *Indefinite Kaehler manifolds*, Math. Ann., **261**(1982), 55–62.

[2] C. L. Bejan and K. L. Duggal, *Global lightlike manifolds and harmonicity*, Kodai Math. J., **28**(2005), 131-145.

[3] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, and M. Fernandez, *Slant submanifolds in Sasakian manifolds*, Glasg. Math. J., **42**(2000), 125–138.

[4] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, and M. Fernandez, *Semi-slant submanifolds of a Sasakian manifold*, Geom. Dedicata., **78**(1999), 183–199.

[5] B. Y. Chen, *Slant immersions*, Bull. Austral. Math. Soc., **41**(1990), 135–147.

[6] B. Y. Chen, *Geometry of Slant Submanifolds*, Katholieke Universiteit, Leuven, (1990).

[7] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of semi-Riemannian Manifolds and Applications*, Vol. 364 of Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, (1996).

[8] K. L. Duggal and D. H. Jin, *Totally umbilical lightlike submanifolds*, Kodai Math. J., **26**(2003), 49–68.

[9] K. L. Duggal and B. Sahin, *Screen Cauchy Riemann lightlike submanifolds*, Acta Math. Hungar., **106**(2005), 125-153.

[10] E. J. Flaherty, *Hermitian and Kaehlerian Geometry in Relativity*, Lecture Notes in Physics, **46**, Springer-Verlag, (1976).

[11] R. S. Gupta, A. Upadhyay and A. Sharfuddin, *Slant lightlike submanifolds of indefinite Cosymplectic manifolds*, Mediterr. J. Math., **8**(2011), 215-227.

[12] A. Lotta, *Slant submanifolds in contact geometry*, Bull. Soc. Sci. Math. Roumanie, **39**(1996), 183–198.

[13] A. Lotta, *Three dimensional slant submanifolds of K-contact manifolds*, Balkan J. Geom. Appl., **3**(1998), 37–51.

[14] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, (1983).

[15] N. Papaghiuc, *Semi-slant submanifolds of a Kaehlerian manifold*, An. Stiint. Univ. Al. I. Cuza Iasi Secct. I a Mat., **40**(1994), 55–61.

[16] Rashmi Sachdeva, Rakesh Kumar and S. S. Bhatia, *Nonexistence of totally contact umbilical slant lightlike submanifolds of indefinite Cosymplectic manifolds*, ISRN Geometry, Volume 2013, Article ID 231869, 8 pages.

[17] Rashmi Sachdeva, Rakesh Kumar and S. S. Bhatia, *Non existence of totally contact umbilical proper slant lightlike submanifold of indefinite Sasakian manifolds*, Bull. Iranian Math. Soc., **40(5)**(2014), 1135–1151.

[18] Rashmi Sachdeva, Rakesh Kumar and S. S. Bhatia, *Warped product slant lightlike submanifolds of indefinite Sasakian manifolds*, Balkan J. Geom. Appl., **20(1)**(2015), 98–108.

[19] Rashmi Sachdeva, Rakesh Kumar and S. S. Bhatia, *Totally contact umbilical slant lightlike submanifolds of indefinite Kenmotsu manifolds*, Tamkang J. Maths., **46(2)**(2015), 179–191.

[20] Rashmi Sachdeva, Rakesh Kumar and S. S. Bhatia, *Totally umbilical hemi-slant lightlike submanifolds*, New York J. Math., **21**(2015), 191–203.

[21] B. Sahin, *Slant lightlike submanifolds of indefinite Hermitian manifolds*, Balkan J. Geom. Appl., **13**(2008), 107–119.

[22] B. Sahin, *Every totally umbilical proper slant submanifolds of a Kaehler manifold is totally geodesic*, Result. Math., **54**(2009), 167–172.