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SOME RESULTS ON UNIQUENESS OF MEROMORPHIC SOLUTIONS OF DIFFERENCE EQUATIONS

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Abstract. In this paper, we investigate the transcendental meromorphic solutions with finite order of two different types of difference equations

$$\sum_{j=1}^{n} a_j f(z+c_j) = \frac{P(z,f)}{Q(z,f)} = \frac{\sum_{k=0}^{p} b_k f^k}{\sum_{l=0}^{q} d_l f^l}$$

and

$$\prod_{j=1}^{n} f(z+c_j) = \frac{P(z,f)}{Q(z,f)} = \frac{\sum_{k=0}^{p} b_k f^k}{\sum_{l=0}^{q} d_l f^l}$$

that share three distinct values with another meromorphic function. Here a_j, b_k, d_l are small functions of f and $a_j \not\equiv 0 (j = 1, 2, \ldots, n)$, $b_p \not\equiv 0$, $d_q \not\equiv 0$. $c_j \neq 0$ are pairwise distinct constants. p, q, n are non-negative integers. $P(z, f)$ and $Q(z, f)$ are two mutually prime polynomials in f.

1. Introduction and results

In this paper, we assume that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, see [4, 6, 9, 10]. We use notation $\rho(f)$ to denote the order of growth of f. In addition, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f)), (r \to \infty, r \notin E)$, where E is an exceptional set with finite logarithmic measure. If a meromorphic function $a(z)$ satisfies $T(r, a) = S(r, f)$, we say $a(z)$ is a small function with respect to $f(z)$.

Let $f(z)$ and $g(z)$ be the non-constant meromorphic functions and let a be a complex number in the complex plane. We say $f(z)$ and $g(z)$ share a CM(IM) provided that $f(z)$ and $g(z)$ have the same a-points counting multiplicities (ignoring multiplicities). If $f(z)$ and $g(z)$ have the same poles, we say $f(z)$ and $g(z)$ share ∞ CM (counting multiplicities) or IM (ignoring multiplicities).

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In 1989, Brosch [1] studied the uniqueness problem that a meromorphic solution of the Malmquist-type ordinary differential equation $(w')^n = \sum_{j=0}^{2n} a_j w^j$ shares three values with another meromorphic function. Recently, Lü, Han and Lü [8] proved a similar uniqueness theorem which is related to Malmquist-type difference equation. The difference equation is

$$\sum_{j=1}^{n} a_j f(z+c_j) = \frac{P(z,f)}{Q(z,f)} = \frac{\sum_{k=0}^{p} b_k f^k}{\sum_{l=0}^{q} d_l f^l}. \tag{1.1}$$

Here a_j, b_k, d_l are small functions of f and $a_j \not\equiv 0 (j = 1, 2, \ldots, n)$, $b_p \not\equiv 0$, $d_q \not\equiv 0$. $c_j \neq 0 (j = 1, 2, \ldots, n)$ are pairwise distinct constants. p, q, n are non-negative integers. $P(z,f)$ and $Q(z,f)$ are co-prime polynomials in f.

Denote $I_1(z,f) = \sum_{j=1}^{n} a_j f(z+c_j)$, $H_1(z,f) = I_1(z,f)Q(z,f) - P(z,f)$. Now, we recall the following theorem.

Theorem A ([8]). *Let $f(z)$ be a finite-order transcendental meromorphic solution of* (1.1) *and let e_1, e_2 be two distinct finite numbers such that $H_1(z,e_1) \not\equiv 0$, $H_1(z,e_2) \not\equiv 0$, $p \le q = n$. If $f(z)$ and another meromorphic function $g(z)$ share e_1, e_2 and ∞ CM, then $f \equiv g$.*

In this paper, we make some further investigation on this problem and obtain some results as follows.

Theorem 1.1. *Let $f(z)$ be a finite-order transcendental meromorphic solution of* (1.1) *and let a, b be two distinct finite numbers such that $H_1(z,a) \not\equiv 0$, $H_1(z,b) \not\equiv 0$. p, q, n satisfy* $\max\{p,q\} = n \ge 2$ *or* $p \le q = n = 1$. *If $f(z)$ and another meromorphic function $g(z)$ share a, b and ∞ CM, then $f \equiv g$.*

Remark 1.1. In Theorem 1.1, if $q = 0$, $p = n = 1$, the result may not be true. For example, $f(z) = e^z + 1$ is a finite-order meromorphic solution of equation $f(z+1) = ef(z) + 1 - e$. $f(z)$ and $g(z) = e^{-z} + 1$ share 0, 2 and ∞ CM with $H_1(z,0) \neq 0$, $H_1(z,2) \neq 0$, but $f(z) \not\equiv g(z)$.

Remark 1.2. In Theorem 1.1, the conditions $H_1(z,a), H_1(z,b) \not\equiv 0$ are necessary. For example, $f(z) = \tan z$ is a solution of the difference equation

$$f\left(z + \frac{\pi}{4}\right) + f\left(z - \frac{\pi}{4}\right) = \frac{4f(z)}{1 - f^2(z)}.$$

$f(z)$ and $g(z) = -\tan z$ share the values 0, $\pm$i, ∞ CM with $H_1(z,0) = H_1(z,\pm\mathrm{i}) = 0$, but $f(z) \not\equiv g(z)$.

We also get some similar results for another type of difference equation

$$\prod_{j=1}^{n} f(z+c_j) = \frac{P(z,f)}{Q(z,f)} = \frac{\sum_{k=0}^{p} b_k f^k}{\sum_{l=0}^{q} d_l f^l}. \tag{1.2}$$

Denote $I_2(z,f) = \prod_{j=1}^{n} f(z+c_j)$, $H_2(z,f) = I_2(z,f)Q(z,f) - P(z,f)$ with restrictions on c_j, b_k, d_l unchanged.

Theorem 1.2. *Let $f(z)$ be a finite-order transcendental meromorphic solution of* (1.2) *and let a, b be two distinct finite complex numbers such that $H_2(z,a) \not\equiv 0$, $H_2(z,b) \not\equiv 0$. We have the following results*:

(i) *When p, q, n satisfy* $\max\{p,q\} \leq n$ *and $q \geq 1$, if $f(z)$ and another meromorphic function $g(z)$ share a, b and ∞ CM, then $f \equiv g$.*

(ii) *When p, q, n satisfy $q = 0$ and $p < n$, if $f(z)$ and another meromorphic function $g(z)$ share a, b and ∞ CM, then $f \equiv g$.*

Theorem 1.3. *Let $f(z)$ be a finite-order transcendental meromorphic solution of* (1.2) *and let a, b be two distinct finite non-zero complex numbers such that $H_2(z,a) \not\equiv 0$, $H_2(z,b) \not\equiv 0$. p, q satisfy* $\max\{p,q\} \leq n$. *We have the following results*:

(i) *When $p < n$, if $f(z)$ and another meromorphic function $g(z)$ share a, b and* 0 *CM, then $f \equiv g$.*

(ii) *When $p = n$ and $b_k \not\equiv 0$ for at least one $k(0 \leq k \leq p-1)$, if $f(z)$ and another meromorphic function $g(z)$ share a, b and* 0 *CM, then $f \equiv g$.*

Theorem 1.4. *Let $f(z)$ be a finite-order transcendental meromorphic solution of* (1.1) *or* (1.2). *Let a, b, c be three distinct finite complex numbers such that $H_i(z,a) \not\equiv 0$, $H_i(z,b) \not\equiv 0$, $H_i(z,c) \not\equiv 0 (i = 1,2)$. When* $\max\{p,q\} \leq n$, *if $f(z)$ and another meromorphic function $g(z)$ share a, b, c CM, then $f \equiv g$.*

Remark 1.3. In Theorem 1.2, the result is not true for $q = 0$, $p = n$. For example, $f(z) = e^z + 1$ is a finite-order meromorphic solution of

$$f(z+1)f(z-1) = f^2(z) + (e + e^{-1} - 2)f(z) - (e + e^{-1} - 2).$$

$f(z)$ and $g(z) = e^{-z} + 1$ share 0, 2 and ∞ CM with $H_2(z,0) \neq 0$, $H_2(z,2) \neq 0$, but $f(z) \not\equiv g(z)$.

Remark 1.4. In Theorem 1.3, the result is not true for $p = n$, $b_k \equiv 0$ for all $k(0 \leq k \leq p-1)$. For example, $f(z) = e^{z^2}$ is a solution of the equation

$$f(z+1)f(z-1) = e^2 f^2(z).$$

Here $b_0, b_1 \equiv 0$. $f(z)$ and $g(z) = e^{-z^2}$ share the values 0, 1, -1 CM with $H_2(z,1) \not\equiv 0$ and $H_2(z,-1) \not\equiv 0$, but $f(z) \not\equiv g(z)$.

Remark 1.5. In Theorems 1.2 and 1.3, the conditions $H_2(z,a) \not\equiv 0$ and $H_2(z,b) \not\equiv 0$ are necessary. For example, $f(z) = \tan \frac{\pi}{4} z$ is a finite-order meromorphic solution of the equation $f(z+1)f(z-1) = -1$. $f(z)$ and $g(z) = -f(z)$ share 0, $\pm$i, ∞ CM with $H_2(z,0) = 1 \neq 0$, $H_2(z,\pm \mathrm{i}) \equiv 0$, but $f(z) \not\equiv g(z)$.

Remark 1.6. It is natural to suppose that $\max\{p,q\} \leq n$ according to the results such as Propositions 8, 9, Theorem 12 in [5].

2. Some lemmas

In this section, we will give some important lemmas for the proof of theorems.

Lemma 2.1 ([2, Corollary 2.6]). *Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite-order meromorphic function. Let σ be the order of $f(z)$, then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.2 ([3, Theorem 3.2], [7, Theorem 2.4]). *Let $f(z)$ be a transcendental meromorphic solution of finite order ρ of the difference equation $P(z,f) = 0$, where $P(z,f)$ is a difference polynomial in $f(z)$ and its shifts. If $P(z,a) \not\equiv 0$ for a small function a, then*

$$m\left(r, \frac{1}{f-a}\right) = S(r,f)$$

outside of a possible exceptional set of finite logarithmic measure.

Lemma 2.3 ([7, Theorem 2.3]). *Let f be a transcendental meromorphic solution of finite order of a difference equation of the form*

$$U(z,f)P(z,f) = Q(z,f),$$

where $U(z,f)$, $P(z,f)$ and $Q(z,f)$ are difference polynomials such that the total degree $\deg U(z,f) = n$ in f and its shifts, and $\deg Q(z,f) \leq n$. Moreover, we assume that $U(z,f)$ contains just one term of maximal total degree in f and its shifts. Then for each $\varepsilon > 0$,

$$m(r, P(z,f)) = O(r^{\rho-1+\varepsilon}) + S(r,f),$$

possibly outside of an exceptional set of finite logarithmic measure.

Lemma 2.4 ([6, Theorem 2.2.5]). *Let $f(z)$ be a meromorphic function. Then for irreducible rational functions in f,*

$$R(z, f(z)) = \frac{\sum_{i=0}^{m} a_i(z) f(z)^i}{\sum_{j=0}^{n} b_j(z) f(z)^j}$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$, the characteristic function of $R(z,f)$ satisfies

$$T(r, R(z, f(z))) = \max\{m,n\} T(r,f) + O(\max\{T(r,a_i), T(r,b_j)\}).$$

Lemma 2.5 ([2, Theorem 2.2]). *Let f be a meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < +\infty$, $\eta \neq 0$ be fixed. Then for each $\varepsilon > 0$,*

$$N(r, f(z+\eta)) = N(r,f) + O(r^{\lambda-1+\varepsilon}) + O(\log r).$$

Lemma 2.6. *Let $f(z)$ be a non-constant meromorphic function, $p(f) = a_0 f^n + a_1 f^{n-1} + \cdots + a_n$, $a_0 (\not\equiv 0), a_1, \ldots, a_n$ are small functions with respect to f. Then*

$$n \cdot m(r,f) \leq m(r, p(f)) + S(r,f).$$

Proof. Without loss of generality, suppose $a_0 \equiv 1$.

On circle $|z| = r$, let $A(z) = \max_{1\leq i\leq n} |a_i(z)|^{\frac{1}{i}}$ $(i = 1, 2, \ldots, n)$. For fixed r, let $E_1 = \left\{\theta \in [0, 2\pi) : |f(re^{i\theta})| \geq 2A(re^{i\theta})\right\}$ and $E_2 = \{\theta \in [0, 2\pi)\} - E_1$. In set E_1,

$$
\begin{aligned}
|p(f)| &= |f|^n \cdot \left|1 + \frac{a_1}{f} + \cdots + \frac{a_n}{f^n}\right| \\
&\geq |f|^n \left\{1 - \left|\frac{a_1}{f}\right| - \cdots - \left|\frac{a_n}{f^n}\right|\right\} \geq |f|^n \left\{1 - \frac{1}{2} - \cdots - \frac{1}{2^n}\right\} = \frac{1}{2^n}|f|^n.
\end{aligned}
$$

So

$$
\begin{aligned}
n \cdot m(r, f) &= m(r, f^n) \\
&= \frac{1}{2\pi}\int_{E_1} \log^+ |f^n| \, d\theta + \frac{1}{2\pi}\int_{E_2} \log^+ |f^n| \, d\theta \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \log^+ |2^n p(f)| \, d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log^+ |2A|^n \, d\theta \\
&\leq m(r, p(f)) + S(r, f).
\end{aligned}
$$

□

Lemma 2.7 ([9, Theorem 1.51]). *Let $f_j(z)(j = 1, \ldots, n)(n \geq 2)$ be meromorphic functions, $g_j(z)$, $j = 1, \ldots, n$, be entire functions satisfying*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(ii) *when $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ is not a constant*;

(iii) *when $1 \leq j \leq n$, $1 \leq h < k \leq n$,*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}, \ (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure. Then $f_j(z) \equiv 0 (j = 1, \ldots, n)$.

Lemma 2.8. *Let $f(z)$ be a finite-order transcendental meromorphic solution of (1.2). If p, q, n satisfy $\max\{p, q\} \leq n$ and $q \geq 1$ or p, q, n satisfy $q = 0$ and $p < n$, then*

$$m(r, f) = S(r, f).$$

Proof. (i) If p, q, n satisfy $\max\{p, q\} \leq n$ and $q \geq 1$, by Lemma 2.3 and $I_2(z, f)Q_2(z, f) = P_2(z, f)$, we have $m(r, Q(z, f)) = S(r, f)$. By Lemma 2.6, we have $m(r, f) = S(r, f)$.

(ii) If p, q, n satisfy $q = 0$ and $p < n$, by Lemma 2.3 and $\left(\prod_{j=1}^{n-1} f(z + c_j)\right)$ $f(z + c_n) = P(z, f)$, we have $m(r, f(z + c_n)) = S(r, f)$, then by Lemma 2.1

$$
\begin{aligned}
m(r, f) &= m\left(r, f(z + c_n) \cdot \frac{f(z)}{f(z + c_n)}\right) \\
&\leq m(r, f(z + c_n)) + m\left(r, \frac{f(z)}{f(z + c_n)}\right) = S(r, f).
\end{aligned}
$$

□

3. Proof of theorems

3.1. Proof of Theorem 1.2

Suppose f is a finite-order transcendental meromorphic solution to equation (1.2) and p, q, n satisfy $\max\{p,q\} \leq n$ and $q \geq 1$ or p, q, n satisfy $q = 0$ and $p < n$, f and g share a, b, ∞ CM. By Nevanlinna's second main theorem, we have

$$\begin{aligned} T(r,f) &\leq N(r,f) + N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) + S(r,f) \\ &= N(r,g) + N\left(r, \frac{1}{g-a}\right) + N\left(r, \frac{1}{g-b}\right) + S(r,f) \\ &\leq 3T(r,g) + S(r,f). \end{aligned}$$

Similarly, $T(r,g) \leq 3T(r,f) + S(r,g)$. So $\rho(g) = \rho(f) \leq \infty$ which means g is a finite-order meromorphic function.

According to $H_2(z,a) \not\equiv 0$ and Lemma 2.2, we have $m\left(r, \frac{1}{f-a}\right) = S(r,f)$. Similarly, $m\left(r, \frac{1}{f-b}\right) = S(r,f)$. By Lemma 2.8, $m(r,f) = S(r,f)$.

f and g share a, b, ∞ CM, so there exist two polynomials α and β such that

$$\frac{f-a}{g-a} = e^{\alpha}, \frac{f-b}{g-b} = e^{\beta}. \tag{3.1}$$

Let $\gamma = \beta - \alpha$, if $e^{\alpha} \equiv 1$ or $e^{\beta} \equiv 1$ or $e^{\gamma} \equiv 1$, Obviously we have $f \equiv g$. Suppose $e^{\alpha} \not\equiv 1$ and $e^{\beta} \not\equiv 1$ and $e^{\gamma} \not\equiv 1$.

By equation (3.1), we have

$$f = a + (b-a)\frac{e^{\beta}-1}{e^{\gamma}-1} \tag{3.2}$$

and

$$f = b + (b-a)\left(\frac{e^{\beta}-1}{e^{\gamma}-1} - 1\right) = b + (b-a)\frac{e^{\alpha}-1}{e^{\gamma}-1}e^{\gamma}. \tag{3.3}$$

First, we claim that the degrees of α, β, γ are at least 1. In the following, we discuss four cases.

Case 1. If α, β are both constants ($e^{\alpha} \neq 1$,$e^{\beta} \neq 1$), then γ is also a constant. We deduce that f is a constant by (3.2), which contradicts with the condition that f is a transcendental meromorphic function.

Case 2. If β is a constant ($e^{\beta} \neq 1$) and the degree of α is at least 1, the degree of γ is also at least 1. Let $\tau_1 = (b-a)(e^{\beta}-1)$, we deduce that τ_1 is a non-zero constant. From (3.2), we have $f = a + \frac{\tau_1}{e^{\gamma}-1}$. So

$$\begin{aligned} T(r,f) &= m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) + O(1) \\ &= S(r,f) + N\left(r, \frac{e^{\gamma}-1}{\tau_1}\right) = S(r,f). \end{aligned}$$

This is a contradiction.

Case 3. If α is a constant ($e^{\alpha} \neq 1$) and the degree of β is at least 1, the degree of γ is also at least 1. Let $\tau_2 = (b-a)(e^{\alpha}-1)$, we deduce that τ_2 is a non-zero constant. By (3.3), we have $f = b + \frac{\tau_2 e^{\gamma}}{e^{\gamma}-1}$. So

$$\begin{aligned} T(r,f) &= m\left(r, \frac{1}{f-b}\right) + N\left(r, \frac{1}{f-b}\right) + O(1) \\ &= S(r,f) + N\left(r, \frac{e^{\gamma}-1}{\tau_2 e^{\gamma}}\right) = S(r,f). \end{aligned}$$

This is a contradiction.

Case 4. If α and β are not constants but γ is a constant ($e^{\gamma} \neq 1$). Let $\tau_3 = \frac{b-a}{e^{\gamma}-1}$, then τ_3 is a non-zero constant. By (3.2), we get $f = a + \tau_3(e^{\beta}-1)$. So

$$T(r,f) = m(r,f) + N(r,f) = S(r,f) + N(r, a + \tau_3(e^{\beta}-1)) = S(r,f).$$

This is a contradiction.

From the four cases above we conclude that α, β and γ are all non-constant polynomials.

Substituting (3.2) into (1.2) yields

$$\begin{aligned} &\left\{\sum_{l=0}^{q} d_l \left(a + (b-a)\frac{e^{\beta(z)}-1}{e^{\gamma(z)}-1}\right)^l\right\} \left\{\prod_{j=1}^{n} \left(a + (b-a)\frac{e^{\beta(z+c_j)}-1}{e^{\gamma(z+c_j)}-1}\right)\right\} \\ &= \sum_{k=0}^{p} b_k \left(a + (b-a)\frac{e^{\beta(z)}-1}{e^{\gamma(z)}-1}\right)^k. \end{aligned} \tag{3.4}$$

α, β, γ are non-constant polynomials, denote $e^{\beta(z+c_j)} = e^{\beta(z)+s_j(z)}$ and $e^{\gamma(z+c_j)} = e^{\gamma(z)+t_j(z)}$. Clearly, $s_j(z)$ and $t_j(z)$ are polynomials of degrees at most $\deg\beta(z)-1$ and $\deg\gamma(z)-1$, respectively.

Multiply both sides of equation (3.4) by $(e^{\gamma}-1)^n \prod_{j=1}^{n}(e^{\gamma(z+c_j)}-1)$, we get

$$\begin{aligned} &\left\{\sum_{l=0}^{q} d_l \left(a(e^{\gamma(z)}-1) + (b-a)(e^{\beta(z)}-1)\right)^l \left(e^{\gamma(z)}-1\right)^{n-l}\right\} \cdot \\ &\left\{\prod_{j=1}^{n} \left(a(e^{\gamma(z+c_j)}-1) + (b-a)(e^{\beta(z+c_j)}-1)\right)\right\} \\ &= \sum_{k=0}^{p} b_k \left(a(e^{\gamma(z)}-1) + (b-a)(e^{\beta(z)}-1)\right)^k \left(e^{\gamma(z)}-1\right)^{n-k} \cdot \prod_{j=1}^{n} \left(e^{\gamma(z+c_j)}-1\right). \end{aligned} \tag{3.5}$$

We can write (3.5) as

$$\sum_{\lambda=0}^{2n}\sum_{\mu=0}^{2n} A_{\lambda,\mu}e^{\lambda\beta+\mu\gamma}=0. \tag{3.6}$$

Here, $A_{\lambda,\mu}$ are either 0 or polynomials in b_k,d_l and e^{s_j},e^{t_j}. By computation,

$$\begin{cases} A_{0,0}=\left(\sum\limits_{l=0}^{q} d_l b^l\right)b^n-\left(\sum\limits_{k=0}^{p} b_k b^k\right)=H_2(z,b)\not\equiv 0,\\ A_{0,2n}=\left(\sum\limits_{l=0}^{q} d_l a^l\right)\left(\prod\limits_{j=1}^{n} ae^{t_j}\right)-\left(\sum\limits_{k=0}^{p} b_k a^k\right)\left(\prod\limits_{j=1}^{n} e^{t_j}\right)\\ \qquad\;\; =\prod\limits_{j=1}^{n} e^{t_j}H_2(z,a)\not\equiv 0. \end{cases} \tag{3.7}$$

Next, we will prove $\deg\beta=\deg\gamma=\deg\alpha=\rho(f)=d$. Here d is a positive integer.

As we known, $T(r,e^{\alpha})\le T(r,f)+T(r,g)+O(1)\le 4T(r,f)+S(r,f)$. Similarly, $T(r,e^{\beta})\le 4T(r,f)+S(r,f)$. So $\rho(f)\ge\max\{\rho(e^{\alpha}),\rho(e^{\beta})\}$. It follows from (3.2) that

$$\rho(f)\le\max\{\rho(e^{\beta}),\rho(e^{\gamma})\}\le\max\{\rho(e^{\beta}),\rho(e^{\alpha})\}\le\rho(f).$$

That is to say $\rho(f)=\max\{\deg(\alpha),\deg(\beta)\}$.

Let $N_0(r)$ be the integrated counting function of common zeros of $e^{\beta}-1$ and $e^{\gamma}-1$(counting multiplicities). If z_0 is a ω_1-order zero of $e^{\beta}-1$ and a ω_2-order zero of $e^{\gamma}-1(\omega_1,\omega_2\ge 1)$, then z_0 will counts $\min\{\omega_1,\omega_2\}$ times in $N_0(r)$. By (3.2), we have

$$\begin{aligned} T(r,f)&=m\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{f-a}\right)+O(1)\\ &=N\left(r,\frac{e^{\gamma}-1}{e^{\beta}-1}\right)+S(r,f)\\ &=N\left(r,\frac{1}{e^{\beta}-1}\right)-N_0(r)+S(r,f). \end{aligned} \tag{3.8}$$

By the Nevanlinna's first and second main theorems, we have

$$\begin{aligned} T(r,e^{\beta})&\le N(r,e^{\beta})+N(r,e^{-\beta})+N\left(r,\frac{1}{e^{\beta}-1}\right)+S(r,e^{\beta})\\ &=N\left(r,\frac{1}{e^{\beta}-1}\right)+S(r,f),\\ T(r,e^{\beta})&\ge N\left(r,\frac{1}{e^{\beta}-1}\right)+S(r,f), \end{aligned}$$

that is, $T(r,e^{\beta})=N\left(r,\frac{1}{e^{\beta}-1}\right)+S(r,f)$. Combining (3.8) yields

$$T(r,e^{\beta})=T(r,f)+N_0(r)+S(r,f).$$

Similarly, we have

$$\begin{aligned} T(r,f) &= m(r,f) + N(r,f) \\ &= N\left(r, \frac{e^{\beta}-1}{e^{\gamma}-1}\right) + S(r,f) \\ &= N\left(r, \frac{1}{e^{\gamma}-1}\right) - N_0(r) + S(r,f), \end{aligned}$$

and

$$T(r,e^{\gamma}) = N\left(r, \frac{1}{e^{\gamma}-1}\right) + S(r,f),$$

that is $T(r,e^{\gamma}) = T(r,f) + N_0(r) + S(r,f)$. Then

$$T(r,e^{\beta}) = T(r,e^{\gamma}) + S(r,f).$$

Similarly by (3.3), we have $T(r,e^{\alpha}) = T(r,e^{\gamma}) + S(r,f)$.

Then we conclude that $\deg\alpha = \deg\beta = \deg\gamma = \rho(f) = d$.

In what follows, we will prove $\deg(\lambda\beta+\mu\gamma) = \deg(\lambda\beta-\mu\gamma) = d$, $1 \le \lambda \le 2n$, $1 \le \mu \le 2n$.

Suppose that $\deg(\lambda\beta+\mu\gamma) < d$, obviously $e^{\lambda\beta+\mu\gamma}$ is a small function of f and $e^{-\alpha}$. So

$$T(r, e^{\lambda\beta+\mu\gamma}\cdot e^{-\lambda\alpha}) = T(r,e^{-\lambda\alpha}) + S(r,f) = \lambda T(r,e^{\alpha}) + S(r,f).$$

On the other hand,

$$\begin{aligned} T(r, e^{\lambda\beta+\mu\gamma}\cdot e^{-\lambda\alpha}) &= T(r, e^{(\lambda+\mu)\gamma}) \\ &= (\lambda+\mu)T(r,e^{\gamma}) + S(r,f) \\ &= (\lambda+\mu)T(r,e^{\alpha}) + S(r,f). \end{aligned}$$

Since $\mu \ne 0$, we obtain a contradiction.

Suppose that $\deg(\lambda\beta-\mu\gamma) < d$, $e^{\lambda\beta-\mu\gamma}$ is a small function of f and $e^{-\alpha}$.

If $\lambda \ge \mu$,

$$T(r, e^{\lambda\beta-\mu\gamma}\cdot e^{-\lambda\alpha}) = \lambda T(r,e^{\alpha}) + S(r,f).$$

On the other hand,

$$\begin{aligned} T(r, e^{\lambda\beta-\mu\gamma}\cdot e^{-\lambda\alpha}) &= T(r, e^{(\lambda-\mu)\gamma}) \\ &= (\lambda-\mu)T(r,e^{\gamma}) + S(r,f) \\ &= (\lambda-\mu)T(r,e^{\alpha}) + S(r,f). \end{aligned}$$

We obtain a contradiction.

If $\lambda < \mu$,

$$T(r, e^{-(\lambda\beta-\mu\gamma)}\cdot e^{\mu\alpha}) = \mu T(r,e^{\alpha}) + S(r,f).$$

On the other hand,

$$T(r, e^{-(\lambda\beta-\mu\gamma)}\cdot e^{\mu\alpha}) = T(r, e^{(\mu-\lambda)\beta}) = (\mu-\lambda)T(r,e^{\alpha}) + S(r,f).$$

It is a contradiction.

It is easy to find $\{A_{\lambda,\mu}\,|\,0 \le \lambda \le 2n, 0 \le \mu \le 2n\}$ are small functions of $e^{\lambda\beta+\mu\gamma}$ and $e^{\lambda\beta-\mu\gamma}$ for each λ, μ satisfied $0 \le \lambda \le 2n$, $0 \le \mu \le 2n$, $\lambda + \mu \ne 0$.

By Lemma 2.7, we deduce that $A_{\lambda,\mu} \equiv 0$, which contradicts with (3.7). So $f \equiv g$.

3.2. Proof of Theorem 1.1

The result of case $p \le q = n = 1$ of equation (1.1) follows immediately from Theorem A.

If $\max\{p, q\} = n \ge 2$, using Lemmas 2.1, 2.4 and 2.5, we have

$$\begin{aligned} T\left(r, \sum_{j=1}^{n} a_j f(z+c_j)\right) &= m\left(r, \sum_{j=1}^{n} a_j f(z+c_j)\right) + N\left(r, \sum_{j=1}^{n} a_j f(z+c_j)\right) \\ &\le m\left(r, f(z)\sum_{j=1}^{n} \frac{f(z+c_j)}{f(z)}\right) + nN(r,f) + S(r,f) \\ &= m(r,f) + nN(r,f) + S(r,f) \end{aligned}$$

and

$$\begin{aligned} T\left(r, \sum_{j=1}^{n} a_j f(z+c_j)\right) &= T\left(r, \frac{P(z,f)}{Q(z,f)}\right) = n \cdot T(r,f) + S(r,f) \\ &= n \cdot m(r,f) + n \cdot N(r,f) + S(r,f). \end{aligned}$$

So $(n-1)m(r,f) = S(r,f)$. When $n \ge 2$, we have $m(r,f) = S(r,f)$. Then use the same method as the proof of Theorem 1.2, we obtain $f \equiv g$.

3.3. Proof of Theorem 1.3

Let $F(z) = \frac{1}{f}$, $G(z) = \frac{1}{g}$, then $F(z)$ and $G(z)$ share $\frac{1}{a}$, $\frac{1}{b}$ and ∞ CM since f and g share a, b, ∞ CM.

If $p < q$, according to equation (1.2), we have

$$\prod_{j=1}^{n} F(z+c_j) = \frac{d_q f^q + \cdots + d_0}{b_p f^p + \cdots + b_0} = \frac{d_q + \cdots + d_0 F^q}{b_p F^{q-p} + \cdots + b_0 F^q}. \tag{3.9}$$

$H_2(z,a) = a^n(d_q a^q + \cdots + d_0) - (b_p a^p + \cdots + b_0) \not\equiv 0$, we multiply both sides by $-\frac{1}{a^{n+q}}$, then

$$\left(\frac{1}{a^n}\right)\left(\frac{b_p}{a^{q-p}} + \cdots + \frac{b_0}{a^q}\right) - \left(d_q + \cdots + \frac{d_0}{a^q}\right) \not\equiv 0.$$

Similarly, $\left(\frac{1}{b^n}\right)\left(\frac{b_p}{b^{q-p}} + \cdots + \frac{b_0}{b^q}\right) - \left(d_q + \cdots + \frac{d_0}{b^q}\right) \not\equiv 0$. By Theorem 1.2(i), $F \equiv G$, so $f \equiv g$.

If $p \geq q$, we have

$$\prod_{j=1}^{n} F(z+c_j) = \frac{d_q f^q + \cdots + d_0}{b_p f^p + \cdots + b_0} = \frac{d_q F^{p-q} + \cdots + d_0 F^p}{b_p + \cdots + b_0 F^p}. \tag{3.10}$$

When $q \leq p < n$, (3.10) satisfies the case (i) or (ii) of Theorem 1.2, then $f \equiv g$. When $p = n$ and $b_k \not\equiv 0$ for at least one $k(0 \leq k \leq p-1)$, (3.10) satisfies the case (i) of Theorem 1.2, then $f \equiv g$.

3.4. Proof of Theorem 1.4

Suppose that f is a finite-order transcendental meromorphic solution of (1.2). Let $F(z) = \frac{1}{f-c}$, $G(z) = \frac{1}{g-c}$, $A = \frac{1}{a-c}$, $B = \frac{1}{b-c}$, then $F(z)$ and $G(z)$ share A, B, ∞ CM, since $f(z)$ and $g(z)$ share a, b, c CM.

$H_2(z,c) \not\equiv 0$, applying Lemmas 2.2, 2.4, $m(r,F) = m(r,\frac{1}{f-c}) = S(r,f) = S(r,F)$.

Substituting $f = \frac{1}{F} + c$ into equation (1.2), we have

$$\prod_{j=1}^{n} \left(\frac{1}{F(z+c_j)} + c \right) = \frac{\sum\limits_{k=0}^{p} b_k (\frac{1}{F} + c)^k}{\sum\limits_{l=0}^{q} d_l (\frac{1}{F} + c)^l}.$$

It can be transformed into

$$\begin{aligned} &\left\{ \prod_{j=1}^{n} [1 + cF(z+c_j)] \right\} \cdot \left\{ \sum_{l=0}^{q} d_l (1+cF)^l F^{n-l} \right\} \\ &- \left\{ \sum_{k=0}^{p} b_k (1+cF)^k F^{n-k} \right\} \cdot \prod_{j=1}^{n} F(z+c_j) = 0. \end{aligned} \tag{3.11}$$

Use $H_3(z,F)$ to denote the left of (3.11). By calculation,

$$H_3(z,A) = \frac{H_2(z,a)}{(a-c)^{2n}} \not\equiv 0, \quad H_3(z,B) = \frac{H_2(z,b)}{(b-c)^{2n}} \not\equiv 0.$$

According to Lemma 2.2, $m(r,\frac{1}{F-A}) = S(r,F)$, $m(r,\frac{1}{F-B}) = S(r,F)$.

F and G share A, B, ∞ CM, so

$$\frac{F-A}{G-A} = e^{\alpha}, \frac{F-B}{G-B} = e^{\beta}. \tag{3.12}$$

Denote $\gamma = \beta - \alpha$

$$F = A + (B-A)\frac{e^{\beta}-1}{e^{\gamma}-1}, \quad \frac{1}{F} = \frac{e^{\gamma}-1}{Ae^{\gamma} + (B-A)e^{\beta} - B}. \tag{3.13}$$

Substituting (3.13) into equation (3.11), and eliminate the denominator, we obtain

$$\sum_{\lambda=0}^{2n}\sum_{\mu=0}^{2n} A_{\lambda,\mu} e^{\lambda\beta+\mu\gamma} = 0.$$

By calculation, we have

$$A_{00} = \frac{H_2(z,b)}{(b-c)^{2n}} \not\equiv 0.$$

Using the method in proof of Theorem 1.2, we can prove $F \equiv G$ easily, then $f \equiv g$.

Similarly, we can obtain the corresponding result for equation (1.1).

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