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A FAMILY OF SERIES AND INTEGRALS INVOLVING WHITTAKER, BESSEL FUNCTIONS, AND THEIR PRODUCTS DERIVABLE FROM THE REPRESENTATION OF THE GROUP $SO(2,1)$

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Abstract. By mainly using certain properties arising from the semisimple Lie group $SO(2,1)$, we aim to show how a family of some interesting formulas for bilateral series and integrals involving Whittaker, Bessel functions, and their product can be obtained.

1. Introduction and preliminaries

It is well known that the irreducible representations of the three-dimensional Lorentz group $SO(2,1)$ can be constructed in the space of σ-homogeneous infinitely differentiable functions ($\sigma \in \mathbb{C} \setminus \mathbb{Z}$) defined on the cone

$$\Lambda := \left\{ (x_1,\, x_2,\, x_3) \in \mathbb{R}^3 \,\middle|\, x_1^2 - x_2^2 - x_3^2 = 0 \right\}.$$

Here and in the following, we denote $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ by the sets of integers, real and complex numbers, respectively. For the fixed σ with $-1 < \Re(\sigma) < 0$, we denote this space by $\mathfrak{D}$ and the analogous space consisting of ς-homogeneous functions when $\varsigma := -\sigma - 1$ by $\mathfrak{D}^\bullet$. The linear operator of the space $\mathfrak{D}$, arising in this case and corresponding to the Lorentz group element g, can be written in the form $T(g)[f(x)] = f(g^{-1}x)$. Since the functions belonging to $\mathfrak{D}$ are homogeneous, we can realize the representation T_σ in the space of the restrictions $\hat{f}$ of functions $f \in \mathfrak{D}$ to any conic section γ, because $\hat{f}(\xi) = t^{-\sigma} f(t\xi) = t(x)$ for $\xi \in \gamma$. Vilenkin and Shleinikova [15] considered the following three sections of Λ: The circle $\gamma_1 : x_1 = 1$; The hyperbola $\gamma_2 : x_2 = \pm 1$; The parabola $\gamma_3 : x_1 + x_2 = 1$. They introduced three bases in $\mathfrak{D}$, which consist of the eigenfunctions of the restrictions of the representation T to the subgroups H_i acting transitively on the above conic sections, respectively, and obtained formulas for 'matrix

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elements' of the linear operator, which transforms the 'circle' basis into the 'parabolic' basis, and some 'matrix elements' of the linear operator, which transforms the 'parabolic' basis into the 'hyperbolic' basis.

Here, we denote the 'circle' and 'parabolic' bases by B_1 and B_2, respectively. Namely,

$$B_1 := \{f_k(x) := x_1^{\sigma-k}\,(x_2 + \mathbf{i}x_3)^k \mid k \in \mathbb{Z}\}$$

and

$$B_2 := \left\{ f_\lambda^*(x) := (x_1 + x_2)^\sigma \exp\left(\frac{\mathbf{i}\,\lambda x_3}{x_1 + x_2}\right) \,\middle|\, \lambda \in \mathbb{R} \right\}.$$

Just as matrix elements of a linear operator can be computed by using a scalar product, the matrix elements of the operators $B_1^\bullet \leftrightarrow B_2^\bullet$ and $T(g)$, $g \in SO(2,1)$, can be derived via the functionals

$$\mathsf{F}_i : \mathfrak{D} \times \mathfrak{D}^\bullet \longrightarrow \mathbb{C},\ (f,g) \longmapsto \int_{\gamma_i} f(x)\,g(x)\,(\mathrm{d}x)_i \quad (i \in \{1,2\}),$$

where $(\mathrm{d}x)_i$ is a H_i-invariant measure. Shilin and Choi [9] computed the matrix elements of the operator, which acts in the $SO(3,1)$-representation space and transforms the 'spherical' basis into the 'hyperbolic' basis, and obtained the corresponding integral formulas for Legendre functions arising around these matrix elements. Shilin and Choi [12] computed the matrix elements of the operator $T^\bullet(a) \in GL(SO(2,1))$ for some block-diagonal matrices and evaluated some integrals involving a product of two Bessel functions (basic or modified). Shilin and Choi [10, 11] have employed the same idea which was in [12] for $SO(2,2)$ and $SO(3,1)$, respectively. In particular, Shilin and Choi [11] evaluated some integrals involving a product of the Macdonald function and the modified hyper-Bessel function of the first or second kind.

In this paper, we reconsider the group $SO(2,1)$ and compute the matrix elements of the operator $T^\bullet(g_i)$ with respect to the basis $B_{3-i}^\bullet$. By using the matrix elements of the transformations $B_1^\bullet \leftrightarrow B_2^\bullet$ and decompositions $G = KAN$ (Iwasawa decomposition) and $G = KAK$ (Cartan decomposition) of Lie groups, we obtain some formulas for series and integrals involving Whittaker or Bessel functions or their product. In particular, in Theorem 1, we consider series involving product of two Whittaker functions $W_{k,\mathbf{i}\tau}$, where k or $-k$ is the parameter of series. Some similar integrals involving the same product, where the variable of integration is τ, are pointed out to be considered in [1].

2. Some series involving one or two Whittaker functions of the second order

Ceratin bilateral series involving a product of two Whittaker functions $W_{k,\mu}$ and one Whittaker function are evaluated as in Theorem 1. In the following, $\Gamma(z)$ and $B(\alpha,\beta)$ are the familiar Gamma and Beta functions (see, *e.g.*, [13, Section 1.1]), respectively, $J_\nu(z)$ and $K_\nu(z)$ are Bessel functions of the first kind and modified Bessel functions (Macdonald functions) (see, *e.g.*, [4, Sections 8.4-8.5, p. 900]), respectively, $W_{\lambda,\mu}(z)$ are the Whittaker functions (see, *e.g.*,

[4, Sections 9.22-9.23, p. 1014]), and ${}_2F_1$ is the hypergeometric function (see, *e.g.*, [13, Section 1.5]).

Theorem 1. *Let $\theta \neq \pi n$, where $n \in \mathbb{Z}$ with $n \equiv 1 \pmod 2$. Then the following formulas hold true*:

- For $-1 < \Re(\sigma) < 0$,

$$\sum_{k\in\mathbb{Z}} \exp(-\mathbf{i}k\theta)\,\{\Gamma(1+\sigma+k\operatorname{sign}\hat{\lambda})\Gamma(-\sigma-k\operatorname{sign}\lambda)\}^{-1}$$
$$\times\, W_{k\operatorname{sign}\hat{\lambda},\,\sigma+\frac{1}{2}}(2|\hat{\lambda}|)\,W_{-k\operatorname{sign}\lambda,\,\sigma+\frac{1}{2}}(2|\lambda|) = \Theta\left(\operatorname{sign}(\lambda\hat{\lambda})\right),$$

where

$$\Theta(1) = -\frac{1}{\pi}\,\sec^2\frac{\theta}{2}\,(\lambda\hat{\lambda})^{\frac{1}{2}}\,\sin(\pi\sigma)\,K_{2\sigma+1}\left(2\left|\sec\frac{\theta}{2}\right|\sqrt{\lambda\hat{\lambda}}\right)$$

and

$$\Theta(-1) = \frac{1}{2}\sec^2\frac{\theta}{2}\,|\lambda\hat{\lambda}|^{\frac{1}{2}}\,\cos(\pi\sigma)$$
$$\times\left[J_{-2\sigma-1}\left(2\left|\sec\frac{\theta}{2}\right|\sqrt{|\lambda\hat{\lambda}|}\right) - J_{2\sigma+1}\left(2\left|\sec\frac{\theta}{2}\right|\sqrt{|\lambda\hat{\lambda}|}\right)\right].$$

- *For $-\frac{1}{2} < \Re(\sigma) < 0$ and $\lambda \neq 0$,*

$$\sum_{k\in\mathbb{Z}} \exp(-\mathbf{i}k\theta)\,\{\Gamma(-\sigma-k\operatorname{sign}\lambda)\,\mathrm{B}(k-\sigma,-k-\sigma)\}^{-1}\,W_{-k\operatorname{sign}\lambda,\sigma+\frac{1}{2}}(2|\lambda|)$$
$$= 2^{-\sigma-1}\,\pi^{-1}\,(-2\sigma-1)\,|\lambda|^{-\sigma}\left(\cos\frac{\theta}{2}\right)^{4\sigma}\sin(\pi\sigma)\,\Gamma(2\sigma+1).$$

- For $-\frac{1}{2} < \Re(\sigma) < 0$ *and* $\hat{\lambda} \neq 0$,

$$\sum_{k\in\mathbb{Z}} \exp(-\mathbf{i}k\theta)\,\{\Gamma(1+\sigma+k\operatorname{sign}\hat{\lambda})\,\mathrm{B}(k-\sigma,-k-\sigma)\}^{-1}\,W_{k\operatorname{sign}\lambda,\sigma+\frac{1}{2}}(2|\hat{\lambda}|)$$
$$= 2^{-\sigma-1}\,\pi^{-1}\,(2\sigma+1)\,|\hat{\lambda}|^{\sigma+1}\left(\sec\frac{\theta}{2}\right)^{4\sigma+4}\sin(\pi\sigma)\,\Gamma(-2\sigma-1).$$

Proof. Let us consider the matrix elements $t_{\lambda\hat{\lambda}}(g)$ of the representation $T^\bullet$ with respect to the basis $B_2^\bullet$. Since

$$T^\bullet(g)[f_\lambda^{*\bullet}(x)] = \int_{-\infty}^{+\infty} t_{\lambda\hat{\lambda}}(g) f_{\hat{\lambda}}^{*\bullet}(x)\,\mathrm{d}\hat{\lambda}, \tag{2.1}$$

we have

$$\mathsf{F}_i\,(T^\bullet(g)[f_\lambda^{*\bullet}], f_\tau^*) = \int_{-\infty}^{+\infty} t_{\lambda\mu}(g)\,\mathsf{F}_j(f_{\hat{\lambda}}^{*\bullet}, f_\tau^*)\,\mathrm{d}\hat{\lambda}$$

for any i and j. Taking

$$\mathsf{F}_3(f_{\hat{\lambda}}^{*\bullet}, f_\tau^*) = \int_{-\infty}^{+\infty} \exp(\mathbf{i}[\mu+\tau]y)\,\mathrm{d}y = \delta(\mu+\tau),$$

where $\delta(\mu+\tau)$ is the $(-\tau)$-delayed Dirac delta function, we obtain

$$\mathsf{F}_i\,(T^\bullet(g)[f_\lambda^{*\bullet}], f_\tau^*) = \int_{-\infty}^{+\infty} t_{\lambda\mu}(g)\,\delta(\mu+\tau)\,\mathrm{d}\hat\lambda = 2\pi\, t_{\hat\lambda,-\tau}(g).$$

Thus we have

$$t_{\lambda\hat\lambda}(g) = \frac{1}{2\pi}\,\mathsf{F}_i(T^\bullet(g)[f_\lambda^{*\bullet}], f_{-\hat\lambda}^*). \tag{2.2}$$

By using the connection $f_\lambda^{*\bullet}(x) = \sum\limits_{k\in\mathbb{Z}} c_{\lambda k}\, f_k^\bullet(x)$ between the bases $B_1^\bullet$ and $B_3^\bullet$, we have

$$t_{\lambda\hat\lambda}(g) = \frac{1}{2\pi}\,\mathsf{F}_i\left(\sum_{k\in\mathbb{Z}} c_{\lambda k}\, T^\bullet(g)[f_k^\bullet], \sum_{\tilde k\in\mathbb{Z}} c_{-\tilde\lambda\tilde k}\, f_{\tilde k}\right).$$

Let g_1 be the rotation through an angle θ in the plane generated by axes Ox_2 and Ox_3. The matrix representation of g_1 is

$$g_1 \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

$g_1 \in H_1$ and f_k is the eigenfunction of the operator $T^\bullet(g_1)$ associated with the eigenvalue $\exp(-\mathbf{i}k\theta)$. The matrix elements of the linear operator that transforms $B_1^\bullet$ into $B_2^\bullet$ were computed for $\lambda \neq 0$ in [15] and for $\lambda = 0$ in [12]:

$$c_{\lambda k} \equiv c_{\lambda k}(\sigma) = \begin{cases} |\lambda|^{-\sigma-1}\,\{\Gamma(-\sigma-k\operatorname{sign}\lambda)\}^{-1}\,W_{-k\operatorname{sign}\lambda,\sigma+\frac12}(2|\lambda|), & \text{if } \lambda\neq 0,\\ 2^{\sigma+1}\,\{(-2\sigma-1)\,\mathrm{B}(k-\sigma,-k-\sigma)\}^{-1}, & \text{if } \lambda = 0.\end{cases}$$

Thus, choosing $i=1$ in (2.2) and taking the basis $B_1^\bullet$ which is orthonormal with respect to the bilinear functional $\frac{1}{2\pi}\mathsf{F}_1$, we obtain

$$\begin{aligned} t_{\lambda\hat\lambda}(g_1) = {} & |\hat\lambda|^\sigma\,|\lambda|^{-\sigma-1}\sum_{k\in\mathbb{Z}}\exp(-\mathbf{i}k\theta)\,\{\Gamma(1+\sigma+k\operatorname{sign}\hat\lambda)\}^{-1} \\ & \times\{\Gamma(-\sigma-k\operatorname{sign}\lambda)\}^{-1}\,W_{k\operatorname{sign}\hat\lambda,\,\sigma+\frac12}(2|\hat\lambda|)\,W_{-k\operatorname{sign}\lambda,\,\sigma+\frac12}(2|\lambda|). \end{aligned}$$

Let us represent g_1 as a product

$$g_1 = h_1\,\mathrm{diag}(1,-1,-1)\,h_2\,h_1,$$

where h_1 belongs to the maximal nilpotent subgroup H_3 and depends on the only parameter θ^*, and h_2 is the hyperbolic rotation through an angle θ^{**} in the plane generated by axes Ox_1 and Ox_2. Taking the matrix representations

$$h_1 \longleftrightarrow \begin{pmatrix} 1+\frac{\theta^{*2}}{2} & \frac{\theta^{*2}}{2} & \theta^* \\ -\frac{\theta^{*2}}{2} & 1-\frac{\theta^{*2}}{2} & -\theta^* \\ \theta^* & \theta^* & 1 \end{pmatrix}, \quad h_2 \longleftrightarrow \begin{pmatrix} \cosh\theta^{**} & \sinh\theta^{**} & 0 \\ \sinh\theta^{**} & \cosh\theta^{**} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$\begin{cases} \frac{1}{2}\theta^{*4}\exp(-\theta^{**}) + \theta^{*2}\exp(-\theta^{**}) + \cosh\theta^{**} - \theta^{*2} = 1, \\ \frac{1}{2}\theta^{*4}\exp(-\theta^{**}) + \sinh\theta^{**} - \theta^{*2} = 0, \\ \theta^{*3}\exp(-\theta^{**}) + \theta^{*}\exp(-\theta^{**}) - \theta^{*} = 0, \\ \theta^{*2} + \theta^{*2}\exp(-\theta^{**}) - \frac{1}{2}\theta^{*4}\exp(-\theta^{**}) - \cosh\theta^{**} = \cos\theta, \\ \theta^{*3}\exp(-\theta^{**}) - \theta^{*}\exp(-\theta^{**}) - \theta^{*} = -\sin\theta, \end{cases}$$

that is, $\frac{2\theta^*}{\theta^{*2}+1} = \sin\theta$ and, therefore, $\theta^* = \tan\frac{\theta}{2}$ and $\exp(-\theta^{**}) = (\theta^{*2}+1)^{-1} = \cos^2\frac{\theta}{2}$. Since $T^\bullet$ is a group homomorphism and for any $g \in SO(2,1)$ the functional F_3 is invariant with respect to the pair $(T(g), T^\bullet(g))$ of operators arising in the corresponding spaces [12], we derive

$$\begin{aligned} t_{\lambda\hat{\lambda}}(g_1) &= \frac{1}{2\pi}\mathsf{F}_3(T^\bullet(h_2h_1)\,[f^{*\bullet}_\lambda], T(\mathrm{diag}(1,-1,-1)h_1^{-1})[f^*_{-\hat{\lambda}}]) \\ &= \frac{1}{2\pi}\,\sec^2\frac{\theta}{2}\int_{-\infty}^{+\infty} y^{2\sigma}\,\exp\left(\mathbf{i}\,\sec^2\frac{\theta}{2}\left[\lambda y - \frac{\hat{\lambda}}{y}\right]\right)\mathrm{d}y \end{aligned}$$

and use the following formulas:

- $\lambda\hat{\lambda} > 0$ (see, *e.g.*, [7, Entry 2.5.24.4]),

$$\int_0^{+\infty} x^{\alpha-1}\cos\left(ax - \frac{b}{x}\right)\mathrm{d}x = 2\left(\frac{b}{a}\right)^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{2}\right)K_\alpha(2\sqrt{ab})$$
$$(a,\ b > 0;\ \ |\Re(\alpha)| < 1).$$

- $\lambda\hat{\lambda} < 0$ (see, *e.g.*, [7, Entry 2.5.24.7]).

$$\int_0^{+\infty} x^{\alpha-1}\cos\left(ax + \frac{b}{x}\right)\mathrm{d}x = \frac{\pi}{2}\left(\frac{b}{a}\right)^{\frac{\alpha}{2}}\sin\left(\frac{\alpha\pi}{2}\right)\left[J_{-\alpha}(2\sqrt{ab}) - J_\alpha(2\sqrt{ab})\right]$$
$$(a,\ b > 0;\ \ |\Re(\alpha)| < 1).$$

- $\lambda \neq 0$ and $\hat{\lambda} = 0$ (see, *e.g.*, [7, Entry 2.5.3.10]).

$$\int_0^{+\infty} x^{\alpha-1}\cos(bx)\,\mathrm{d}x = b^{-\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha) \tag{2.3}$$
$$(b > 0;\ \ 0 < \Re(\alpha) < 1).$$

- $\lambda = 0$ and $\hat{\lambda} \neq 0$.
  Substitute $x = z^{-1}$ and use the formula (2.3).

By comparing the results for $t_{\lambda\hat{\lambda}}(g_1)$ and $t_{\lambda\hat{\lambda}}(h_1\,\mathrm{diag}(1,-1,-1)h_2h_1)$, we completes the proof. □

3. Some integrals involving a product of Bessel and Whittaker functions

Throughout this section, we assume that $\theta \neq \pi n$ ($n \in \mathbb{Z}$ with $n \equiv 1 \pmod 2$). Here we establish a formula for certain integrals involving Bessel functions of the first kind, Macdonald functions, and Whittaker functions, which is asserted by Theorem 2.

Theorem 2. *For $-1 < \Re(\sigma) < 0$ and $\lambda > 0$, the following formula holds true*:

$$\begin{aligned}
&\frac{1}{2}\cos(\pi\sigma)\,\{\Gamma(k-\sigma)\}^{-1}\int_0^{+\infty}\left[J_{-2\sigma-1}\left(2\left|\sec\frac{\theta}{2}\right|\sqrt{\lambda\hat{\lambda}}\right)\right.\\
&\quad\left.-\,J_{2\sigma+1}\left(2\left|\sec\frac{\theta}{2}\right|\sqrt{\lambda\hat{\lambda}}\right)\right]W_{k,\sigma+\frac{1}{2}}(2\hat{\lambda})\,\mathrm{d}\hat{\lambda}\\
&\quad-\frac{1}{\pi}\sin(\pi\sigma)\,\Gamma(-k-\sigma)\}^{-1}\int_0^{+\infty}K_{2\sigma+1}\left(2\left|\sec\frac{\theta}{2}\right|\sqrt{\lambda\hat{\lambda}}\right)W_{-k,\sigma+\frac{1}{2}}(2\hat{\lambda})\,\mathrm{d}\hat{\lambda}\\
&=\lambda^{-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^2\exp(-\mathbf{i}k\theta)\,\{\Gamma(-k-\sigma)\}^{-1}\,W_{-k,\sigma+\frac{1}{2}}(2\lambda).
\end{aligned}$$

Proof. Taking the connection between the bases $B_1^\bullet$ and $B_2^\bullet$, we rewrite (2.1) for the case $g = g_1$ in the following manner:

$$T^\bullet(g_1)[f_\lambda^{*\bullet}(x)] = \sum_{k\in\mathbb{Z}}\left(\int_{-\infty}^{+\infty} t_{\lambda\hat{\lambda}}(g_1)\,c_{\hat{\lambda}k}\,\mathrm{d}\hat{\lambda}\right)f_k^\bullet(x). \tag{3.1}$$

Since $f_k^\bullet$ is an eigenfunction of the operator $T^\bullet(g_1)$ associated with the eigenvalue $\exp(-\mathbf{i}k\theta)$, we have

$$T^\bullet(g_1)[f_\lambda^{*\bullet}(x)] = \sum_{k\in\mathbb{Z}} c_{\lambda k}\,\exp(-\mathbf{i}k\theta)\,f_k^\bullet(x). \tag{3.2}$$

By using formulas for $c_{\lambda k}$ and $t_{\lambda\hat{\lambda}}(g_1)$ given in the proof of Theorem 1, from (3.1) and (3.2) we derive the desired result. □

A similar integral formula can be established for the case $\lambda < 0$. For the case $\lambda = 0$, we obtain an integral formula asserted by Theorem 3.

Theorem 3. *For $-\frac{1}{2} < \Re(\sigma) < 0$, the following formula holds true*:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\hat{\lambda}^{2\sigma+1}\,|\hat{\lambda}|^{-\sigma-1}\,\{\Gamma(-\sigma-k\,\mathrm{sign}\,\hat{\lambda})\}^{-1}\,W_{-k\,\mathrm{sign}\,\hat{\lambda},\sigma+\frac{1}{2}}(2|\hat{\lambda}|)\,\mathrm{d}\hat{\lambda}\\
&=2^{\sigma+1}\,\frac{\pi\,(2\sigma+1)}{\Gamma(-2\sigma-1)\,\Gamma(k-\sigma)\,\Gamma(-k-\sigma)\,\cos(\sigma\pi)}.
\end{aligned} \tag{3.3}$$

Proof. The formula (3.3) follows from (3.1) and (3.2), where g_1 is the identity matrix, that is, $\theta = 0$. □

Setting here $k=0$ and using a known formula (see, *e.g.*, [4, Entry 8.476.5]):

$$K_\nu(\mathrm{e}^{m\pi\mathbf{i}}z) = \mathrm{e}^{-m\nu\pi\mathbf{i}}\,K_\nu(z) - \mathbf{i}\pi\frac{\sin(m\nu\pi)}{\sin(\nu\pi)}I_\nu(z), \tag{3.4}$$

we obtain

$$\begin{aligned}&\int_0^{+\infty}\hat\lambda^{\sigma+\frac12}\left(K_{\sigma+\frac12}(\hat\lambda)+\mathbf{i}\pi\cos(\sigma\pi)\,I_{\sigma+\frac12}(\hat\lambda)\right)\mathrm{d}\hat\lambda\\ &= 2^\sigma\,\frac{\pi^{\frac32}\,(2\sigma+1)}{\Gamma(-2\sigma-1)\,\Gamma(-\sigma)\,\cos(\sigma\pi)}.\end{aligned}$$

4. The cosine-Fourier transform of squared Macdonald function

Here we present a formula for the cosine-Fourier transform of squared Macdonald function given in Theorem 4.

Theorem 4. *For $-1<\Re(\sigma)<0$, the following formula holds true*:

$$\begin{aligned}&\int_0^{+\infty}\left\{K_{\sigma+\frac12}(\lambda)\right\}^2\cos(b\lambda)\,\mathrm{d}\lambda\\ &= \frac{\pi^2}{4}\operatorname{cosec}(\pi\sigma)\,\exp(-\mathbf{i}b_1k)\,\exp(\sigma b_2)\,{}_2F_1\left(-\sigma,\,\frac12;\,1;\,1-\exp(-2b_2)\right),\end{aligned}$$

where

$$b_1 = b_3-\pi = \arccos\frac{b}{\sqrt{b^2+4}} \quad \textit{and} \quad \cosh b_2 = \frac{b^2}{2}+1. \tag{4.1}$$

Proof. Let $g_2(b)\in H_3$, that is, there is $b\in\mathbb{R}$ such that

$$g_2 = \begin{pmatrix} 1+\frac{b^2}{2} & \frac{b^2}{2} & b\\ -\frac{b^2}{2} & 1-\frac{b^2}{2} & -b\\ b & b & 1\end{pmatrix}.$$

Let

$$f_k^\bullet(x) = \int_{-\infty}^{+\infty}\tilde c_{k\lambda}\,f_\lambda^{*\bullet}(x)\,\mathrm{d}\lambda \quad\text{and}\quad T^\bullet(g)[f_k^\bullet(x)] = \sum_{k\in\mathbb{Z}} t_{k\hat k}(g)f_{\hat k}^\bullet(x).$$

We readily obtain

$$\tilde c_{k\lambda}(\sigma) = \frac{1}{2\pi}\mathsf{F}_i(f_k^\bullet, f_{-\lambda}^*) = c_{-\lambda,-k}(-\sigma-1)$$

and, therefore,

$$\begin{aligned}t_{k\hat k}(g_2) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde c_{k\lambda}\,\tilde c_{\hat k,-\lambda}\,\exp(-\mathbf{i}b\lambda)\,\mathrm{d}\lambda\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}c_{-\lambda,-k}(-\sigma-1)\,c_{\lambda,-\hat k}(\sigma)\,\exp(-\mathbf{i}b\lambda)\,\mathrm{d}\lambda.\end{aligned} \tag{4.2}$$

Let us represent g_2 as a product $g_2 = \tilde{h}_1(b_1)\tilde{h}_2(b_2)\tilde{h}_3(b_3)$, where $\tilde{h}_1, \tilde{h}_3 \in H_1$ and $\tilde{h}_2$ belongs to the subgroup of hyperbolic rotations in the plane generated by axes Ox_1 and Ox_2. By solving the system of equations

$$\begin{pmatrix} 1+\frac{b^2}{2} & \frac{b^2}{2} & b \\ -\frac{b^2}{2} & 1-\frac{b^2}{2} & -b \\ b & b & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos b_1 & -\sin b_1 \\ 0 & \sin b_1 & \cos b_1 \end{pmatrix} \begin{pmatrix} \cosh b_2 & \sinh b_2 & 0 \\ \sinh b_2 & \cosh b_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos b_3 & -\sin b_3 \\ 0 & \sin b_3 & \cos b_3 \end{pmatrix},$$

we obtain the relations (4.1), that is,

$$g_2 = \tilde{h}_1(b_1)\tilde{h}_2(b_2)\,\mathrm{diag}(1,-1,-1)\,\tilde{h}_1(b_1).$$

Since $T^\bullet$ is a group homomorphism, $(\tilde{h}_i(b_i))^{-1} = \tilde{h}_i(-b_i)$ for any i, and F_1 is invariant with respect to the pair $(T^\bullet, T)$, we have

$$\begin{aligned} t_{k\hat{k}}(g_2) &= \frac{1}{2\pi}\mathsf{F}_1(T^\bullet(h_1)T^\bullet(h_2)[f_k^\bullet], T(h_3^{-1})T(\mathrm{diag}(1,-1,-1))[f_{-\hat{k}}]) \\ &= \frac{1}{2\pi\,\exp(\mathbf{i}(\pi+b_1)\hat{k})}\int_{-\pi}^{\pi} \exp(\mathbf{i}(k+\hat{k})\varphi)\,(\cosh b_2 - \sinh b_2\cos\varphi)^{\sigma-k} \\ &\quad \times (\cosh b_2\cos\varphi - \sinh b_2 + \mathbf{i}\sin\varphi)^k\,\mathrm{d}\varphi. \end{aligned}$$

By using 'universal' substitution $t = \tan\frac{\varphi}{2}$, we obtain

$$t_{00}(g_2) = \frac{\exp(\sigma b_2)}{2\pi^2\,\exp(\mathbf{i}(\pi+b_1)\hat{k})}\int_0^{+\infty}(1+t^2)^{-\sigma-1}\,(\exp(-2b_2)+t^2)^\sigma\,\mathrm{d}t. \tag{4.3}$$

The integral (4.3) can be evaluated by using the following known formula (see [4, Entry 3.259.3]):

$$\begin{aligned} &\int_0^{+\infty} x^{\lambda-1}(1+\alpha x^p)^{-\mu}\,(1+\beta x^p)^{-\nu}\,\mathrm{d}x \\ &= \frac{1}{p}\,\alpha^{-\frac{\lambda}{p}}\,\mathrm{B}\left(\frac{\lambda}{p}, \mu+\nu-\frac{\lambda}{p}\right)\,{}_2F_1\left(\nu,\,\frac{\lambda}{p};\,\mu+\nu;\,1-\frac{\beta}{\alpha}\right) \\ &\qquad (|\arg(\alpha)|,\ |\arg(\beta)|<\pi,\ p>0,\ 0<\Re(\lambda)<2\Re(\mu+\nu)). \end{aligned}$$

Finally comparing formulas (4.2) for $k = \hat{k} = 0$ and (4.3), we obtain the desired result. □

5. Concluding remarks

Theorem 2 is a generalization of [12, Theorem 1] which was derived by using the restriction of the representation $T^\bullet$ to the diagonal matrix $\mathrm{diag}(1,-1,-1)$.

Theorem 4 can be considered as a new approach that differs from the formula (see [8, Entry 2.16.36.2]):

(5.1)
$$\int_0^{+\infty} \cos(ax)\, K_\nu(bx)\, K_\nu(cx)\, \mathrm{d}x = \frac{\pi^2}{4\sqrt{bc}\,\cos(\pi\nu)}\, P_{\nu-\frac{1}{2}}\left(\frac{a^2+b^2+c^2}{2bc}\right)$$
$$\left(\Re(b+c) > |\Im(a)|,\ \Re(\nu) < \frac{1}{2}\right),$$

which does not use functions $a_1(a)$ and $a_2(a)$ in the right side.

The approach employed in this paper looks a good starting point for further research: By using various decompositions of the group $SO(2,1)$, formulas of the transformations $B_1 \leftrightarrow$ 'hyperbolic basis' B_3 and $B_2 \leftrightarrow B_3$, formulas for the matrix elements of the operator $T^\bullet(h)$ with respect to the $B_1^\bullet$ or $B_2^\bullet$, where h belongs to the transitive subgroup of a hyperbolic section of Λ, and formulas for the matrix elements of the $T^\bullet(g_1)$ and $T^\bullet(g_2)$ with respect to the $B_3^\bullet$, it seems possible to obtain new formulas for the special functions related to the group $SO(2,1)$.

Bessel functions play an important role, in particular, in investigating solutions of various differential equations, and are associated with a wide range of problems in many research areas of mathematical physics. So, Bessel functions, and their extensions and variants have been investigated by many authors (see, e.g., [2, 3, 5, 6, 14]).

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