ON t-ALMOST DEDEKIND GRADED DOMAINS

GYU WHAN CHANG AND DONG YEOL OH

Abstract. Let Γ be a nonzero torsionless commutative cancellative monoid with quotient group ⟨Γ⟩, \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \) be a graded integral domain graded by Γ such that \( R_\alpha \neq \{0\} \) for all \( \alpha \in \Gamma \), \( H \) be the set of nonzero homogeneous elements of \( R \), \( C(f) \) be the ideal of \( R \) generated by the homogeneous components of \( f \in R \), and \( N(H) = \{ f \in R \mid C(f) = R \} \). In this paper, we introduce the notion of graded \( t \)-almost Dedekind domains. We then show that \( R \) is a \( t \)-almost Dedekind domain if and only if \( R \) is a graded \( t \)-almost Dedekind domain and \( RH \) is a \( t \)-almost Dedekind domain. We also show that if \( R = D[\Gamma] \) is the monoid domain of \( \Gamma \) over an integral domain \( D \), then \( R \) is a graded \( t \)-almost Dedekind domain if and only if \( D \) and \( \Gamma \) are \( t \)-almost Dedekind, if and only if \( R_{N(H)} \) is an almost Dedekind domain. In particular, if \( (\Gamma) \) satisfies the ascending chain condition on its cyclic subgroups, then \( R = D[\Gamma] \) is a \( t \)-almost Dedekind domain if and only if \( R \) is a graded \( t \)-almost Dedekind domain.

1. Introduction

An almost Dedekind domain \( D \) is an integral domain in which \( D_M \) is a rank-one discrete valuation ring (DVR) for all maximal ideals \( M \) of \( D \). As in [15], we say that \( D \) is a \( t \)-almost Dedekind domain if \( D_P \) is a rank-one DVR for all maximal \( t \)-ideals \( P \) of \( D \). (Definitions related with the \( t \)-operation and graded integral domains will be reviewed in Section 2.) Clearly, \( D \) is an almost Dedekind domain if and only if \( D \) is a \( t \)-almost Dedekind domain whose nonzero maximal ideals are \( t \)-ideals. Also, a Dedekind domain is an almost Dedekind domain, while an almost Dedekind domain need not be a Dedekind domain (see, for example, [16]). It is clear that \( D \) is a Dedekind domain (resp., Krull domain) if and only if \( D \) is an almost Dedekind domain (resp., a \( t \)-almost Dedekind domain) in which each nonzero nonunit is contained in only finitely many maximal ideals (resp., maximal \( t \)-ideals) of \( D \). Note that a rank-one DVR has (Krull) dimension one; so if \( D \) is an almost (resp., a \( t \)-almost) Dedekind domain, then \( \dim(D) \leq 1 \) (resp., \( t\dim(D) \leq 1 \)), i.e., each nonzero prime ideal (resp., prime \( t \)-ideal) of \( D \) is a maximal ideal (resp., maximal \( t \)-ideal).

Received August 9, 2016; Accepted December 21, 2016.
2010 Mathematics Subject Classification. 13A02, 13A15, 13F05, 20M25.
Key words and phrases. graded integral domain, (t-)almost Dedekind domain, (graded) \( t \)-almost Dedekind domain.

©2017 Korean Mathematical Society

1969
Let $D$ be an integral domain, $\Gamma$ be a nonzero torsionless commutative cancellative monoid, and $D[\Gamma]$ be the monoid domain of $\Gamma$ over $D$. If $\Gamma = \mathbb{N}_0$ is the additive monoid of nonnegative integers, then $D[\Gamma] = D[X]$, the polynomial ring over $D$. Clearly, $D[X]$ is an almost Dedekind domain if and only if $D$ is a field, if and only if $D[\Gamma]$ is a Dedekind domain. Also, it is known that $D[\Gamma]$ is an almost Dedekind domain if and only if $D$ is a field and $\Gamma$ is isomorphic to either $\mathbb{Z}_+$ or a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ such that if char$(D) = p$ is nonzero, then $\frac{1}{p} \not\in \Gamma$ for some integer $k \geq 1$ [11, Corollary 20.15]. However, note that $D$ is a $t$-almost Dedekind domain if and only if $D[\Gamma]$ is a $t$-almost Dedekind domain, if and only if $D[X]\[\Gamma\]$ is an almost Dedekind domain, where $N = \{f \in D[X]\mid (A_f)_v = D\}$ and $A_f$ is the ideal of $D$ generated by the coefficients of $f \in D[X]$ [15, Theorems 4.2 and 4.4]. Hence, it is natural to ask when $D[\Gamma]$ is a $t$-almost Dedekind domain.

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain graded by $\Gamma$ such that $R_\alpha \neq \{0\}$ for all $\alpha \in \Gamma$, $H$ be the set of nonzero homogeneous elements of $R$, $C(f)$ be the ideal of $R$ generated by the homogeneous components of $f \in R$, and $N(H) = \{f \in R \mid C(f)_v = R\}$. In Section 2, we review definitions related with the $t$-operation and graded integral domains. In Section 3, we first introduce the concept of graded $t$-almost Dedekind domains. We then show that $R$ is a graded $t$-almost Dedekind domain if and only if every nonzero homogeneous ideal of $R$ is a $v$-cancellation ideal. We also prove that $R$ is a $t$-almost Dedekind domain if and only if $R$ is a graded $t$-almost Dedekind domain and $R_H$ is a $t$-almost Dedekind domains. In particular, if $R$ satisfies property $(\#)$, then $R$ is a graded $t$-almost Dedekind domain if and only if $R_{N(H)}$ is an almost Dedekind domain. In Section 4, we study (graded) $t$-almost Dedekind domain properties of $R$ when $R = D[\Gamma]$. Among other things, we prove that $R = D[\Gamma]$ is a graded $t$-almost Dedekind domain if and only if $D$ and $\Gamma$ are $t$-almost Dedekind, if and only if $R_{N(H)}$ is an almost Dedekind domain. As a corollary, we have that $R = D[\Gamma]$ is a $t$-almost Dedekind domain if and only if $R_{N(H)}$ is an almost Dedekind domain and $K[\Gamma]$ is a $t$-almost Dedekind domain, where $\Gamma$ is the quotient group of $\Gamma$. In particular, if $\Gamma$ satisfies the ascending chain condition on its cyclic subgroups, then $D[\Gamma]$ is a graded $t$-almost Dedekind domain if and only if $D[\Gamma]$ is a $t$-almost Dedekind domain.

2. The $t$-operation and graded integral domains

Let $D$ be an integral domain with quotient field $K$ and $F(D)$ (resp., $f(D)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of $D$. For $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$; then $I^{-1} \in F(D)$. The $v$-operation on $D$ is defined by $I_v = (I^{-1})^{-1}$; the $t$-operation by $I_t = \bigcup\{J_v \mid J \subseteq I\}$; and the $w$-operation by $I_w = \{x \in K \mid xJ \subseteq I \text{ for some } J \subseteq f(D) \text{ with } J_v = D\}$ for all $I \in F(D)$. We say that $I \in F(D)$ is a $v$-ideal (resp.,
$t$-ideal, $w$-ideal) if $I_v = I$ (resp., $I_t = I$, $I_w = I$), and a $v$-ideal (resp., $t$-ideal, $w$-ideal) $I$ is a maximal $v$-ideal (resp., maximal $t$-ideal, maximal $w$-ideal) if $I$ is maximal (under inclusion) among proper integral $v$-ideals (resp., $t$-ideals, $w$-ideals). Let $v$-Max$(D)$ (resp., $t$-Max$(D)$, $w$-Max$(D)$) be the set of maximal $v$-ideals (resp., $t$-ideals, $w$-ideals) of $D$. As in the case of rank-one nondiscrete valuation domains, $v$-Max$(D)$ can be empty even when $D$ is not a field. However, it is well known that if $v = t$ or $w$, then $v$-Max$(D) \neq \emptyset$ when $D$ is not a field; each prime ideal minimal over a $v$-ideal is a $v$-ideal (hence a height-one prime ideal is a $v$-ideal); each proper integral $v$-ideal is contained in a maximal $v$-ideals; $D = \bigcap_{P \in v$-Max$(D)} D_P$; and $t$-Max$(D) = w$-Max$(D)$. An $I \in F(D)$ is said to be $t$-invertible if $(11^{-1})_I = D$, and $D$ is called a Prüfer $v$-multiplication domain (PrMD) if each nonzero finitely generated ideal of $D$ is $t$-invertible. It is known that $D$ is a PrMD if and only if $D_P$ is a valuation domain for all maximal $t$-ideals $P$ of $D$ [13, Theorem 5]. We say that a nonzero ideal $I$ of $D$ is a cancellation ideal (resp., $w$-cancellation ideal) if $IA = IB$ (resp., $(IA)_w = (IB)_w$) for nonzero ideals $A$ and $B$ of $D$ implies $A = B$ (resp., $A_w = B_w$). The $v$, $t$, and $w$-operations are the most well-known examples of so-called star-operations. For more on basic properties of star-operations, see [12, Sections 32 and 34].

Let $\Gamma$ be a torsionless grading (i.e., commutative, cancellative) monoid (written additively) and $\Gamma = \{a - b \mid a, b \in \Gamma\}$ be the quotient group of $\Gamma$; so $(\Gamma)$ is a torsionfree abelian group. It is well known that a commutative monoid $\Gamma$ is torsionless if and only if $\Gamma$ can be given a total order compatible with the monoid operation [17, page 123]. A $(\mathcal{G})$-graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ is an integral domain graded by $\Gamma$. For every $\alpha \in \Gamma$, a nonzero element $x \in R_{\alpha}$ is called a homogeneous element of degree $\alpha$, i.e., $\deg(x) = \alpha$, and $\deg(0) = 0$. Thus, every $0 \neq f \in R$ can be written uniquely as $f = x_{\alpha_1} + \cdots + x_{\alpha_n}$ with $\deg(x_{\alpha_i}) = \alpha_i$ and $\alpha_1 < \cdots < \alpha_n$. Let $H = \bigcup_{\alpha \in \Gamma} R_{\alpha} \setminus \{0\}$. Then $H$ is the saturated multiplicative set of nonzero homogeneous elements of $R$, and $R_H = \bigoplus_{\alpha \in \Gamma} (R_H)_\alpha$, called the homogeneous quotient field of $R$, is a $(\mathcal{G})$-graded integral domain. Clearly, $(R_H)_\alpha = \{\frac{a}{b} \mid a \in R_B, 0 \neq b \in R_{\alpha},$ and $\alpha = \beta - \gamma\}$ for all $\alpha \in \Gamma$, $(R_H)_0$ is a field, and every nonzero homogeneous element of $R_H$ is a unit. For a fractional ideal $I$ of $R$ with $I \subseteq R_H$, let $I^*$ denote the fractional ideal of $R$ generated by the homogeneous elements in $I$. We say that $I$ is homogeneous if $I^* = I$. A graded integral domain $R$ is a graded DVR if $R$ has a unique nonzero prime homogeneous ideal and the prime homogeneous ideal is principal. It is easily shown that a graded DVR is a graded valuation ring. (R is a graded valuation ring if for each nonzero homogeneous element $x \in R_H$, either $x \in R$ or $x^{-1} \in R$.) For more on basic properties of graded integral domains, the reader can refer to [5] or [17].

For $f \in R_H$, let $C(f)$ denote the fractional ideal of $R$ generated by the homogeneous components of $f$. Dedekind-Mertens Lemma says that if $f, g \in R$, then $C(f)^{n+1}C(g) = C(f)^nC(fg)$ for some integer $n \geq 1$ [5, Lemma 1.2].
For an ideal $I$ of $R$, let $C(I) = \sum_{f \in I} C(f)$; so $I$ is homogeneous if and only if $C(f) \subseteq I$ for all $f \in I$. A homogeneous ideal of $R$ is called a maximal homogeneous ideal (resp., maximal homogeneous $t$-ideal) of $R$ if it is maximal among proper integral homogeneous ideals (resp., homogeneous $t$-ideals) of $R$. Let $N(H) = \{0 \neq f \in R \mid C(f)_v = R\}$ and $\Omega$ be the set of maximal $t$-ideals $Q$ of $R$ with $Q \cap H \neq \emptyset$. Note that if $Q$ is a maximal $t$-ideal of $R$, then $Q \cap H \neq \emptyset$ if and only if $Q$ is homogeneous [3, Lemma 1.2]. Hence, $\Omega$ is the set of maximal homogeneous $t$-ideals of $R$.

Let $\text{Max}(R_{N(H)}) = \{Q \cap H \neq \emptyset \mid Q \in \Omega\}$ and $\Omega$ be the set of maximal $t$-ideals $Q$ of $R$ with $Q \cap H \neq \emptyset$. Note that if $Q$ is a maximal $t$-ideal of $R$, then $Q \cap H \neq \emptyset$ if and only if $Q$ is homogeneous [3, Lemma 1.2]. Hence, $\Omega$ is the set of maximal homogeneous $t$-ideals of $R$. As in [4], we say that $R$ satisfies property ($\#\#$) if $I \cap N(H) \neq \emptyset$ when $I$ is a nonzero ideal of $R$ with $C(I)_v = R$; equivalently, $\text{Max}(R_{N(H)}) = \{Q \cap H \neq \emptyset \mid Q \in \Omega\}$ [4, Proposition 1.4]. It is known that $R$ satisfies property ($\#\#$) if $R = D[\Gamma]$ or $R$ contains a unit of nonzero degree [4, Example 1.6]. For any undefined definition and notation, see [11].

3. Graded $t$-almost Dedekind domains

Let $\Gamma$ be a nonzero torsionless commutative cancellative monoid with quotient group $\langle \Gamma \rangle$, $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain graded by $\Gamma$ such that $R_{\alpha} \neq \{0\}$ for all $\alpha \in \Gamma$, and $H$ be the saturated multiplicative set of nonzero homogeneous elements of $R$.

**Definition 1.** A graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ is a graded almost Dedekind domain (resp., graded $t$-almost Dedekind domain) if $R_Q$ is a rank-one DVR for all maximal homogeneous ideals (resp., maximal homogeneous $t$-ideals) $Q$ of $R$.

It is clear that $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ is a graded almost Dedekind domain if and only if $R$ is a graded $t$-almost Dedekind domain in which each nonzero maximal homogeneous ideal is a $t$-ideal.

**Proposition 2.** A graded $t$-almost Dedekind domain is a PeMD.

**Proof.** Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$. Then $R$ is a graded PeMD if and only if $R_Q$ is a valuation domain for all nonzero maximal homogeneous $t$-ideals $Q$ of $R$ [7, Lemma 2.7]. (A graded PeMD is a graded integral domain in which each nonzero finitely generated homogeneous ideal is $t$-invertible.) Hence, a graded $t$-almost Dedekind domain $R$ is a graded PeMD, and thus $R$ is a PeMD [1, Theorem 6.4].

However, a PeMD need not be a graded $t$-almost Dedekind domain. (For example, let $V[\mathbb{X}]$ be the polynomial ring over a rank-one nondiscrete valuation domain $V$. Then $V[\mathbb{X}]$ is a $\mathbb{N}_0$-graded integral domain with $\deg(a\mathbb{X}^n) = n$ for $0 \neq a \in V$ and $n \geq 0$, $V[\mathbb{X}]$ is a PeMD, but $V[\mathbb{X}]$ is not a graded $t$-almost
Dedekind domain.) By definitions, we have the following implications:

Almost Dedekind domain $\implies$ Graded almost Dedekind domain

$t$-Almost Dedekind domain $\implies$ Graded $t$-almost Dedekind domain,

while the next examples show that the reverse implications don’t hold.

**Example 3.** (1) Since $\mathbb{Z}$ is a PID, $\mathbb{Z}$ is a $t$-almost Dedekind domain. Thus, $\mathbb{Z}[X]$ is a $t$-almost Dedekind domain (and hence a graded $t$-almost Dedekind domain) [15, Theorems 4.2]. However, note that $(2, X)$ is a maximal homogeneous ideal but $\mathbb{Z}[X]_{(2, X)}$ is not a rank-one DVR. Thus, $\mathbb{Z}[X]$ is neither a graded almost Dedekind domain nor an almost Dedekind domain.

(2) Let $K$ be a field with $\text{char}(K) = p > 0$ and $\mathbb{Q}$ be the additive group of rational numbers. Then $K[\mathbb{Q}]$ is a Prüfer domain [11, Theorem 13.6] but not an almost Dedekind domain [11, Corollary 20.15]; hence $K[\mathbb{Q}]$ is not a $t$-almost Dedekind domain. (Note that a Prüfer domain $D$ is an almost Dedekind domain if and only if $D$ is a $t$-almost Dedekind domain.) Since all of nonzero homogeneous elements of $K[\mathbb{Q}]$ are unit, $K[\mathbb{Q}]$ is a graded almost Dedekind domain (hence a graded $t$-almost Dedekind domain). Thus, a graded almost (resp., graded $t$-almost) Dedekind domain need not be an almost (resp., $t$-almost) Dedekind domain.

The next result is the graded $t$-almost Dedekind domain analog of [9, Theorem 2.14] that an integral domain $D$ is a $t$-almost Dedekind domain if and only if each nonzero ideal of $D$ is a $w$-cancellation ideal. We recall that $I$ is a cancellation (resp., $w$-cancellation) ideal of $D$ if and only if $ID_P$ is principal for all maximal ideals (resp., maximal $t$-ideals) $P$ of $D$ [6, Corollary 2.4].

**Proposition 4.** The following statements are equivalent for $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$.

1. $R$ is a graded $t$-almost Dedekind domain.
2. Each nonzero homogeneous ideal of $R$ is a $w$-cancellation ideal.
3. Each nonzero prime homogeneous ideal of $R$ is a $w$-cancellation ideal.
4. Each prime homogeneous $w$-ideal of $R$ is a $w$-cancellation ideal.
5. Each prime homogeneous $t$-ideal of $R$ is a $w$-cancellation ideal.
6. $RH \setminus M$ is a graded DVR for every maximal homogeneous $t$-ideal $M$ of $R$.

**Proof.** (1) $\Rightarrow$ (2) Let $I$ be a nonzero homogeneous ideal of $R$. If $Q$ is a maximal $t$-ideal of $R$, then $IR_Q = R_Q$ if $Q$ is not homogeneous, and $IR_Q$ is principal if $Q$ is homogeneous by assumption. Thus, $I$ is a $w$-cancellation ideal.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) Clear.

(5) $\Rightarrow$ (1) Let $M$ be a maximal homogeneous $t$-ideal of $R$. Then $M$ is a $w$-cancellation ideal by assumption, and thus $MR_M$ is principal. So it suffices to show that $ht M = 1$. Assume $ht M \geq 2$, and let $Q$ be a prime ideal of $R$
with (0) ⊆ Q ⊆ M. We may assume that Q is a t-ideal. If Q is homogeneous, then Q is a w-cancellation ideal by assumption, and hence QRM is principal, a contradiction because MRM is principal. Hence, Q is not homogeneous, and we may assume that M does not contain a nonzero prime homogeneous ideal.

Let 0 ≠ f = xα1 + ⋯ + xαn ∈ Q with α1 < ⋯ < αn. Then fRM = xα,RM for some αi (because MRM is principal and MRM = √xαiRM for j = 1, ⋯ , n) and \( \frac{1}{x_{\alpha_i}} \) is a unit in RM. Hence, xαi ∈ Q, and so if P is a minimal prime ideal of xα,R such that P ⊆ Q, then P is homogeneous and P ⊆ M, a contradiction. Thus, htM = 1.

(1) ⇔ (6) This follows because RH\M is a graded DVR if and only if RM = (RH\M)MRH\M is a rank-one DVR [8, Theorem 9]. □

The next result is the graded almost Dedekind domain analog of [12, Theorem 36.5] that an integral domain D is an almost Dedekind domain if and only if each nonzero ideal of D is a cancellation ideal.

**Corollary 5.** The following statements are equivalent for \( R = \bigoplus_{\alpha \in \Gamma} R_{\alpha} \).

(1) \( R \) is a graded almost Dedekind domain.
(2) Each nonzero homogeneous ideal of \( R \) is a cancellation ideal.
(3) Each nonzero prime homogeneous ideal of \( R \) is a cancellation ideal.
(4) \( RH\M \) is a graded DVR for every maximal homogeneous ideal \( M \) of \( R \).

**Proof.** (1) ⇒ (2) Let \( I \) be a nonzero homogeneous ideal of \( R \) and \( M \) be a maximal ideal of \( R \). It suffices to show that \( IR_M \) is principal [6, Corollary 2.4]. If \( I \not\subseteq M \), then \( IR_M = R_M \). Next, assume that \( I \subseteq M \), and let \( P = M^* \). Then \( P \) is a nonzero prime homogeneous ideal of \( R \) such that \( I \subseteq P \subseteq M \). Hence, \( RP \) is a rank-one DVR by assumption, and thus \( IR_P = xR_P \) for some \( x \in I \). Clearly, we can choose \( x \) in \( H \) because \( I \) is homogeneous. Let \( a \in I \cap H \). Then \( a = x\frac{f}{g} \) for some \( f \in R \setminus P \) and \( g \in R \), and since \( f \not\in P \), at least one of the homogeneous components of \( f \) is not in \( P \). So if \( \alpha \) is such a homogeneous element, then \( \alpha a = x\beta \) for some \( \beta \in H \), and since \( P = M^* \), \( \alpha \not\in M \). Thus, \( a = x\frac{f}{g} \in xRM \). Again, since \( I \) is homogeneous, \( IR_M \subseteq xRM \), and thus \( IR_M = xRM \).

(2) ⇒ (3) Clear.

(3) ⇒ (1) Note that a cancellation ideal is a w-cancellation t-ideal [6, Corollary 2.5 and Theorem 4.1]; so \( R \) is a graded t-almost Dedekind domain whose nonzero maximal homogeneous ideals are t-ideals by Proposition 4. Thus, \( R \) is a graded almost Dedekind domain.

(1) ⇔ (4) See the proof of (1) ⇔ (6) in Proposition 4. □

It is known that if \( D \) is an almost (resp., a t-almost) Dedekind domain, then \( DS \) is an almost (resp., a t-almost) Dedekind domain for a multiplicative subset \( S \) of \( D \) [12, Corollary 36.3] (resp., [15, Proposition 4.3]). We next give the graded integral domain analog.
Proposition 6. Let \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \) be a graded integral domain and \( S \) be a saturated multiplicative set of nonzero homogeneous elements of \( R \).

1. If \( \Delta = \{ \alpha \in \langle \Gamma \rangle \mid \alpha = \beta - \gamma \text{ for some } \beta, \gamma \in \Gamma \text{ with } S \cap R_\gamma \neq \emptyset \} \), then \( \Delta \) is a monoid with \( \Gamma \subseteq \Delta \subseteq \langle \Gamma \rangle \).
2. \( R_S \) is a \( \Delta \)-graded integral domain.
3. If \( R \) is a graded almost (resp., graded \( t \)-almost) Dedekind domain, then \( R_S \) is a graded almost (resp., graded \( t \)-almost) Dedekind domain.

Proof. (1) and (2) This follows because \( R_S \) is an integral domain.

(3) Let \( M \) be a nonzero maximal homogeneous ideal (resp., nonzero maximal homogeneous \( t \)-ideal) of \( R_S \), and let \( P = M \cap R \). Then \( P \) is a nonzero prime homogeneous ideal (resp., nonzero prime homogeneous \( t \)-ideal) of \( R \) and \( M = PR_S \); hence \( R_M = (R_S)_P R_S = R_P \) is a rank-one DVR by assumption. Thus, \( R_S \) is a graded almost (resp., graded \( t \)-almost) Dedekind domain. \( \square \)

Theorem 7. Let \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \) be a graded integral domain. Then \( R \) is a \( t \)-almost Dedekind domain if and only if \( R_H \) is a \( t \)-almost Dedekind domain.

Proof. Assume that \( R \) is a graded \( t \)-almost Dedekind domain and \( R_H \) is a \( t \)-almost Dedekind domain. Let \( Q \) be a maximal \( t \)-ideal of \( R \). If \( Q \cap H \neq \emptyset \), then \( Q \) is homogeneous [3, Lemma 1.2], and thus \( R_Q \) is a rank-one DVR. Next, if \( Q \cap H = \emptyset \), then \( Q_H \) is a \( t \)-ideal of \( R_H \) because \( R \) is a P$v$MD by Proposition 2. Hence, \( R_Q = (R_H)_{Q_H} \) is a rank-one DVR. Thus, \( R \) is a \( t \)-almost Dedekind domain. The converse is clear. \( \square \)

Corollary 8. Let \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \) be a graded integral domain, and assume that \( R \) satisfies property (\#).

1. \( R \) is a graded \( t \)-almost Dedekind domain if and only if \( R_{N_H} \) is an almost Dedekind domain.
2. \( R \) is a \( t \)-almost Dedekind domain if and only if \( R_{N_H} \) is an almost Dedekind domain and \( R_H \) is a \( t \)-almost Dedekind domain.

Proof. (1) Recall that \( R \) satisfies property (\#) if and only if \( \operatorname{Max}(R_{N_H}) = \{ Q_{N_H} \mid Q \in \Omega \} \) [4, Proposition 1.4]. Thus, the result follows directly from the definition of graded \( t \)-almost Dedekind domains.

(2) This is an immediate consequence of (1) and Theorem 7. \( \square \)

Let \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \) be a graded integral domain. If \( R_H \) is a UFD, then \( R_H \) is a \( t \)-almost Dedekind domain, and hence by Corollary 8(2), we have:

Corollary 9. Let \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \) be a graded integral domain that satisfies property (\#). If \( R_H \) is a UFD, then the following statements are equivalent.

1. \( R \) is a \( t \)-almost Dedekind domain.
2. \( R_{N_H} \) is an almost Dedekind domain.
3. \( R \) is a graded \( t \)-almost Dedekind domain.
We end this section with two examples of graded integral domains $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ such that $R_H$ is a UFD.

**Example 10.** (1) If $\langle \Gamma \rangle = \mathbb{Z}$, then $R_H \cong k[ X, X^{-1} ]$, where $k$ is a field and $X$ is an indeterminate over $k$. Hence, $R_H$ is a PID.

(2) If $\langle \Gamma \rangle$ satisfies the ascending chain condition on its cyclic subgroups, then $R_H$ is a UFD [2, Proposition 3.5].

### 4. $t$-Almost Dedekind domains as monoid domains

Let $\Gamma$ be a nonzero torsionless commutative cancellative monoid with quotient group $\langle \Gamma \rangle$, $D$ be an integral domain with quotient field $K$, and $D[\Gamma]$ be the monoid domain of $\Gamma$ over $D$. Clearly, $D[\Gamma]$ is a $\Gamma$-graded integral domain with $\deg(aX^\alpha) = \alpha$ for $0 \neq a \in D$ and $\alpha \in \Gamma$. Also, $D[\Gamma]$ satisfies property (\#). In this section, we study when $D[\Gamma]$ is a graded $t$-almost Dedekind domain.

Let $A_f$ be the ideal of $D$ generated by the coefficients of $f \in D[\Gamma]$; so $C(f) \subseteq A_fD[\Gamma]$. For a proper prime ideal $S$ of $\Gamma$, let $\Gamma_S = \{ \alpha - s \mid \alpha \in \Gamma$ and $s \in \Gamma \setminus S \}$; then $\Gamma_S$ is a monoid with $\Gamma \subseteq \Gamma_S \subseteq \langle \Gamma \rangle$. We say that $\Gamma$ is a $t$-almost Dedekind monoid if $\Gamma_S$ is a principal monoid for all maximal $t$-ideals $S$ of $\Gamma$. Clearly, torsionfree abelian groups and unique factorization monoids are $t$-almost Dedekind monoids. (The $t$-operation on $\Gamma$ is defined by the same way as in the case of integral domains. For more on definitions related with monoids, see [14].)

**Lemma 11.** Let $D[\Gamma]$ be the monoid domain of $\Gamma$ over $D$ and $D(\Gamma) = \{ \frac{f}{g} \mid f, g \in D[\Gamma], g \neq 0, \text{ and } A_g = D \}$.

1) If $D$ is a valuation domain, then $D(\Gamma)$ is a valuation domain whose value group is the same as that of $D$.

2) If $\Gamma$ is a valuation monoid with maximal ideal $S$, then $D[\Gamma]|_{D[S]}$ is a valuation domain whose value group is the same as that of $S$.

**Proof.** (1) Let $f = a_1X^{\alpha_1} + \cdots + a_nX^{\alpha_n} \in D[\Gamma]$ with $\alpha_1 < \cdots < \alpha_n$. Since $D$ is a valuation domain, $A_f = a_iD$ for some $i$, and thus $fD(\Gamma) = a_iD(\Gamma)$. Thus, $D(\Gamma)$ is a valuation domain whose value group is the same as that of $D$.

(2) Let $f$ be as in (1). Then, since $\Gamma$ is a valuation monoid, $\bigcup_{i=1}^n (\alpha_i + \Gamma) = \alpha_i + \Gamma$ for some $i$, and hence $fD[\Gamma]|_{D[S]} = X^{\alpha_i}D[\Gamma]|_{D[S]}$. Thus, $D[\Gamma]|_{D[S]}$ is a valuation domain whose value group is the same as that of $S$.

**Lemma 12.** Let $D[\Gamma]$ be the monoid domain of $\Gamma$ over $D$ and $H$ be the set of nonzero homogeneous elements of $D[\Gamma]$. Then

\[
t-Max(D[\Gamma]) = \{ P[\Gamma] \mid P \in t-Max(D) \} \cup \{ D[S] \mid S \in t-Max(\Gamma) \} \cup \{ Q \in t-Max(D[\Gamma]) \mid Q \cap H = \emptyset \}.
\]

**Proof.** ($\subseteq$) [3, Lemma 1.2 and Corollary 1.3]. ($\supseteq$) Let $P \in t-Max(D)$ and $S \in t-Max(\Gamma)$. Then $P[\Gamma]$ and $D[S]$ are both $t$-ideals of $D[\Gamma]$ [10, Corollary 2.4], and thus $P[\Gamma]$ and $D[S]$ are maximal $t$-ideals [3, Corollary 1.3].
We next give the main result of this section.

**Theorem 13.** Let $D[\Gamma]$ be the monoid domain of $\Gamma$ over $D$. Then the following statements are equivalent.

1. $D[\Gamma]$ is a graded $t$-almost Dedekind domain.
2. $D$ is a $t$-almost Dedekind domain and $\Gamma$ is a $t$-almost Dedekind monoid.
3. $D[\Gamma]_{N(H)}$ is an almost Dedekind domain.

**Proof.** (1) $\Rightarrow$ (2) Let $P$ be a maximal $t$-ideal of $D$. Then $P[\Gamma]$ is a maximal $t$-ideal of $D[\Gamma]$, and hence $D_P(\Gamma) = D[\Gamma]_{[P]_1}$ is a rank-one DVR. Note that $D_P(\Gamma) \cap K = D_P$; thus $D_P$ is a rank-one DVR [12, Theorem 19.16]. Next, let $S$ be a maximal $t$-ideal of $\Gamma$. Then, by Lemma 12, $D[S]$ is a maximal $t$-ideal of $D[\Gamma]$, and hence $D[D[S]]_P = D[\Gamma]_{D[S]}$ is a rank-one DVR. Note that $\{ \beta \in (\Gamma) \mid X^\beta \in D[\Gamma]_{D[S]} \} = \Gamma_S$, Thus, $\Gamma_S$ is a rank-one discrete valuation monoid.

(2) $\Rightarrow$ (1) Let $Q$ be a nonzero maximal homogeneous $t$-ideal of $D[\Gamma]$. If $Q \cap D \neq (0)$, then $Q \cap D$ is a maximal $t$-ideal of $D$ and $Q = (Q \cap D)[\Gamma]$ by Lemma 12. Hence, $D[\Gamma]_Q = D[\Gamma]_{[Q \cap D]_1} = D_{Q \cap D}(\Gamma)$, and since $D_{Q \cap D}$ is a rank-one DVR by (2), $D_{Q \cap D}(\Gamma)$ is a rank-one DVR by Lemma 11(1). Next, assume that $Q \cap D = (0)$, and let $S = \{ \alpha \in (\Gamma) \mid X^\alpha \in Q \}$. Then $S \neq \emptyset$, and hence by Lemma 12, $S$ is a maximal $t$-ideal of $\Gamma$ and $Q = D[S]$; so $\Gamma_S$ is a rank-one discrete valuation monoid. Thus, $D[\Gamma]_Q = D[\Gamma]_{D[S]}$ is a rank-one DVR by Lemma 11(2).

(1) $\Leftrightarrow$ (3) This follows directly from Corollary 8(1) because $D[\Gamma]$ satisfies property (#). □

**Corollary 14.** Let $D[\Gamma]$ be the monoid domain of $\Gamma$ over $D$, and assume that $\Gamma$ is a group. Then $D[\Gamma]$ is a graded $t$-almost Dedekind domain if and only if $D$ is a $t$-almost Dedekind domain.

**Proof.** Clearly, a torsionfree abelian group is a $t$-almost Dedekind monoid. Hence, the result follows directly from Theorem 13. □

**Corollary 15.** Let $D[\Gamma]$ be the monoid domain of $\Gamma$ over $D$. Then the following statements are equivalent.

1. $D[\Gamma]$ is a $t$-almost Dedekind domain.
2. (i) $D$ is a $t$-almost Dedekind domain, (ii) $\Gamma$ is a $t$-almost Dedekind monoid, and (iii) $K[[\Gamma]]$ is a $t$-almost Dedekind domain.
3. $D[\Gamma]_{N(H)}$ is an almost Dedekind domain and $K[[\Gamma]]$ is a $t$-almost Dedekind domain.
4. $D[\Gamma]$ is a graded $t$-almost Dedekind domain and $K[[\Gamma]]$ is a $t$-almost Dedekind domain.

**Proof.** (1) $\Rightarrow$ (2) Since a $t$-almost Dedekind domain is a graded $t$-almost Dedekind domain, by Theorem 13, (i) and (ii) are satisfied. For (iii), note that $K[[\Gamma]] = D[\Gamma]_{N(H)}$. Thus, $K[[\Gamma]]$ is a $t$-almost Dedekind domain by Corollary 8.

(2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (1) Theorem 13 and Corollary 8. □
Corollary 16. Assume that \((\Gamma)\) satisfies the ascending chain condition on its cyclic subgroups (e.g., \((\Gamma) \cong \mathbb{Z}\)). Then the following statements are equivalent.

1. \(D[\Gamma]\) is a \(t\)-almost Dedekind domain.
2. \(D\) is a \(t\)-almost Dedekind domain and \(\Gamma\) is a \(t\)-almost Dedekind monoid.
3. \(D[\Gamma]_{N(H)}\) is an almost Dedekind domain.
4. \(D[\Gamma]\) is a graded \(t\)-almost Dedekind domain.

\textbf{Proof.} By Example 10, \(K[[\Gamma]]\) is a UFD. Thus, the result follows directly from Corollary 15. \(\square\)

Corollary 17. Let \(D\) be a Krull domain and \(\mathbb{Q}\) be the additive group of rational numbers.

1. If \(\text{char}(D) = 0\), then \(D[\mathbb{Q}]\) is a \(t\)-almost Dedekind domain.
2. If \(\text{char}(D) \neq 0\), then \(D[\mathbb{Q}]\) is a graded \(t\)-almost Dedekind domain but not a \(t\)-almost Dedekind domain.

\textbf{Proof.} Let \(K\) be the quotient field of \(D\). Then \(K[[\mathbb{Q}]]\) is a \(t\)-almost Dedekind domain if and only if \(K[\mathbb{Q}]\) is an almost Dedekind domain, if and only if \(\text{char}(D) = 0\) [11, Theorem 13.6 and Corollary 20.15]. Thus, the result follows directly from Theorem 13 and Corollary 15. \(\square\)

Let \(\{X_\alpha\}\) be a nonempty set of indeterminates over an integral domain \(D\) and \(D[\{X_\alpha\}]\) be the polynomial ring over \(D\). Let \(N_\alpha\) be the additive monoid of nonnegative integers for all \(\alpha\), and let \(\Gamma = \bigoplus_\alpha N_\alpha\). Clearly, \(\Gamma\) is a unique factorization monoid and \(D[\{X_\alpha\}] = D[\Gamma]\). Also, if \(N_\alpha = \{f \in D[\{X_\alpha\}] \mid (Af)_\alpha = D\}\), then \(N_\alpha\) is a saturated multiplicative subset of \(D[\{X_\alpha\}]\).

Corollary 18. Let \(D[\{X_\alpha\}]\) be the polynomial ring over \(D\). Then the following statements are equivalent.

1. \(D\) is a \(t\)-almost Dedekind domain.
2. \(D[\{X_\alpha\}]\) is a \(t\)-almost Dedekind domain.
3. \(D[\{X_\alpha\}]\) is a graded \(t\)-almost Dedekind domain.
4. \(D[\{X_\alpha, X_\alpha^{-1}\}]\) is a \(t\)-almost Dedekind domain.
5. \(D[\{X_\alpha, X_\alpha^{-1}\}]\) is a graded \(t\)-almost Dedekind domain.
6. \(D[\{X_\alpha\}]_{N_\alpha}\) is an almost Dedekind domain.

\textbf{Proof.} This result follows from Corollary 15 and the following observation: Let \(N_\alpha = \mathbb{N}_0\) be the additive monoid of nonnegative integers and \(\Gamma = \bigoplus_\alpha N_\alpha\). Then \(D[\Gamma] = D[\{X_\alpha\}], D[\Gamma'] = D[\{X_\alpha, X_\alpha^{-1}\}]\), \(\Gamma\) is a unique factorization monoid, \(K[\Gamma] = K[\{X_\alpha, X_\alpha^{-1}\}]\) is a UFD, and \(D[\{X_\alpha\}]_{N_\alpha} = R_{N(H)}\) where \(R = D[\Gamma']\). \(\square\)

Corollary 19. Let \(G_i\) be either \(\mathbb{Z}\) or \(\mathbb{Q}\) for \(i = 1, 2\) and \(G = G_1 \bigoplus G_2\). If \(\text{char}(D) = 0\), then \(D\) is a \(t\)-almost Dedekind domain if and only if \(D[G]\) is a \(t\)-almost Dedekind domain.
Proof. By Corollary 15, it suffices to show that $K[G]$ is a $t$-almost Dedekind domain. If $G_1 = G_2 = \mathbb{Z}$, then $K[G] \cong K[X, X^{-1}, Y, Y^{-1}]$, a Laurent polynomial ring, and thus $K[G]$ is a $t$-almost Dedekind domain by Corollary 18. Next, assume that $G_1 = \mathbb{Q}$ and $G_2 = \mathbb{Z}$ or $\mathbb{Q}$. By [11, Theorem 7.1], $K[G] = K[\mathbb{Q} \oplus G_2] \cong (K[\mathbb{Q}])(G_2)$. Hence, if $R = K[\mathbb{Q}]$ and $L$ is the quotient field of $R$, then $R$ and $L[G_2]$ are $t$-almost Dedekind domains by Corollaries 17(1) and 18. Thus, by Corollary 15, $K[G]$ is a $t$-almost Dedekind domain. □

By Corollary 15, if $K[\langle \Gamma \rangle]$ is a $t$-almost Dedekind domain, then $D[\Gamma]$ is a $t$-almost Dedekind domain if and only if $D$ and $\Gamma$ are both $t$-almost Dedekind.

We end this article with the following question.

Question 20. When $K[\langle \Gamma \rangle]$ is a $t$-almost Dedekind domain ?

Acknowledgements. We would like to thank the referee for several valuable suggestions. D. Y. Oh was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2014R1A1A2054132).

References


**Gyu Whan Chang**  
Department of Mathematics Education  
Incheon National University  
Incheon 22012, Korea  
E-mail address: whan@inu.ac.kr

**Dong Yeol Oh**  
Department of Mathematics Education  
Chosun University  
Gwangju 61452, Korea  
E-mail address: dyoh@chosun.ac.kr, dongyeol70@gmail.com