FLAG-TRANSITIVE POINT-PRIMITIVE SYMMETRIC DESIGNS AND THREE DIMENSIONAL PROJECTIVE SPECIAL UNITARY GROUPS

ASHRAF DANESHKHAH AND SHEYDA ZANG ZARIN

Abstract. The main aim of this article is to study symmetric \((v, k, \lambda)\) designs admitting a flag-transitive and point-primitive automorphism group \(G\) whose socle is \(\text{PSU}(3, q)\). We indeed show that such designs must be complete.

1. Introduction

A symmetric \((v, k, \lambda)\) design is an incidence structure \(\mathcal{D} = (\mathcal{V}, \mathcal{B})\) consisting of a set \(\mathcal{V}\) of \(v\) points and a set \(\mathcal{B}\) of \(v\) blocks such that every point is incident with exactly \(k\) blocks, and every pair of blocks is incident with exactly \(\lambda\) points. A nontrivial symmetric design is one in which \(2 < k < v - 1\). A symmetric \((v, v - 1, v - 2)\) design is called complete. A flag of \(\mathcal{D}\) is an incident pair \((\alpha, B)\) where \(\alpha\) and \(B\) are a point and a block of \(\mathcal{D}\), respectively. An automorphism of a symmetric design \(\mathcal{D}\) is a permutation of the points permuting the blocks and preserving the incidence relation. An automorphism group \(G\) of \(\mathcal{D}\) is called flag-transitive if it is transitive on the set of flags of \(\mathcal{D}\). If \(G\) is primitive on the point set \(\mathcal{V}\), then \(G\) is said to be point-primitive. A group \(G\) is said to be almost simple with socle \(X\) if \(X \triangleleft G \leq \text{Aut}(X)\) where \(X\) is a nonabelian simple group.

Further notation and definitions in both design theory and group theory are standard and can be found, for example, in [5, 10, 13].

Symmetric designs with \(\lambda\) small have been of most interest. Kantor [11] classified flag-transitive symmetric \((v, k, 1)\) designs (projective planes) of order \(n\) and showed that either \(\mathcal{D}\) is a Desarguesian projective plane and \(\text{PSL}(3, n) \triangleleft G\), or \(G\) is a sharply flag-transitive Frobenius group of odd order \((n^2 + n + 1)(n+1)\), where \(n\) is even and \(n^2 + n + 1\) is prime. Regueiro [17] gave a complete classification of biplanes \((\lambda = 2)\) with flag-transitive automorphism groups apart from those admitting a 1-dimensional affine group (see also [18, 19, 20, 21]). Zhou and Dong studied nontrivial symmetric \((v, k, 3)\) designs (triplanes) and
proved that if $D$ is a nontrivial symmetric $(v,k,3)$ design with a flag-transitive and point-primitive automorphism group $G$, then $D$ has parameters $(11, 6, 3)$, $(15, 7, 3)$, $(45, 12, 3)$ or $G$ is a subgroup of $A\Gamma L(1,q)$ where $q = p^m$ with $p \geq 5$ prime $[7, 27, 28, 29, 30]$. Nontrivial symmetric $(v,k,4)$ designs admitting flag-transitive and point-primitive almost simple automorphism group whose socle is an alternating group or $PSL(2,q)$ have also been investigated $[6, 31]$. It is known $[24]$ that if a nontrivial symmetric $(v,k,\lambda)$ design $D$ with $\lambda \leq 100$ admitting a flag-transitive, point-primitive automorphism group $G$ must be an affine or almost simple group. Therefore, it is interesting to study such designs whose socle is of almost simple type or affine type.

In this paper, however, we are interested in large $\lambda$. In this direction, it is recently shown in $[2]$ that there are only four possible symmetric $(v,k,\lambda)$ designs admitting a flag-transitive and point-primitive automorphism group $G$ satisfying $X \triangleleft G \leq \text{Aut}(X)$ where $X = PSL(2,q)$, see also $[26]$. In the case where $X$ is a sporadic simple group, there also exist four possible parameters (see $[25]$). This study for $X := PSL(3,q)$ gives rise to one nontrivial design (up to isomorphism) which is a Desarguesian projective plane $PG(2,q)$ and $PSL(3,q) \leq G$ (see $[1]$). This paper is devoted to studying symmetric designs admitting a flag-transitive and point-primitive almost simple automorphism group $G$ whose socle is $X := PSU(3,q)$. Indeed, the situation for $PSU(3,q)$ is rather different and trivial design is the only symmetric design admitting such automorphism group $G$. We prove Theorem 1.1 below in Section 3.

**Theorem 1.1.** Let $D$ be a symmetric $(v,k,\lambda)$ design, and let $G$ be an automorphisms group of $D$ with socle $X = PSU(3,q)$. If $G$ is flag-transitive and point-primitive, then $D$ is a complete design.

In order to prove Theorem 1.1, we need to know the complete list $[3, Table 8.5]$ of maximal subgroups of almost simple groups with socle $PSU(3,q)$ (see Lemma 2.4 below). We frequently apply Lemma 2.1 below as a key tool and use GAP $[8]$ for computations.

In the case where $G$ is imprimitive, Praeger and Zhou $[22]$ studied point-imprimitive symmetric $(v,k,\lambda)$ designs, and determined all such possible designs for $\lambda \leq 10$. This motivates Praeger and Reichard $[14]$ to classify flag-transitive symmetric $(96,20,4)$ designs. As a result of their work, the only examples for flag-transitive, point-imprimitive symmetric $(v,k,4)$ designs are $(15,8,4)$ and $(96,20,4)$ designs. In a recent study of imprimitive flag-transitive designs $[4]$, Cameron and Praeger gave a construction of a family of designs with a specified point-partition, and determine the subgroup of automorphisms leaving invariant the point-partition. They gave necessary and sufficient conditions for a design in the family to possess a flag-transitive group of automorphisms preserving the specified point-partition. Consequently, they gave examples of flag-transitive designs in the family, including a new symmetric $(1480,336,80)$ design with automorphism group $2^{12} : ((3 \cdot M_{22}) : 2)$, and a construction of
one of the families of the symmetric designs exhibiting a flag-transitive, point-imprimitive automorphism group.

2. Preliminaries

In this section, we state some useful facts in both design theory and group theory. The following Lemma 2.1 is a key result in our approach to prove Theorem 1.1:

**Lemma 2.1.** Let \( D \) be a symmetric \((v, k, \lambda)\) design, and let \( G \) be a flag-transitive automorphism group of \( D \). If \( \alpha \) is a point in \( V \) and \( M := G_\alpha \), then

(a) \( k(k-1) = \lambda(v-1) \);
(b) \( k \mid |M| \) and \( \lambda v < k^2 \);
(c) \( k \mid \gcd(\lambda(v-1), |M|) \);
(d) \( k \mid \lambda d \), for all subdegrees \( d \) of \( G \).

**Proof.** The proof follows from [2, Lemma 2.1], see also [31, Lemma 2.2]. \( \square \)

Recall that a group \( G \) is called almost simple if \( X \trianglelefteq G \leq \text{Aut}(X) \) where \( X \) is a (nonabelian) simple group. If \( M \) is a maximal subgroup of an almost simple group \( G \) with socle \( X \), then \( G = MX \), and since we may identify \( X \) with \( \text{Inn}(X) \), the group of inner automorphisms of \( X \), we also conclude that \( |M| \) divides \( |\text{Out}(X)| \cdot |X \cap M| \). This implies the following elementary and useful fact:

**Lemma 2.2.** Let \( G \) be an almost simple group with socle \( X \), and let \( M \) be maximal in \( G \) not containing \( X \). Then

(a) \( G = MX \);
(b) \( |M| \) divides \( |\text{Out}(X)| \cdot |X \cap M| \).

**Lemma 2.3.** Suppose that \( D \) is a symmetric \((v, k, \lambda)\) design admitting a flag-transitive and point-primitive almost simple automorphism group \( G \) with socle \( X \) of Lie type in odd characteristic \( p \). Suppose also that the point-stabiliser \( G_\alpha \), not containing \( X \), is not a parabolic subgroup of \( G \). Then \( \gcd(p, v-1) = 1 \).

**Proof.** Note that \( G_\alpha \) is maximal in \( G \), then by Tits’ Lemma [23, (1.6)], \( p \) divides \( |G : G_\alpha| = v \), and so \( \gcd(p, v-1) = 1 \). \( \square \)

If a group \( G \) acts primitively on a set \( V \) and \( \alpha \in V \) (with \( |V| \geq 2 \)), then the point-stabiliser \( G_\alpha \) is maximal in \( G \) [5, Corollary 1.5A]. Therefore, in our study, we need a list of all maximal subgroups of almost simple group \( G \) with socle \( X := \text{PSU}(3, q) \). Note that if \( M \) is a maximal subgroup of \( G \), then \( M_0 := M \cap X \) is not necessarily maximal in \( X \) in which case \( M \) is called a novelty. By [3, Tables 8.5 and 8.6], the complete list of maximal subgroups of an almost simple group \( G \) with socle \( \text{PSU}(3, q) \) are known, and in this case, there arose only four novelties, see [3, 9, 12, 16].
Lemma 2.4 ([3, Tables 8.5 and 8.6]). Let $G$ be a group such that $X = \text{PSU}(3, q) \leq G \leq \text{Aut}(X)$, and let $M$ be a maximal subgroup of $G$ not containing $X$. Then $M_0 = X \cap M$, is (isomorphic to) one of the following subgroups:

(a) $[q]^{1+2} : (q^2 - 1)$;
(b) $\text{GU}(2, q)$;
(c) $(q^2 - q + 1) : 3$ with $q \neq 3, 5$ (novelty if $q = 5$);
(d) $(q + 1)^2 : S_3$ (novelty if $q = 5$);
(e) $\text{SO}_3(q)$ with $q \geq 7$, $q$ odd;
(f) $\text{SU}(3, q_0) \cdot \text{gcd}(3, \frac{q_0 + 1}{3q_0 + 1})$, where $q = q_0^r$, $r$ odd and prime;
(g) $3^2 : Q_8$ with $p = q \equiv 2 \pmod{3}$, $q \geq 11$ (novelty if $q = 5$);
(h) $\text{PSL}(2, 7)$ with $q \neq 5$, $p = q \equiv 3, 5, 6 \pmod{7}$ (novelty if $q = 5$);
(i) $\text{A}_6$ with $p = q \equiv 11, 14 \pmod{15}$;
(j) $\text{A}_6 \cdot 2_3$ with $q = 5$;
(k) $\text{A}_7$ with $q = 5$.

3. Proof of Theorem 1.1

In this section, suppose that $D$ is a symmetric $(v, k, \lambda)$ design and $G$ is an almost simple automorphism group $G$ with simple socle $X := \text{PSU}(3, q)$, where $q = p^f$ ($p$ prime), that is to say, $X < G \leq \text{Aut}(X)$. Suppose also that $V$ is the underlying vector space of $X$ over the finite field $F_q^2$.

Let now $G$ be a flag-transitive and point-primitive automorphism group of $D$. Then the point-stabiliser $M := G_\alpha$ is maximal in $G$ [5, Corollary 1.5A]. Set $M_0 := X \cap M$. So $M_0$ is (isomorphic to) one of the subgroups listed in Lemma 2.4(a)-(k). Moreover, by Lemma 2.2,

$$v = \frac{|X|}{|M_0|} = \frac{q^3(q^2 - 1)(q^3 + 1)}{\text{gcd}(3, q + 1) \cdot |M_0|}. \tag{3.1}$$

Note that $|\text{Out}(X)| = 2f \cdot \text{gcd}(3, q + 1)$. Therefore, by Lemma 2.1(b) and Lemma 2.2(b),

$$k | 2f \cdot \text{gcd}(3, q + 1) \cdot |M_0|. \tag{3.2}$$

In what follows, considering possible structure for the subgroup $M_0$ as in Lemma 2.4(b)-(k), we prove that none of these cases could occur.

Lemma 3.1. The subgroup $M_0$ cannot be $\text{GU}(2, q)$.

Proof. Let $V$ be the underlying vector space of $X = \text{PSU}(3, q)$ over the finite field $F_q^2$. By (3.1), we have that $v = q^2(q^2 - q + 1)$. It follows from [15, Lemma 3.9] and Lemma 2.1(d) that $k$ divides $\lambda(q^2 - 1)(q + 1)$, and by Lemma 2.1(c), $k$ divides $\lambda \cdot \text{gcd}(q(q^2 - 1)(q + 1), q^4 - q^3 + q^2 - 1) = 16\lambda(q - 1)$, so $k | 16\lambda(q - 1)$.

Let now $m$ be a positive integer such that $mk = 16\lambda(q - 1)$. By Lemma 2.1(a), $k(k - 1) = \lambda(v - 1)$, and so

$$k = \frac{m(q^3 + q + 1)}{16} + 1. \tag{3.3}$$
Moreover, $k \mid 6fq(q^2 - 1)(q + 1)$, and by (3.3), we have $k \mid 6f(q + 1)(m(q^3 + q + 1) + 16)$. Thus
\begin{equation}
(3.4) \quad k \mid 6fm(2q^2 + 3q + 1) + 96f(q + 1),
\end{equation}
and so $16k < 96fm(2q^2 + 3q + 1) + 16 \cdot 96f(q + 1)$. By (3.3), we have that $m(q^3 + q + 1) + 16 < 96fm(2q^2 + 19q + 17)$. Therefore $q/2 < 96f + 5$. This inequality holds when
\begin{itemize}
  \item $p = 2$, $f \leq 6$;
  \item $p = 3$, $f \leq 3$;
  \item $p = 5$, $f \leq 2$;
  \item $p \in \{11, 13, 17, 19\}$, $f \leq 1$.
\end{itemize}
\begin{equation}
(3.5)
\end{equation}

The possible values of $k$ and $v$ are listed in Table 1 below. For such parameters $k$ and $v$ as in Table 1, by straightforward calculation, we observe that Lemma 2.1(a) does not hold, which is a contradiction. □

**Lemma 3.2.** The subgroup $M_0$ cannot be $(q^2 - q + 1) : 3$.

**Proof.** Here, by (3.1), we have $v = q^3(q^2 - 1)(q + 1)/3$. Note that $|\text{Out}(X)| = 2 \cdot \gcd(3, q + 1) \cdot f$. Then by (3.2), we conclude that $k$ divides $6f(q^2 - q + 1)$. By [20, 30], we may assume that $\lambda \geq 4$, and so Lemma 2.1(b) yields
\begin{equation}
\begin{split}
\frac{4q^3(q^2 - 1)(q + 1)}{3} \leq \lambda v < k^2 \leq 36f^2(q^2 - q + 1)^2. 
\end{split}
\end{equation}

Then $q^3(q^2 - 1)(q + 1) < 27f^2(q^2 - q + 1)^2$. It is easy to observe that $q^2 < \frac{q^3(q^2 - 1)(q + 1)}{(q^2 - q + 1)^2}$ for $q \geq 2$. Then $q^2 < 27f^2$. This inequality holds when
\begin{itemize}
  \item $p = 2$, $f \leq 4$;
  \item $p = 3$, $f \leq 2$;
  \item $p = 5$, $f = 1$.
\end{itemize}
\begin{equation}
(3.6)
\end{equation}

Recall that $k$ is a divisor of $6f(q^2 - q + 1)$. Then, for each $q = p^f$ with $p$ and $f$ as in (3.6), the possible values of $k$ and $v$ are listed in Table 2 below. It is a contradiction as for each $k$ and $v$ as in Table 2, $v - 1$ does not divide $k(k - 1)$. □

**Lemma 3.3.** The subgroup $M_0$ cannot be $(q + 1)^2 : S_3$.

**Proof.** The argument here is the same as proof of Lemma 3.2. By (3.1), we have $v = q^3(q - 1)(q^2 - q + 1)/6$, and since $|\text{Out}(X)| = 2f \cdot \gcd(3, q + 1)$, it follows from (3.2) that $k$ divides $12f(q + 1)^2$. As $\lambda$ is at least 4, by Lemma 2.1(b), we must have
\begin{equation}
\begin{split}
\frac{4q^3(q - 1)(q^2 - q + 1)}{6} \leq \lambda v < k^2 \leq 144f^2(q + 1)^4. 
\end{split}
\end{equation}
Table 1. Possible value for $k$ and $v$ when $p$ and $f$ are as in (3.5).

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This implies that $q^3(q-1)(q^2-q+1) < 216f^2(q+1)^4$, and since $q+1 \leq (3/2)q$, we have

$$q^3(q-1)(q^2-q+1) < 216f^2(q+1)^4 \leq \left(\frac{3}{2}\right)f^2q^4.$$ 

It follows that

$$(q-1)(q^2-q+1) < \left(\frac{3}{2}\right)f^2q.$$
Table 2. Possible value for $k$ and $v$ when $q = p^f$ with $p$ and $f$ as in (3.6).

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and so $q^2 - 2q + 2 - 1/q < (3^7/2)f^2$. Therefore, $(q - 1)^2 < (3^7/2)f^2$, and since $(16/81)(q - 1)^2 < q^3(q - 1)(q^2 - q + 1)/(q + 1)^4$, we must have $q = p^f < (27\sqrt{6}/2)f + 1$. This is true only when

$$p = 2,\quad f \leq 8;$$

$$p = 3,\quad f \leq 4;$$

$$p = 5, 7,\quad f \leq 2;$$

$$11 \leq p \leq 31,\quad f = 1.$$ (3.7)

Table 3. Possible value for $k$ and $v$ when $q = p^f$ is as in (3.7).

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Since $k$ is a divisor of $12f(q+1)^2$, for each $q = p^f$ with $p$ and $f$ as in (3.7), the possible values of $k$ and $v$ are listed in Table 3. This leads us to a contradiction as, for parameters $k$ and $v$ as in Table 3, the fraction $k(k - 1)/(v - 1)$ is not integer.

Lemma 3.4. The subgroup $M_0$ cannot be $SO_3(q)$ with $q \geq 7$, odd.

Proof. By (3.1), we have that $v = q^2(q^2 + 1)/d$ with $d = \gcd(3, q + 1)$. It follows from (3.2) that $k$ divides $2df(q^2 - 1)$, and so $k$ is a divisor of $6f(q^2 - 1)$.

Moreover, Lemma 2.1(a) implies that $k$ divides $\lambda(v - 1)$. Note by Lemma 2.3 that $v - 1$ is coprime to $q$. Thus $k$ divides $6\lambda f \gcd(q^2 - 1, v - 1)$. Let $d = 1$, then $v - 1 = q^2 + q^2 - 1$, and so $\gcd(q^2 - 1, v - 1) = \gcd(q^2 - 1, q) = 1$. Thus, in this case $k$ divides $6\lambda f$. Let now $d = 3$. Then $v - 1 = (q^2 + q^2 - 3)/3$, and so $\gcd(q^2 - 1, 3(v - 1)) = \gcd(q^2 - 1, q - 2) = \gcd(q - 2, 3) = 1$ or 3. Since $k$
Lemma 2.1(b), we must have \( c \) by [20, 30]. Moreover, note by (3.2) that (3.10)

\[
\text{Proof.} \quad k = \frac{m(q^5 + q^2 - d)}{18df} + 1.
\]

Since \( d = 1 \) or 3, we have by (3.2) that \( k | 6fq(q^2 - 1) \). Then (3.8) yields \( m(q^5 + q^2 - d) \leq 108df^2q(q^2 - 1) \). Since also \( m \geq 1 \) and \( d \leq 3 \), we have that \( q^2 < q^5 + q^2 - 3/q(q^2 - 1) \leq 324f^2 \). This inequality only holds for

\[
q \in \{7, 9, 11, 13, 17, 25, 27\}.
\]

For these values of \( q \), since \( k \) divides \( 2dfq(q^2 - 1) \), the possible values of \( k \) can be found as in Table 4. This leads us to a contradiction as for each value of \( v \) and \( k \) as in Table 4, the fraction \( k/(k - 1)/(v - 1) \) is not integer. □

**Lemma 3.5.** The subgroup \( M_0 \) cannot be \( \text{SU}(3, q_0) \cdot c \), where \( q = q_0^r \), \( r \) odd prime and \( c := \gcd\left(3, \frac{q+1}{q_0+1}\right) \).

**Proof.** In this case, \( |M_0| = c \cdot q_0^3(q_0^2 - 1)(q_0^3 + 1)/\gcd(3, q + 1) \). It follows from (3.1) that

\[
v = \frac{1}{c} \cdot \frac{q_0^{3r}(q_0^{2r} - 1)(q_0^{3r} + 1)}{q_0^3(q_0^2 - 1)(q_0^3 + 1)}.
\]

Note by (3.2) that \( k \) divides \( 6f^2q_0^3(q_0^2 - 1)(q_0^3 + 1) \). We may assume that \( \lambda \geq 4 \) by [20, 30]. Moreover, \( c \in \{1, 3\} \), and \( f^2 \leq q_0^3 \) as \( q = q_0^r \). Since \( \lambda v < k^2 \) by Lemma 2.1(b), we must have

\[
\frac{4q_0^{3r}(q_0^{2r} - 1)(q_0^{3r} + 1)}{3q_0^3(q_0^2 - 1)(q_0^3 + 1)} \leq \lambda v < k^2 \leq 36f^2q_0^3(q_0^2 - 1)^2(q_0^3 + 1)^2
\]

\[
\leq 36q_0^{6+2r}(q_0^2 - 2)^2(q_0^3 + 1)^2.
\]

Therefore \( q_0^{3r}(q_0^{2r} - 1)(q_0^{3r} + 1) < 27q_0^{9+r}(q_0^2 - 1)^3(q_0^3 + 1)^3 \). Since \( q_0^{sr-1} \leq q_0^{3r}(q_0^{2r} - 1)(q_0^{3r} + 1) \) and \( q_0^{9+r}(q_0^2 - 1)^3(q_0^3 - 1)^3 \leq q_0^{24+r} \), we have that \( q_0^{sr-1} < \)
27, \(q_0^{24+r} \), and so \(q_0^{7r-25} < 27 \). But \(q_0 \geq 2 \) and \( r \) is odd. Then \( r = 3 \). Therefore, by (3.10), we have that

\[
(3.11) \quad v = \frac{1}{c} \cdot \frac{q_0^6(q_0^3 - 1)(q_0^3 + 1)}{q_0^3 - 1},
\]

where \( c := \gcd \left( \frac{3}{q_0^3 + 1} \right) \). By (3.2), \( k \) divides \( 6f(q_0^3 - 1)(q_0^3 + 1) \). It follows from Lemma 2.1(b), that

\[
\lambda \frac{q_0^6(q_0^3 - 1)(q_0^3 + 1)}{c(q_0^3 - 1)} < k^2 \leq 36f^2q_0^6(q_0^3 - 1)^2(q_0^3 + 1)^2.
\]

Therefore

\[
(3.12) \quad \lambda < 36cf^2 \frac{(q_0^3 - 1)^3(q_0^3 + 1)^2}{(q_0^3 - 1)(q_0^3 + 1)} \leq 108f^2.
\]

We now observe by Lemma 2.3 that \( v - 1 \) and \( q \) are coprime, and since \( k \) divides both \( 6f(q_0^3 - 1)(q_0^3 + 1) \) and \( \lambda(v - 1) \), again by Lemma 2.1(b), we must have

\[
\lambda \frac{q_0^6(q_0^3 - 1)(q_0^3 + 1)}{c(q_0^3 - 1)} < k^2 \leq 36\lambda^2f^2(q_0^3 - 1)^2(q_0^3 + 1)^2,
\]

and so

\[
(3.13) \quad \frac{q_0^6(q_0^3 - 1)(q_0^3 + 1)}{(q_0^3 - 1)^2(q_0^3 + 1)^2} < 36\lambda^2f^2c.
\]

Since \( c \leq 3 \) and \( \lambda \leq 108f^2 \) by (3.12), it follows that \( q_0^6 < 23328f^4 \). Since also \( q_0 \) is at least 2, we conclude that \( 2^{2f} < 23328 \cdot f^4 \), and this holds for \( f \leq 15 \). Then \( q_0 \leq 32 \). Considering (3.13), \( q_0 \) is one of the numbers: 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 25, 27, 29, 31, 32, 1923. Inspecting each such value of \( q_0 \), we observe by (3.10) that \( v = 896 \) is the only possible value when \( q_0 = 2 \). In this case, \( k \) divides \( 6f(q_0^3 - 1)(q_0^3 + 1) = 1296 \). But by straightforward calculation, \( v - 1 = 895 \) is not a divisor of \( k(k - 1) \), for each divisor \( k \) of 1296, contradicting Lemma 2.1(a).

\[\Box\]

**Lemma 3.6.** The subgroup \( M_0 \) cannot be \( 3^2 : Q_8 \) with \( q \geq 11 \) and \( p = q \equiv 2 \pmod{3} \).

**Proof.** By (3.1), we have that

\[
(3.14) \quad v = \frac{q^3(q^2 - 1)(q^3 + 1)}{72 \cdot \gcd(3, q + 1)}.
\]

Note that \( |\text{Out}(X)| = 2f \cdot \gcd(3, q + 1) \). Then by (3.2), we conclude that \( k \) divides \( 432f \). Since \( \lambda \geq 4 \), Lemma 2.1(b) implies that

\[
\frac{4q^3(q^2 - 1)(q^3 + 1)}{216} \leq \lambda v < k^2 \leq 432f^2.
\]

Therefore \( q^3(q^2 - 1)(q^3 + 1) < 10077696f^2 \) and this implies that \( q \in \{3, 5, 7\} \) but this violates \( q \geq 11 \). \[\Box\]
Lemma 3.7. The subgroup $M_0$ cannot be $\text{PSL}(2,7)$ with $p = q \equiv 3, 5, 6 \pmod{7}$.

Proof. Note that $f = 1$ as $q = p$ by (3.1), we have that

\[ v = \frac{q^3(q^2 - 1)(q^3 + 1)}{168 \cdot \gcd(3, q + 1)}. \]

As $f = 1$, by (3.2) that $k$ divides 1008. Moreover, since $\lambda \geq 4$, by Lemma 2.1(b),

\[ \frac{4q^3(q^2 - 1)(q^3 + 1)}{504} \leq \lambda v < k^2 \leq 1008^2. \]

Then $q^3(q^2 - 1)(q^3 + 1) < 128024064$. Note that $q \equiv 3, 5, 6 \pmod{7}$. Thus $q \in \{3, 11\}$. Note that $q \neq 11$ as $v$ given in (3.15) must be integer. If $q = 3$, then $v = 36$, but $v - 1 = 35$ does not divides $k(k - 1)$ which is also a contradiction. \[\square\]

Lemma 3.8. The subgroup $M_0$ cannot be $A_6$, with $p = q \equiv 11, 14 \pmod{15}$.

Proof. By (3.1), we have that

\[ v = \frac{q^3(q^2 - 1)(q^3 + 1)}{360 \cdot \gcd(3, q + 1)} = 175. \]

Note by (3.2) that $k$ divides 2160. By [20, 30], we may only focus on $\lambda \geq 4$, and so Lemma 2.1(b) yields

\[ \frac{4q^3(q^2 - 1)(q^3 + 1)}{1080} \leq \lambda v < k^2 \leq 2160^2 f^2. \]

This follows that

\[ q^3(q^2 - 1)(q^3 + 1) < 1259712000 f^2. \]

Since $q^8 < 2q^3(q^2 - 1)(q^3 + 1)$ and $q = p^f$ is odd, (3.17) implies that $q \in \{3, 5, 7, 9, 11, 13\}$. Since also the fraction (3.16) must be integer, $q \in \{5, 9, 11\}$, and since $p = q \equiv 11, 14 \pmod{15}$, the only acceptable value for $q$ is $q = 11$. So $v = 196988$ and $k$ divides 2160. We then easily observe that, for each divisor $k$ of 2160, the fraction $k(k - 1)/(v - 1)$ is not integer, which is a contradiction. \[\square\]

Lemma 3.9. The subgroup $M_0$ cannot be $A_6 \cdot 2_3$ with $q = 5$.

Proof. By (3.1), we have that

\[ v = \frac{q^3(q^2 - 1)(q^3 + 1)}{720 \cdot \gcd(3, q + 1)} = 175. \]

It follows from (3.2) that $k$ divides 4320. In this case, for each possible value of $k$ the fraction $k(k - 1)/(v - 1)$ is not integer, which is a contradiction. \[\square\]

Lemma 3.10. The subgroup $M_0$ cannot be $A_7$ with $q = 5$.
Proof. By (3.1), we have that
\[ v = \frac{q^3(q^2 - 1)(q^3 + 1)}{2520 \cdot \gcd(3, q + 1)} = 50. \]

Note by (3.2) that \( k \) divides 15120. Moreover, Lemma 2.1(a) implies that \( k \) divides \( \lambda(v-1) \). Then \( k \) divides \( \gcd(15120, \lambda(v-1)) \), and so \( k \) divides 7\( \lambda \). Thus there exists a positive integer \( m \) such that \( mk = 7\lambda \). Since \( k(k-1) = \lambda(v-1) \), it follows that \( k = 7n + 1 \). Since \( k \) divides 15120 and \( k < v \), we have \( k = 15 \). This is a contradiction as \( v - 1 = 49 \) does not divide \( k(k-1) \).

\[ \Box \]

3.1. Proof of Theorem 1.1

Suppose that \( D \) is a symmetric \((v, k, \lambda)\) design and \( G \) is an almost simple automorphism group with simple socle \( X = \text{PSU}(3, q) \). If \( G \) is a flag-transitive and point-primitive automorphism group of \( D \), then the point-stabiliser \( M := G_\alpha \) is maximal in \( G \), and so \( M_0 := X \cap M \) is isomorphic to one of the subgroups in Lemma 2.4. It follows from Lemmas 3.1–3.10 that \( M_0 = [q]^{1+2} : (q^2 - 1) \). In this case, by (3.1), we have that \( v = q^3 + 1 \). Then by [15, Lemma 3.9] and Lemma 2.1(c), \( k \) divides \( \lambda q^3 \). Let now \( m \) be a positive integer such that \( mk = \lambda q^3 \). By Lemma 2.1(a), \( k(k-1) = \lambda(v-1) \), and so \( \lambda q^3(k-1)/m = \lambda q^3 \). Thus, \( k = m + 1 \) and \( \lambda = (m^2 + m)/q^3 \) which the latter statement implies that \( q^3 | m^2 + m \). Thus, \( q^3 \) divides either \( m \), or \( m + 1 \). Since \( m < q^3 \), it follows that \( q^3 \) divides \( m + 1 \), and so \( q^3 = m + 1 \). Therefore, \( \lambda = q^3 - 1 = k - 1 \) and \( v = q^3 + 1 \), that is to say, \( D \) is a \((v, v-1, v-2)\) design with \( b = \binom{v}{k} \), which is a complete design.

References

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