SOME RESULTS OF $f$-BIHARMONIC MAPS INTO A RIEMANNIAN MANIFOLD OF NON-POSITIVE SECTIONAL CURVATURE

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ABSTRACT. The authors investigate $f$-biharmonic maps $u : (M, g) \to (N, h)$ from a Riemannian manifold into a Riemannian manifold with non-positive sectional curvature, and derive that if $\int_M f^p |\tau(u)|^p dv_g < \infty$, $\int_M |\tau(u)|^2 dv_g < \infty$ and $\int_M |du|^2 dv_g < \infty$, then $u$ is harmonic. When $u$ is an isometric immersion, the authors also get that if $u$ satisfies some integral conditions, then it is minimal. These results give an affirmative partial answer to conjecture 4 (generalized Chen’s conjecture for $f$-biharmonic submanifolds).

1. Introduction

In the past several decades harmonic maps have played a central role in geometry and analysis. Let $(M^m, g)$ and $(N^n, h)$ be Riemannian manifolds of dimensions $m, n$ and $u : (M^m, g) \to (N^n, h)$ be a smooth map. The energy of $u$ is defined by $E(u) = \int_M |du|^2 dv_g$, where $dv_g$ is the volume element on $(M^m, g)$. Harmonic maps are the critical maps of $E(\cdot)$. The Euler-Lagrange equation of harmonic maps is $\tau(u) = 0$, where $\tau(u)$ is called the tension field of $u$. $p$-harmonic maps [19], exponentially harmonic maps [16], $F$-harmonic maps and $f$-harmonic maps are extensions to harmonic maps and many results have been carried out (for instance, see [1–3, 10, 24, 33]).

In 1983, J. Eells and L. Lemaire [13] proposed the problem to consider biharmonic maps which are critical points of the bi-energy functional $E_2(u) = \int_M |\tau(u)|^2 dv_g$. We see that biharmonic maps are a generalization of harmonic maps. In 1986, G. Y. Jiang [21] studied the first and the second variational formulas of the bi-energy. There have been many studies on biharmonic maps (for instance, see [4–6, 11, 20, 25, 26, 32]). To further generalize the notion of

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harmonic maps, Y. B. Han and S. X. Feng [17] introduced the $F$-bienergy functional $E_{F,2}(u) = \int_M F(|\tau(u)|^2)\,dv_g$. The critical points of $F$-bienergy $E_{F,2}(u)$ are called $F$-biharmonic maps. If $F(u) = (2u)^2$, we have $p$-biharmonic functional $E_{p,2}(u) = \int_M |\tau(u)|^p\,dv_g$. If $F(u) = e^u$, we have exponential bienergy functional $E_{e,2}(u) = \int_M e^{|\tau(u)|^2}\,dv_g$.

A. Lichnerowicz [23] (see also [12]) introduced and studied $f$-harmonic maps between Riemannian manifolds. The study of $f$-harmonic maps comes from a physical motivation, since in physics $f$-harmonic maps can be viewed as stationary solutions to the inhomogeneous Heisenberg spin system (see [22]).

W. J. Lu [27] introduced the following functional:

$$E_{2,f}(u) = \int_M f|\tau(u)|^2\,dv_g,$$

where $f : (M, g) \to (0, +\infty)$ is a smooth function. A map $u$ is called an $f$-biharmonic map if it is a critical point of the $f$-bienergy functional.

Recently, N. Nakauchi et al. [31] showed that every biharmonic map of a complete Riemannian manifold into a Riemannian manifold of non-positive curvature whose energy and bi-energy are finite must be harmonic. S. Maeta [29] obtained that biharmonic maps from a complete Riemannian manifold into a non-positive curved manifold with finite $(a+2)$-bienergy $\int_M |\tau(u)|^{a+2}\,dv_g < \infty$ ($a \geq 0$) and energy are harmonic. Y. B. Han and W. Zhang [18] obtained that $p$-biharmonic maps from a complete manifold into a non-positive curved manifold with finite $(a+p)$-bienergy $\int_M |\tau(u)|^{a+p}\,dv_g < \infty$ and energy are harmonic. In this paper, we first obtain the following results:

**Theorem 1.1** (cf. Theorem 3.1). Let $u : (M^m, g) \to (N^n, h)$ be an $f$-biharmonic map from a compact Riemannian manifold $(M^m, g)$ without boundary into a Riemannian manifold $(N^n, h)$ with non-positive sectional curvature, then $u$ is harmonic.

**Theorem 1.2** (cf. Theorem 3.3). Let $u : (M^m, g) \to (N^n, h)$ be an $f$-biharmonic map from a complete Riemannian manifold $(M^m, g)$ into a Riemannian manifold $(N^n, h)$ with non-negative real constant.

(i) If

$$\int_M f^p|\tau(u)|^p\,dv_g < \infty, \quad \int_M |\tau(u)|^2\,dv_g < \infty, \quad \text{and} \quad \int_M |du|^2\,dv_g < \infty,$$

then $u$ is harmonic.

(ii) If $Vol(M, g) = \infty$, and $\int_M f^p|\tau(u)|^p\,dv_g < \infty$, then $u$ is harmonic.

Chen’s conjecture is the most interesting problem in the biharmonic theory. In 1988, Chen [9] raised the following problem:

**Conjecture 1.** Any biharmonic submanifold in $E^n$ is minimal.
There are some affirmative partial answers to Conjecture 1.

Then Chen’s conjecture was generalized as follows ([8]): Any biharmonic submanifolds in a Riemannian manifold with non-positive sectional curvature is minimal. There are also some affirmative partial answers to this Conjecture (for instance, see [7,17,30,31]).

Motivated by Chen’s conjecture, Y. B. Han [15] proposed the following conjecture:

**Conjecture 2.** Any $p$-biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.

Some affirmative partial answers to Conjecture 2 were proved in [15,18,28].

Y. B. Han [16] also proposed the following conjecture:

**Conjecture 3.** Any exponentially biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.

Some affirmative partial answers to Conjecture 3 were proved in [16].

For $f$-biharmonic submanifolds, it is natural to consider the following conjecture.

**Conjecture 4.** Any $f$-biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.

For $f$-biharmonic submanifolds, we obtain some results:

**Theorem 1.3** (cf. Theorem 4.1). Let $u : (M,g) \to (N,h)$ be an $f$-biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold $(N,h)$ with non-positive sectional curvature and let $p, q$ be two real constants satisfying $2 \leq p < \infty$ and $0 < q \leq p < \infty$. If

$$\int_M f^p |\vec{H}|^q \, dv_g < \infty,$$

then $u$ is minimal.

**Theorem 1.4** (cf. Theorem 4.2). Let $u : (M,g) \to (N,h)$ be an $f$-biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold $(N,h)$ with non-positive sectional curvature. If

$$\int_{B_r(x_0)} f^p \, dv_g \leq C_0 (1 + r)^s$$

for some positive integer $s$, $C_0$ independent of $r$ and $p \geq 2$, then $u$ is minimal.

**Theorem 1.5** (cf. Theorem 4.3). Let $u : (M,g) \to (N,h)$ be an $f$-biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold $(N,h)$ whose sectional curvature is smaller than $-\varepsilon$ for some constant $\varepsilon > 0$ and $\int_{B_r(x_0)} |f \vec{H}|^p \, dv_g (p \geq 2)$ is of at most polynomial growth of $r$. Then $u$ is minimal.
Theorem 1.6 (cf. Theorem 4.4). Let $u : (M, g) \to (N, h)$ be a complete $\varepsilon$-supper $f$-biharmonic submanifold in $(N, h)$ for $\varepsilon > 0$. If
\[ \int_M |f\tilde{H}|^p dv_g < \infty, \]
where $p \geq 2$, then $u$ is minimal.

2. Preliminaries

In this section we give some necessary notations and terminologies about harmonic maps, biharmonic maps, $f$-biharmonic maps and $f$-biharmonic submanifolds.

Let $u : (M^m, g) \to (N^n, h)$ be a smooth map from an $m$-dimensional Riemannian manifold $(M^m, g)$ to an $n$-dimensional Riemannian manifold $(N^n, h)$. The energy of $u$ is defined by
\[ E(u) = \int_M \frac{|du|^2}{2} dv_g, \]
where $dv_g$ is the volume element on $(M^m, g)$.

The Euler-Lagrange equation of harmonic maps is $\tau(u) = 0$ where $\nabla$ is the Levi-Civita connection on $(M^m, g)$ and $\tilde{\nabla}$ is the induced Levi-Civita connection of the pullback bundle $u^{-1}TN$. \{e_i\}_{i=1}^m is an orthonormal frame field on $(M^m, g)$. If $\tau(u) = 0$, then $u$ is called a harmonic map.

In 1983, J. Eells and L. Lemaire [13] proposed the problem to consider the bi-energy functional:
\[ E_2(u) = \int_M \frac{\left|\tau(u)\right|^2}{2} dv_g. \]
Then, in 1986, G. Y. Jiang [21] obtained the first and the second variational formulas of the bi-energy functional. The Euler-Lagrange equation of the bi-energy functional is
\[ \tau_2(u) = -\tilde{\Delta}(\tau(u)) - \sum_i R^N(\tau(u), du(e_i)) du(e_i) = 0, \]
where $R^N(X, Y) = [^N N \nabla_X, ^N N \nabla_Y] - ^N N \nabla_{[X,Y]}$ is the curvature operator on $(N, h)$. If $\tau_2(u) = 0$, then $u$ is called a biharmonic map.

To generalize the notation of biharmonic maps, W. J. Lu [27] studied the $f$-bienergy functional
\[ E_{2,f}(u) = \int_M f(x)\left|\tau(u)\right|^2 dv_g, \]
where $f : (M, g) \to (0, +\infty)$ is a smooth function. The Euler-Lagrange equation of $E_{2,f}$ is
\[ \tau_{2,f}(u) = -\tilde{\Delta}(f\tau(u)) - \sum_i R^N(f\tau(u), du(e_i)) du(e_i) = 0. \]
If \( \tau_2 f(u) = 0 \), then \( u \) is called an \( f \)-biharmonic map.

Now we briefly recall the submanifold theory. Let \( u : (M^m, g) \to (N^{m+t}, h) \) be an isometric immersion from an \( m \)-dimensional Riemannian manifold \((M^m, g)\) into an \((m + t)\)-dimensional Riemannian manifold \((N^{m+t}, h)\). The second fundamental form \( B : TM \otimes TM \to NM \) is defined by
\[
B(X, Y) = N \nabla_X Y - \nabla_X Y, \quad X, Y \in \Gamma(TM).
\]
The shape operator \( A_\xi : TM \to TM \) for a unit normal vector field \( \xi \) on \( M \) is defined by
\[
N \nabla_X \xi = -A_\xi X + \nabla_X \xi, \quad X \in \Gamma(TM), \xi \in \Gamma(T^\perp M),
\]
where \( \nabla^\perp \) denotes the normal connection on the normal bundle of \( M \) in \( N \).

It’s well known that \( B \) and \( A_\xi \) are related by
\[
\langle B(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.
\]
For any \( x \in M \), the mean curvature vector field \( \vec{H} \) of \( M \) at \( x \) is given by
\[
\vec{H} = \frac{1}{m} \sum_i B(e_i, e_i).
\]
If an isometric immersion \( u : (M, g) \to (N, h) \) is \( f \)-biharmonic, then \( M \) is called an \( f \)-biharmonic submanifold in \( N \). In this case, \( \tau(u) = m \vec{H} \). We know that \( M \) is an \( f \)-biharmonic submanifold in \( N \) if and only if
\[
(1) -\tilde{\Delta}(f \vec{H}) - \sum_i R^N(f \vec{H}, e_i)e_i = 0.
\]
From (2), we obtain the sufficient and necessary condition for \( M \) to be an \( f \)-biharmonic submanifold in \( N \) as follows:
\[
(2) \quad \triangle^\perp (f \vec{H}) - \sum_i B(e_i, A_f \vec{H} e_i) + \left[ \sum_i R^N(f \vec{H}, e_i)e_i \right]^\perp = 0,
\]
\[
(3) \quad Tr_g(\nabla(\cdot) A_f \vec{H}(\cdot)) + Tr_g[A_{\nabla^\perp (f \vec{H})}(\cdot)] - \left[ \sum_i R^N(f \vec{H}, e_i)e_i \right]^\perp = 0.
\]
We also need the following lemma.

**Lemma 2.1.** (Gaffney [14]). Let \((M, g)\) be a complete Riemannian manifold. If a \( C^1 \) 1-form \( \alpha \) satisfies that \( \int_M |\alpha| dv_g < \infty \) and \( \int_M (\delta \alpha) dv_g < \infty \), or equivalently, a \( C^1 \) vector \( X \) defined by \( \alpha(Y) = \langle X, Y \rangle \) satisfies that \( \int_M |X| dv_g < \infty \) and \( \int_M \text{div}(X) dv_g < \infty \), then \( \int_M (\delta \alpha) dv_g = \int_M \text{div}(X) dv_g = 0 \).

3. \( f \)-biharmonic maps in a Riemannian manifold of non-positive sectional curvature

In this section, we obtain some results as follows:

**Theorem 3.1.** Let \( u : (M^m, g) \to (N^n, h) \) be an \( f \)-biharmonic map from a compact Riemannian manifold \((M^m, g)\) without boundary into a Riemannian manifold \((N^n, h)\) with non-positive sectional curvature, then \( u \) is harmonic.
Proof. From (1), we have
\[
\frac{1}{2} \Delta|f \tau(u)|^2 = |\tilde{\nabla}(f \tau(u))|^2 + \langle \tilde{\Delta}[f \tau(u)], f \tau(u) \rangle
\]
\[
= |\tilde{\nabla}(f \tau(u))|^2 - \sum_i \langle R^N(f \tau(u), du(e_i)) du(e_i), f \tau(u) \rangle
\]
\[
\geq |\tilde{\nabla}(f \tau(u))|^2.
\]
From Green theorem and the compactness of \((M, g)\), we have
\[
0 = \int_M \frac{1}{2} \Delta|f \tau(u)|^2 dv_g = \int_M |\tilde{\nabla}(f \tau(u))|^2 dv_g.
\]
Then, for every \(X \in \Gamma(TM)\), we have
\[
\tilde{\nabla}X|f \tau(u)| = 0.
\]
Let \(Y = \sum_i h(du(e_i), f \tau(u)) e_i\), we have
\[
\text{div}(Y) = \sum_k g(\nabla e_k Y, e_k)
\]
\[
= \sum_k [h(\nabla e_k du(e_k), f \tau(u)) - h(du(\nabla e_k e_k), f \tau(u))]
\]
\[
= h(\tau(u), f \tau(u)) = f|\tau(u)|^2.
\]
From (6), we have
\[
0 = \int_M \text{div}(Y) dv_g = \int_M f|\tau(u)|^2 dv_g.
\]
Since \(f > 0\) in \(M\), so we have \(\tau(u) = 0\). \(\Box\)

Corollary 3.2. Any \(f\)-biharmonic function in a compact manifold \((M, g)\) without boundary is constant.

Proof. From Theorem 3.1, \(u\) is an \(f\)-biharmonic function if and only if \(u\) is a harmonic function. On the other hand, any harmonic function in a compact manifold \((M, g)\) is constant, so we have \(u = C\). \(\Box\)

Theorem 3.3. Let \(u : (M^m, g) \to (N^n, h)\) be an \(f\)-biharmonic map from a complete Riemannian manifold \((M^m, g)\) into a Riemannian manifold \((N^n, h)\) with non-positive sectional curvature and let \(p \geq 2\) be a non-negative real constant.

(i) If
\[
\int_M f^p|\tau(u)|^p dv_g < \infty, \int_M |\tau(u)|^2 dv_g < \infty \text{ and } \int_M |du|^2 dv_g < \infty,
\]
then \(u\) is harmonic.

(ii) If \(Vol(M, g) = \infty\) and \(\int_M f^p|\tau(u)|^p dv_g < \infty\), then \(u\) is harmonic.
Proof. Take a fixed point \( x_0 \in M \) and for every \( r > 0 \), let us consider the following cut off function \( \lambda(x) \) on \( M \):

\[
\begin{cases}
0 \leq \lambda(x) \leq 1, & x \in M, \\
\lambda(x) = 1, & x \in B_r(x_0), \\
\lambda(x) = 0, & x \in M - B_2r(x_0), \\
|\nabla \lambda| \leq \frac{C}{r}, & x \in M,
\end{cases}
\]

where \( B_r(x_0) = \{ x \in M : d(x, x_0) < r \} \), \( C \) is a positive constant and \( d \) is the distance of \( M \). From (1), we have

\[
\int_M \langle -\tilde{\Delta}(f\tau(u)), \lambda^2|f\tau(u)|^{p-2}f\tau(u) \rangle dv_g
= \int_M \lambda^2 f^p|\tau(u)|^{p-2} \sum_i (R^N(\tau(u), du(e_i))du(e_i), \tau(u))dv_g \leq 0,
\]

where the inequality follows from the sectional curvature of \((N, h)\) is non-positive. From (8), we have

\[
0 \geq \int_M \langle -\tilde{\Delta}(f\tau(u)), \lambda^2|f\tau(u)|^{p-2}f\tau(u) \rangle dv_g
= \int_M \langle \tilde{\nabla}(f\tau(u)), \tilde{\nabla}(\lambda^2|f\tau(u)|^{p-2}f\tau(u)) \rangle dv_g
= \int_M \sum_{i=1}^{m} \langle \tilde{\nabla}_{e_i}(f\tau(u)), \tilde{\nabla}_{e_i}(\lambda^2|f\tau(u)|^{p-2}f\tau(u)) \rangle dv_g
= \int_M \sum_{i=1}^{m} \langle \tilde{\nabla}_{e_i}(f\tau(u)), 2\lambda e_i(\lambda)|f\tau(u)|^{p-2}f\tau(u) \rangle
+ \lambda^2 e_i(\lambda)|f\tau(u)|^{p-2}\tilde{\nabla}_{e_i}[f\tau(u)]dv_g
\]

\[
\geq \int_M \sum_{i=1}^{m} 2\lambda e_i(\lambda)|f\tau(u)|^{p-2}\langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u) \rangle dv_g
+ \int_M \sum_{i=1}^{m} (p-2)\lambda^2|f\tau(u)|^{p-4}\langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u) \rangle^2 dv_g
+ \int_M \sum_{i=1}^{m} \lambda^2|f\tau(u)|^{p-2}\langle \tilde{\nabla}_{e_i}[f\tau(u)], \tilde{\nabla}_{e_i}[f\tau(u)] \rangle dv_g
\]
where the inequality follows from
\[
\int_M \sum_{i=1}^{m} \lambda^2 |f \tau(u)|^{p-4} \langle \nabla e_i[f \tau(u)], \nabla e_i[f \tau(u)] \rangle dv_g \geq 0.
\]

From (9), we have
\[
\int_M \sum_{i=1}^{m} \lambda^2 |f \tau(u)|^{p-2} \langle \nabla e_i[f \tau(u)], \nabla e_i[f \tau(u)] \rangle dv_g
\]
\[
\leq - \int_M \sum_{i=1}^{m} 2 \lambda e_i(\lambda) |f \tau(u)|^{p-2} \langle \nabla e_i[f \tau(u)], f \tau(u) \rangle dv_g.
\]

By using Young’s inequality, we have
\[
\int_M \sum_{i=1}^{m} 2 \lambda e_i(\lambda) |f \tau(u)|^{p-2} \langle \nabla e_i[f \tau(u)], f \tau(u) \rangle dv_g
\]
\[
\leq \int_M \sum_{i=1}^{m} \lambda^2 |f \tau(u)|^{p-2} |\nabla e_i[f \tau(u)]|^2 dv_g + 2 \int_M |\nabla \lambda|^2 |f|^p |\tau(u)|^p dv_g.
\]

From (10) and (11), we have
\[
\int_M \sum_{i=1}^{m} \lambda^2 |f \tau(u)|^{p-2} |\nabla e_i[f \tau(u)]|^2 dv_g
\]
\[
\leq 4 \int_M |\nabla \lambda|^2 |f|^p |\tau(u)|^p dv_g \leq \frac{4C^2}{r^2} \int_M |f|^p |\tau(u)|^p dv_g.
\]

By assumption \( \int_M |f|^p |\tau(u)|^p dv_g < \infty \), letting \( r \to \infty \) in (12), we have
\[
\int_M \sum_{i=1}^{m} f^{p-2} |\tau(u)|^{p-2} \langle \nabla e_i[f \tau(u)], \nabla e_i[f \tau(u)] \rangle dv_g = 0.
\]

So we obtain that \( f|\tau(u)| \) is constant. If \( |\tau(u)| \neq 0 \), we get
\[
\int_M f^p |\tau(u)|^p = |f \tau(u)|^p Vol(M) = \infty,
\]
which yields a contradiction. So we have \( |\tau(u)| = 0 \), i.e., \( u \) is harmonic. We derive that (ii) is tenable.

For (i), we assume that
\[
\int_M f^p |\tau(u)|^p dv_g < \infty, \int_M |\tau(u)|^p dv_g < \infty, \int_M |du|^2 dv_g < \infty.
\]

We define a 1-form
\[
\alpha(X) = |f \tau(u)|^{\frac{p}{2}} \langle du(X), f \tau(u) \rangle,
\]
\[
\int_M f^p |\tau(u)|^p dv_g < \infty, \int_M |\tau(u)|^p dv_g < \infty, \int_M |du|^2 dv_g < \infty.
\]

We define a 1-form
\[
\alpha(X) = |f \tau(u)|^{\frac{p}{2}} \langle du(X), f \tau(u) \rangle,
\]
where $X \in \Gamma(TM)$. We note that
\begin{equation}
\int_{M} |\alpha| dv_g = \int_{M} \left( \sum_{i=1}^{m} |\alpha(e_i)|^2 \right)^{\frac{1}{2}} dv_g
\end{equation}
\begin{equation}
= \int_{M} \left( \sum_{i=1}^{m} [f^p u]^q du(e_i), f^p u \right) dv_g
\end{equation}
\begin{equation}
\leq \int_{M} |f^p u|^q|du|dv_g \leq \left[ \int_{M} |f^p u|^p dv_g \right]^{\frac{1}{p}} \left[ \int_{M} |du|^2 dv_g \right]^{\frac{1}{2}} < \infty.
\end{equation}
We compute
\begin{equation}
-\delta\alpha = \sum_{i=1}^{m} (\nabla_{e_i} \alpha)(e_i) = \sum_{i=1}^{m} \nabla_{e_i} \alpha(e_i) - \alpha(\nabla_{e_i} e_i)
\end{equation}
\begin{equation}
= \sum_{i=1}^{m} \nabla_{e_i} [f^p u]^q du(e_i), f^p u] - \sum_{i=1}^{m} [f^p u]^q du(\nabla_{e_i} e_i), f^p u]
\end{equation}
\begin{equation}
= \sum_{i=1}^{m} [f^p u]^q du(\nabla_{e_i} e_i), f^p u) = |f^p u|^q |f^p u|
\end{equation}
where the third equality follows from that $|f^p u|$ is constant and $\tilde{\nabla}_X f^p u = 0$, for all $X \in \Gamma(TM)$. We have
\begin{equation}
\int_{M} (-\delta\alpha) dv_g = \int_{M} |f^p u|^q |f^p u| dv_g \leq \left[ \int_{M} f^p |f^p u|^p dv_g \right]^{\frac{1}{p}} \left[ \int_{M} |f^p u|^q dv_g \right]^{\frac{1}{q}} < \infty.
\end{equation}
From $\int_{M} f^p |f^p u|^p dv_g < \infty$ and $\int_{M} |f^p u|^q dv_g < \infty$, we know the function $-\delta\alpha$ is also integrable over $M$. From this and (14), applying Lemma 2.1 for the 1-form $\alpha$, we have
\begin{equation}
0 = \int_{M} (-\delta\alpha) dv_g = \int_{M} f^p |f^p u|^q + 1 dv_g.
\end{equation}
So we have $\tau(u) = 0$, i.e., $u$ is harmonic. \hfill \Box

4. $f$-biharmonic submanifolds in a Riemannian manifold of non-positive sectional curvature

**Theorem 4.1.** Let $u : (M, g) \rightarrow (N, h)$ be an $f$-biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold $(N, h)$ with non-positive sectional curvature and let $p, q$ be two real constants satisfying $2 \leq p < \infty$ and $0 < q \leq p < \infty$. If
\begin{equation}
\int_{M} f^p |\tilde{H}|^q dv_g < \infty,
\end{equation}
then $u$ is minimal.
Proof. From (3), we have
\[ \Delta|f\vec{H}|^2 = \Delta\langle f\vec{H}, f\vec{H} \rangle = 2(\Delta^\perp(f\vec{H}), f\vec{H}) + 2|\nabla^\perp(f\vec{H})|^2 \]
\[ = 2|\nabla^\perp(f\vec{H})|^2 + 2\sum_{i=1}^{m}\langle B(A_{f\vec{H}}e_i, e_i), f\vec{H} \rangle - 2\sum_{i=1}^{m}\langle R^N(f\vec{H}, e_i)e_i, f\vec{H} \rangle \]
\[ \geq 2|\nabla^\perp(f\vec{H})|^2 + 2\sum_{i=1}^{m}\langle B(A_{f\vec{H}}e_i, e_i), f\vec{H} \rangle, \]  
(14)

where the inequality follows from the sectional curvature of \( N \) is non-positive.

Now we proof the following inequality:
\[ \sum_{i=1}^{m}\langle B(A_{f\vec{H}}e_i, e_i), f\vec{H} \rangle \geq mf^2|\vec{H}|^4. \]
(15)

Let \( x \in M, \) if \( \vec{H} = 0, \) we are done. If \( \vec{H}(x) \neq 0, \) we have at \( x, \)
\[ \sum_{i=1}^{m}\langle B(A_{f\vec{H}}e_i, e_i), f\vec{H} \rangle = \sum_{i=1}^{m}f^2|\vec{H}|^2\langle B(A_{f\vec{H}}e_i, e_i), f\vec{H} \rangle \]
\[ = \sum_{i=1}^{m}f^2|\vec{H}|^2\langle A_{f\vec{H}}e_i, A_{f\vec{H}}e_i \rangle = \sum_{i,j=1}^{m}f^2|\vec{H}|^2|\langle B(e_i, e_j), f\vec{H} \rangle|^2 \geq mf^2|\vec{H}|^4. \]

From (15) and (16), we have
\[ \frac{\Delta|f\vec{H}|^2}{2} \geq 2|\nabla^\perp(f\vec{H})|^2 + 2mf^2|\vec{H}|^4. \]  
(16)

Take a fixed point \( x_0 \in M \) and for every \( r > 0, \) let us consider the following cut off function \( \lambda(x) \) on \( M: \)
\[ \begin{cases} 
0 \leq \lambda(x) \leq 1, & x \in M, \\
\lambda(x) = 1, & x \in B_r(x_0), \\
\lambda(x) = 0, & x \in M - B_{2r}(x_0), \\
|\nabla \lambda| \leq \frac{C}{r}, & x \in M, 
\end{cases} \]  
(17)

where \( B_r(x_0) = \{ x \in M : d(x, x_0) < r \}, \) \( C \) is a positive constant and \( d \) is the distance of \( M. \) From (17), we have
\[ - \int_M \nabla(\lambda^{n+4}|f\vec{H}|^n)\nabla|f\vec{H}|^2dv_g = \int_M \lambda^{n+4}|f\vec{H}|^n\Delta|f\vec{H}|^2dv_g \]
\[ \geq 2\int_M \lambda^{n+4}|f\vec{H}|^n|\nabla^\perp(f\vec{H})|^2dv_g + 2m\int_M \lambda^{n+4}|f\vec{H}|^n|\vec{H}|^4dv_g, \]
(18)
where $a$ is a positive constant to be determined later. On the other hand, we have
\[
- \int_M \nabla (\lambda^{a+4} |f \tilde{H}|^a) \nabla |f \tilde{H}|^2 \, dv_g \\
= - 2(a + 4) \int_M \lambda^{a+3} \nabla |f \tilde{H}|^a \langle \nabla \perp (f \tilde{H}), f \tilde{H} \rangle \, dv_g \\
- 2a \int_M \lambda^{a+4} |f \tilde{H}|^a \langle \nabla \perp (f \tilde{H}), f \tilde{H} \rangle^2 \, dv_g \\
\leq - 2(a + 4) \int_M \lambda^{a+3} \nabla |f \tilde{H}|^a \langle \nabla \perp (f \tilde{H}), f \tilde{H} \rangle \, dv_g.
\]
(19)

From (19) and (20), we have
\[
2 \int_M \lambda^{a+4} |f \tilde{H}|^a |\nabla \perp (f \tilde{H})|^2 \, dv_g + 2m \int_M \lambda^{a+4} |f \tilde{H}|^a f^2 |\tilde{H}|^4 \, dv_g \\
\leq - 2(a + 4) \int_M \lambda^{a+3} \nabla |f \tilde{H}|^a \langle \nabla \perp (f \tilde{H}), f \tilde{H} \rangle \, dv_g \\
\leq (a + 4)^2 \int_M \lambda^{a+2} f^{a+2} |\tilde{H}|^{a+2} |\nabla \lambda|^2 \, dv_g.
\]
(20)

From Young’s inequality, we have
\[
(a + 4)^2 \int_M f^{a+2} \lambda^a |\tilde{H}|^{a+2} |\nabla \lambda|^2 \, dv_g \\
= (a + 4)^2 \int_M f^{a+2} \lambda^a |\tilde{H}|^{a+2} |\tilde{H}|^{a+2-s} |\nabla \lambda|^2 \, dv_g \\
\leq \int_M \lambda^{a+4} \tilde{H}^{a+4} f^{a+2} \, dv_g \\
+ C(a, s) \int_M f^{a+2} \lambda^a (a+2-s) \frac{a+4}{s+4} |\tilde{H}|^{(a+2-s) \frac{a+4}{s+4}} |\nabla \lambda|^{2 \frac{a+4}{s+4}} \, dv_g,
\]
where $s \in (0, a+2)$ and $C(a, s)$ is a constant depending on $a, s$. From (22) and (23), we have
\[
\int_M \lambda^{a+4} |f \tilde{H}|^a |\nabla \perp (f \tilde{H})|^2 \, dv_g + (2m - 1) \int_M f^{a+2} \lambda^a |\tilde{H}|^{a+4} \, dv_g \\
\leq C(a, s) \int_M f^{a+2} \lambda^a (a+2-s) \frac{a+4}{s+4} |\tilde{H}|^{(a+2-s) \frac{a+4}{s+4}} |\nabla \lambda|^2 \frac{a+4}{s+4} \, dv_g,
\]
(23)
We know that when $s$ varies from 0 to $a + 2$, then $(a + 2 - s)\frac{a + 1}{a + 4 - s}$ varies from $a + 2$ to 0. Let $q = (a + 2 - s)\frac{a + 1}{a + 4 - s}$, then $q \in (0, a + 2)$. Let $p = a + 2$, from \(\int_M f^p |\vec{H}|^p dv_g < \infty, 2 \leq p < \infty\) and $0 < q \leq p < \infty$, set $r \to \infty$ in (24), we have
\[
\int_M |f\vec{H}|^a|\nabla^\perp (f\vec{H})|^2 dv_g + (2m - 1) \int_M f^{a+2}|\vec{H}|^{a+4} dv_g = 0.
\]
So we have $\vec{H} = 0$. \(\Box\)

**Theorem 4.2.** Let $u : (M, g) \to (N, h)$ be an $f$-biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold $(N, h)$ with non-positive sectional curvature. If
\[
\int_{B_r(x_0)} f^p dv_g \leq C_0 (1 + r)^s
\]
for some positive integer $s$, $C_0$ independent of $r$ and $p \geq 2$, then $u$ is minimal.

**Proof.** From (21), we have
\[
2 \int_M \lambda^a |f\vec{H}|^a |\nabla^\perp (f\vec{H})|^2 dv_g + 2m \int_M \lambda^{a+4} |f\vec{H}|^a f^2 |\vec{H}|^4 dv_g \leq -2(a + 4) \int_M \lambda^{a+3} \nabla\lambda f\vec{H}^a (\nabla^\perp (f\vec{H}), F'(\frac{m^2 |\vec{H}|^2}{2})f\vec{H}) dv_g.
\]
From Young’s inequality, we have
\[
-2(a + 4) \int_M \lambda^{a+3} \nabla\lambda f\vec{H}^a (\nabla^\perp (f\vec{H}), f\vec{H}) dv_g \leq \int_M \lambda^{a+4} |f\vec{H}|^a |\nabla^\perp (f\vec{H})|^2 dv_g + \int_M \lambda^{a+4} f^{a+2} |\vec{H}|^{a+4} dv_g + C(a) \int_M f^{a+2} |\nabla\lambda|^{a+4} dv_g,
\]
where $C(a)$ is a constant depending on $a$. From (26) and (27), we have
\[
\int_M \lambda^{a+4} f\vec{H}^a |\nabla^\perp (f\vec{H})|^2 dv_g + \int_M (2m - 1) \lambda^{a+4} f^{a+2} |\vec{H}|^{a+4} dv_g \leq C(a) \int_M f^{a+2} |\nabla\lambda|^{a+4} dv_g \leq C(a) \frac{C_0^{a+4}}{r^{a+4}} \int_{B_r(x_0)} f^{a+2} dv_g
\]
(27) \leq C(a) C_0^{a+4} 0 \frac{(1 + 2r)^s}{r^{a+4}}.

Let $a$ be big enough and $r \to \infty$, then we finish the proof. \(\Box\)

**Theorem 4.3.** Let $u : (M, g) \to (N, h)$ be an $f$-biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold $(N, h)$ whose sectional curvature is smaller than $-\varepsilon$ for some constant $\varepsilon > 0$ and $\int_{B_r(x_0)} |f\vec{H}|^p dv_g (p \geq 2)$ is of at most polynomial growth of $r$. Then $u$ is minimal.
Proof. From the equation (3), we have
\[ \Delta |f \vec{H}|^2 = \Delta \langle f \vec{H}, f \vec{H} \rangle = 2\langle \Delta^\perp (f \vec{H}), f \vec{H} \rangle + 2|\nabla^\perp |f \vec{H}|^2 \]
\[ = 2|\nabla^\perp (f \vec{H})|^2 + 2 \sum_{i=1}^{m} \langle R^N(f \vec{H}, e_i), f \vec{H} \rangle - 2 \sum_{i=1}^{m} \langle R^N(f \vec{H}, e_i)e_i, f \vec{H} \rangle \]
\[ \geq 2|\nabla^\perp (f \vec{H})|^2 + 2m|\vec{H}|^4 f^2 + 2m\varepsilon|f \vec{H}|^2 \]
\[ \geq 2|\nabla^\perp (f \vec{H})|^2 + 2m\varepsilon|f \vec{H}|^2, \]
that is
\[ \Delta |f \vec{H}|^2 \geq 2|\nabla^\perp (f \vec{H})|^2 + 2m\varepsilon|f \vec{H}|^2. \]
From (29), we have
\[ -\int_M \nabla [\lambda^2 |f \vec{H}|^a] \nabla |f \vec{H}|^2 dv = \int_M [\lambda^2 |f \vec{H}|^a] \Delta |f \vec{H}|^2 dv \]
\[ \geq 2\int_M \lambda^2 |f \vec{H}|^a |\nabla^\perp (f \vec{H})|^2 dv_g + 2m\varepsilon \int_M \lambda^2 |f \vec{H}|^{a+2} dv_g, \]
where \( \lambda \) is given by (18) and \( a \) is a nonnegative constant. We also have
\[ -\int_M \lambda \nabla [\lambda^2 |f \vec{H}|^a] \nabla |f \vec{H}|^2 dv_g \]
\[ = -4 \int_M \lambda \lambda^2 |f \vec{H}|^a \langle \nabla^\perp (f \vec{H}), f \vec{H} \rangle dv_g \]
\[ - 2a \int_M \lambda^2 |f \vec{H}|^a \langle \nabla^\perp (f \vec{H}), f \vec{H} \rangle dv_g \]
\[ \leq -4 \int_M \lambda \lambda^2 |f \vec{H}|^a \langle \nabla^\perp (f \vec{H}), f \vec{H} \rangle dv_g \]
\[ \leq 2\int_M \lambda^2 |f \vec{H}|^a |\nabla^\perp (f \vec{H})|^2 dv_g + 2 \int_M |f \vec{H}|^{a+2} |\nabla \lambda|^2 dv_g \]
\[ \leq 2\int_M \lambda^2 |f \vec{H}|^a |\nabla^\perp (f \vec{H})|^2 dv_g + 2 \int_M |f \vec{H}|^{a+2} \frac{C^2}{r^2} dv_g \int_{B_r(x_0)} |f \vec{H}|^{a+2} dv_g \]
\[ \leq 2\int_M \lambda^2 |f \vec{H}|^a |\nabla^\perp (f \vec{H})|^2 dv_g + 2 \int_M |f \vec{H}|^{a+2} \frac{C^2}{r^2} dv_g \int_{B_r(x_0)} |f \vec{H}|^{a+2} dv_g. \]
From (30) and (31), we have
\[ 2m\varepsilon \int_{B_{2r}(x_0)} |f \vec{H}|^{a+2} dv_g \leq 2 \int_{B_{2r}(x_0)} |f \vec{H}|^{a+2} dv_g. \]
Letting \( g(r) = \int_{B_r(x_0)} |f \vec{H}|^{a+2} dv_g \), we have \( g(r) \leq \frac{C_1}{r^2} g(2r) \) where \( C_1 = \frac{c^2}{m\varepsilon} \).
Then we know \( g(r) \leq \frac{C_2}{r^2} g(2^n r) \), where \( C_2 \) is a constant independent of \( r \). From the assumption, we know \( g(r) \leq C_2(1 + 2^{an}r^s) \) for some integer \( s > 0 \). When \( r \)
is big enough, we have $g(r) \leq \frac{C^2(1+2^n r^n)}{r^n}$. Set $2n > s$, then $\lim_{r \to \infty} g(r) = 0$, so $\vec{H} = 0$. □

**Definition.** Let $M$ be a submanifold in $N$ with the metric $\langle \cdot, \cdot \rangle$, then we call $M$ a $\varepsilon$-super $f$-biharmonic submanifold, if

$$\langle \Delta (f \vec{H}), f \vec{H} \rangle \geq (\varepsilon - 1) |\nabla (f \vec{H})|^2,$$

where $\varepsilon \in [0,1]$ is a constant.

**Theorem 4.4.** Let $u : (M,g) \to (N,h)$ be a complete $\varepsilon$-super $f$-biharmonic submanifold in $(N,h)$ for $\varepsilon > 0$. If

$$\int_M |f \vec{H}|^p \, dv_g < \infty,$$

then $u$ is minimal, where $p \geq 2$.

**Proof.** From (32), we have

$$(\varepsilon - 1) \int_M \lambda^2 |f \vec{H}|^n |\nabla (f \vec{H})|^2 \, dv_g \leq \int_M \lambda^2 |f \vec{H}|^n \langle \Delta (f \vec{H}), f \vec{H} \rangle \, dv_g$$

$$= - \int_M \lambda^2 |f \vec{H}|^n |\nabla (f \vec{H})|^2 \, dv_g - \int_M 2\lambda \nabla \lambda |f \vec{H}|^n \langle \nabla (f \vec{H}), f \vec{H} \rangle \, dv_g$$

$$- a \int_M \lambda^2 |f \vec{H}|^{n-2} |\nabla (f \vec{H}), f \vec{H}|^2 \, dv_g$$

$$\leq - \int_M \lambda^2 |f \vec{H}|^n |\nabla (f \vec{H})|^2 \, dv_g - \int_M 2\lambda \nabla \lambda |f \vec{H}|^n \langle \nabla (f \vec{H}), f \vec{H} \rangle \, dv_g,$$

where $\lambda$ is defined by (18), $a \geq 0$, we have

$$\varepsilon \int_M \lambda^2 |f \vec{H}|^n |\nabla (f \vec{H})|^2 \, dv_g \leq - \int_M 2\lambda \nabla \lambda |f \vec{H}|^n \langle \nabla (f \vec{H}), f \vec{H} \rangle \, dv_g.$$
and then $\vec{H} = 0$ or $\nabla (f \vec{H}) = 0$.

We will prove that $\nabla (f \vec{H}) = 0$ implies $\vec{H} = 0$.

Set $x \in M$ such that $\nabla (f \vec{H}) = 0$. We take an orthonormal basis $\{e_i\}_{i=1}^m$ of $T_x M$, an orthonormal basis $\{v_\alpha\}_{\alpha=1}^m$ of $(T_x M)^\perp$, then we have

$$0 = \langle \nabla_{e_i} (f \vec{H}), e_j \rangle = -\langle f \vec{H}, B(e_i, e_j) \rangle. \quad (35)$$

From (36), we have

$$0 = \sum_{i=1}^m \langle f \vec{H}, B(e_i, e_i) \rangle = m|\vec{H}|^2 f,$$

so we obtain $\vec{H} = 0$. \hfill $\square$

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