East Asian Math. J.
Vol. 34 (2018), No. 1, pp. 017–020
http://dx.doi.org/10.7858/eamj.2018.002

ON HARDY-BENNETT INEQUALITY

Ern Gun Kwon

Abstract. Hardy-Bennett inequality is an inequality of Hardy-type having logarithmic weight. It appeared in 1973, and several generalizations followed. We, in this note, present another generalization with a simple proof.

1. Hardy-Bennett inequality

The classical Hardy's inequality reads:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p dx \;<\; \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{1.1}$$

where f is a nonnegative function of $L^p(0,\infty)$ and $p > 1$. In 1928, Hardy [1] proved a weighted modification of (1.1):

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^a dx \;<\; \left(\frac{p}{p-1-a}\right)^p \int_0^\infty f^p(x)x^a\,dx, \tag{1.2}$$

where f is a nonnegative measurable function on $(0,\infty)$ and $p > 1,\ a < p-1$. The constant

$$\left(\frac{p}{p-1-a}\right)^p$$

is the best possible.

Due to significance and usefulness of (1.1) and (1.2), there have been quite a lot of researches on their variants, proofs, and extensions. For basic examples, one may see [2].

In 1973, Bennett [3] proved the following:

Theorem A. Let $\alpha > 0$, $1 \le p \le \infty$, and f be a nonnegative and measurable function on $[0,1]$. Then

Received December 11, 2017; Accepted December 13, 2017.
2010 *Mathematics Subject Classification.* 26D15.
Key words and phrases. Hardy inequality, Hardy-Bennett inequality.
This work was supported by a grant from 2017 Research Fund of Andong National University.

©2018 The Youngnam Mathematical Society
(pISSN 1226-6973, eISSN 2287-2833)

$$\left\{ \int_0^1 [\log(e/x)]^{\alpha p - 1} \left(\int_0^x f(y)\ dy \right)^p \frac{dx}{x} \right\}^{1/p} \\ \leq\ \alpha^{-1} \left(\int_0^1 x^p \,[\log(e/x)]^{(1+\alpha)p-1}\, f^p(x)\ \frac{dx}{x} \right)^{1/p} \tag{1.3}$$

and

$$\left\{ \int_0^1 [\log(e/x)]^{-\alpha p - 1} \left(\int_x^1 f(y)\ dy \right)^p \frac{dx}{x} \right\}^{1/p} \\ \leq\ \alpha^{-1} \left(\int_0^1 x^p \,[\log(e/x)]^{(1-\alpha)p-1}\, f^p(x)\ \frac{dx}{x} \right)^{1/p} \tag{1.4}$$

with the usual modification if $p = \infty$.

In 2014, the inequality (1.3) was extended in [4] as

$$\alpha^{p-1} \left(\int_0^1 f(x)\ dx \right)^p + \alpha^p \int_0^1 [\log(e/x)]^{\alpha p - 1} \left(\int_0^x f(y)\ dy \right)^p \frac{dx}{x} \\ \leq \int_0^1 x^p \,[\log(e/x)]^{(1+\alpha)p-1}\, f^p(x)\ \frac{dx}{x}, \tag{1.5}$$

where f is a nonnegative measurable function on $[0, 1]$ and $p > 1,\ \alpha > 0$.

We, in this paper, generalize the inequalities (1.5) and (1.4) to a Hardy-type inequality with general weight by a simple elementary method different from that in [4].

2. A generalization and simple proof

Theorem 2.1. *Let $1 \leq p < \infty$ and $a < b < \infty$. Let H be a monotone function having continuous derivative on $[a, b]$. Then the following inequalities remain valid for all nonnegative measurable functions f on $[a, b]$:*

$$pe^{-H(b)} \left(\int_a^b f(t)dt \right)^p + \int_a^b \left(\int_a^x f(t)dt \right)^p \left| \left(e^{-H(x)} \right)' \right|\ dx \\ \leq\ p^p \int_a^b f^p(x) \left| \left(e^{-H(x)} \right)' \right| |H'(x)|^{-p}\ dx \tag{2.1}$$

if H is increasing;

$$pe^{-H(a)} \left(\int_a^b f(t)dt \right)^p + \int_a^b \left(\int_x^b f(t)dt \right)^p \left| \left(e^{-H(x)} \right)' \right|\ dx \\ \leq\ p^p \int_a^b f^p(x) \left| \left(e^{-H(x)} \right)' \right| |H'(x)|^{-p}\ dx \tag{2.2}$$

if H is decreasing.

Proof. By the density argument, it is sufficient to prove the inequalities for f continuous on $[a,b]$. Fix such a nonnegative function f. Let $\nu = \exp(-H)$.

For (2.1), we find ν decreasing and let $F(x) = \int_a^x f(t)\, dt$. Noting that $F'(x) = f(x)$ and that $F(a) = 0$, integration by parts gives

$$-\int_a^b \nu'(x)F^p(x)\, dx = -\nu(b)F^p(b) + p\int_a^b \nu(x)F^{p-1}(x)f(x)\, dx. \tag{2.3}$$

Applying Hölder inequality and the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} &p\int_a^b \nu(x)F^{p-1}(x)f(x)dx \\ \le& p\left(-\int_a^b \nu'(x)F^p(x)dx\right)^{(p-1)/p}\left(\int_a^b [\nu(x)]^p[-\nu'(x)]^{-p+1}f^p(x)dx\right)^{1/p} \\ \le& \frac{p-1}{p}\left(-\int_a^b \nu'(x)F^p(x)dx\right) + \frac{p^p}{p}\left(\int_a^b [\nu(x)]^p[-\nu'(x)]^{-p+1}f^p(x)dx\right). \end{aligned} \tag{2.4}$$

Now, a simple arrangement and calculation after substituting (2.4) into (2.3) makes

$$p\nu(b)F^p(b) - \int_a^b F^p(x)\nu'(x)dx \le -p^p\int_a^b f^p(x)\nu'(x)\left[\left(\log\frac{1}{\nu(x)}\right)'\right]^{-p} dx. \tag{2.5}$$

Substituting $H = -\log\nu$ in (2.5) yields (2.1).

For (2.2), we see that ν is increasing and let $G(x) = \int_x^b f(t)\, dt$. Noting that $G'(x) = -f(x)$ for $x \in (a,b)$ and that $G(b) = 0$, integration by parts gives

$$\int_a^b \nu'(x)G^p(x)\, dx = -\nu(a)G^p(a) + p\int_a^b \nu(x)G^{p-1}(x)f(x)\, dx. \tag{2.6}$$

It follows as before that

$$\begin{aligned} &p\int_a^b \nu(x)G^{p-1}(x)f(x)\, dx \\ \le\ & p\left(\int_a^b \nu'(x)G^p(x)\, dx\right)^{(p-1)/p}\left(\int_a^b [\nu(x)]^p[\nu'(x)]^{-p+1}f^p(x)\, dx\right)^{1/p} \\ \le\ & \frac{p-1}{p}\left(\int_a^b \nu'(x)G^p(x)\, dx\right) + \frac{p^p}{p}\left(\int_a^b [\nu(x)]^p[\nu'(x)]^{-p+1}f^p(x)\, dx\right) \end{aligned} \tag{2.7}$$

by Hölder inequality and the arithmetic-geometric mean inequality. Now, a simple arrangement and calculation after substituting (2.7) into (2.6) makes

$$p\nu(a)G^p(a) + \int_a^b G^p(x)\nu'(x)dx \le p^p\int_a^b f^p(x)\nu'(x)\left[\left(\log\frac{1}{\nu(x)}\right)'\right]^{-p} dx, \tag{2.8}$$

whence substituting $H = -\log \nu$ in (2.8) gives (2.2). $\square$

Remark 2.2. (1) By taking $\alpha > 0, p > 1, a = 0, b = 1$ and $e^{-H(x)} = v(x) = \frac{1}{\alpha p}\left(\log \frac{e}{x+\epsilon}\right)^{\alpha p}$ in (2.1) and letting $\epsilon \to 0$, it reduces to (1.5).

Also, by taking $\alpha > 0, p > 1, a = 0, b = 1$ and $e^{-H(x)} = v(x) = \frac{1}{\alpha p}\left(\log \frac{e}{x+\epsilon}\right)^{-\alpha p}$ in (2.2) and letting $\epsilon \to 0$, it reduces to (1.4).

(2) Concerning a further extension, the inequality (1.4) was extended in [4] as

$$\begin{aligned}\alpha^{p-1}\left(\int_0^1 f(x)\,dx\right)^p + \alpha^p \int_0^1 [\log(e/x)]^{\alpha p-1}\left(\int_x^1 f(y)\,dy\right)^p \frac{dx}{x} \\ \leq \int_0^1 x^p\,[\log(e/x)]^{(1-\alpha)p-1} f^p(x)\,\frac{dx}{x},\end{aligned} \tag{2.9}$$

where f is a nonnegative measurable function on $[0,1]$ and $p > 1,\ \alpha > 0$.

We can check by taking simply $f = 1$ and $\alpha = 1$ that the right hand side is less than 1 while the left hand side is bigger than 1, whence (2.9) is a mistake.

References

[1] G. H. Hardy, *Notes on some points in the integral calculus, LXIV. Further inequalities between integrals*, Messenger Math. 57 (1928), 12-16.

[2] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, 2nd ed., Cambridge University Press, 1952.

[3] C. Bennett, *Intermediate spaces and the class* $L\log^+ L$, Ark. Math. 11 (1973), 215-228.

[4] Sorina Barza, Lars-Erik Persson and Natasha Samko, *Some new sharp limit Hardy-type inequalities via convexity*, J. Inequal. App. 2014 (2014), Article ID 6.

Ern Gun Kwon
Department of Math-Education, Andong National University, 36729 Korea
E-mail address: egkwon@anu.ac.kr