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SYMBOLS OF MINIMUM TYPE AND OF ZERO CLASS IN EXPONENTIAL CALCULUS

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Abstract. We introduce formal symbols of product type, of zero class, and of minimum type and show that the formal power series representations for e^p and e^q are formal symbols of product type giving the same pseudodifferential operator, where p and q are formal symbols of minimum type and $p - q$ is of zero class.

1. Introduction

Pseudodifferential operators are essential for studying the theory of partial differential equations and itself have been studied in various respects. A pseudodifferential operator can be represented as an equivalent class of some function defined on the cotangent bundle. We call this function a symbol of pseudodifferential operator. We can represent the operations of addition, scalar multiplication, composition, formal adjoint and change of variables of pseudodifferential operators by using these symbols and so can treat pseudodifferential operators concretely. Symbolic calculus means calculating pseudodifferential operators by using symbols. Exponential calculus means symbolic calculus on pseudodifferential operators expressed as symbols of exponential functions. A symbol of exponential function means that the symbol can be expressed as a form of exponential function. We can analyze (pseudo) differential operators of infinite order or (pseudo) differential equations of infinite order by studying the pseudodifferential operators expressed as symbols of exponential functions. In particular, Sato, M., T. Kawai and M. Kashiwara (ref. [17]) won epoch-making results in the theory of transformation of the system of linear partial (pseudo) differential equations by using pseudodifferential operators of infinite order. T. Aoki accomplished exponential calculus of analytic pseudodifferential operators(ref. [1] – [8]). The author considers the case of a kind of the direct product structure introducing symbols, and moreover formal symbols of several types (ref. [12] – [16]). That is, the author introduces and studies calculus of analytic pseudodifferential operators of product type and extends the theory of T. Aoki

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in some sense. This study is deeply connected with exponential calculus of positive definite operators of infinite order which have deep relation to the energy method in the hyperfunction theory(ref. [9] – [11]). In this article, we introduce formal symbols of product type, of zero class, and of minimum type and show that the formal power series representations for e^p and e^q are formal symbols of product type giving the same pseudodifferential operator, where p and q are formal symbols of minimum type and $p-q$ is of zero class. The formal symbols of minimum type play a decisive role in exponential calculus.

2. Product Type and Minimum Type

Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be domains. Then, the cotangent bundles T^*X and T^*Y are identified with $X \times \mathbb{C}^n$ and $Y \times \mathbb{C}^m$, respectively. We set

$$S^*X := (T^*X - X)/\mathbb{R}^+,\ S^*Y := (T^*Y - Y)/\mathbb{R}^+$$

and define the mapping γ as

$$\gamma\colon \overset{\circ\circ}{T^*}(X \times Y) \ni (z, w; \xi, \eta) \longmapsto \left(z; \frac{\xi}{|\xi|}\right) \times \left(w; \frac{\eta}{|\eta|}\right) \in S^*X \times S^*Y,$$

where $|\xi| := \max\{|\xi_i|; \xi_i \in \mathbb{C}, 1 \leq i \leq n\}$, $|\eta| := \max\{|\eta_i|; \eta_i \in \mathbb{C}, 1 \leq i \leq m\}$, and

$$\overset{\circ\circ}{T^*}(X \times Y) := T^*(X \times Y) \backslash \{(T^*X \times Y) \cup (X \times T^*Y)\}.$$

For $d_1, d_2 > 0$ and any nonempty open subset U of $S^*X \times S^*Y$, we use the notation

$$\gamma^{-1}(U; d_1, d_2) := \gamma^{-1}(U) \cap \{|\xi| > d_1, |\eta| > d_2\}.$$

Hereafter we write (z, ξ, w, η) for coordinates $(z, w; \xi, \eta)$.

Let K be a compact subset of $S^*X \times S^*Y$.

Definition 2.1. A function $\Lambda : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ is said to be infra-linear if the following conditions hold:

(1) Λ is continuous.
(2) For each $\alpha > 1$, $\Lambda(\alpha t) \leq \alpha\Lambda(t)$ on $(0, \infty)$.
(3) Λ is increasing.
(4) $\displaystyle\lim_{t\to\infty} \frac{\Lambda(t)}{t} = 0$.

Definition 2.2. A formal series $\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)$ is called a formal symbol of product type on K if the following hold:

(1) There are some constants $d > 0, 0 < A < 1$, and an open set $U \supset K$ in $S^*X \times S^*Y$ such that $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.
(2) There exists an infra-linear function Λ such that

$$|P_{j,k}(z, \xi, w, \eta)| \leq A^{j+k} e^{\Lambda(|\xi|+|\eta|)} \quad \text{on } \gamma^{-1}(U; (j+1)d, (k+1)d) \tag{1}$$

for each $j, k \geq 0$.

We denote by $\widehat{S}(K)$ the set of such formal symbols on K.
We often write a formal power series $\sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z,\xi,w,\eta)$ with indeterminates t_1 and t_2 instead of $\sum_{j,k=0}^{\infty} P_{j,k}(z,\xi,w,\eta)$.

Definition 2.3. We denote by $\widehat{R}(K)$ the set of all $P(t_1,t_2;z,\xi,w,\eta) := \sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z,\xi,w,\eta)$ in $\widehat{S}(K)$ such that there are some constants $d > 0, 0 < A < 1$, an infra-linear function Λ, and an open set $U \supset K$ in $S^*X \times S^*Y$ satisfying the following:

$$\left| \sum_{\substack{0 \le j \le s \\ 0 \le k \le t}} P_{j,k}(z,\xi,w,\eta) \right| \le A^{\min\{s,t\}} e^{\Lambda(|\xi|+|\eta|)} \tag{2}$$

on $\gamma^{-1}(U;(s+1)d,(t+1)d)$ for each $s,t \ge 0$.

We call an element of $\widehat{R}(K)$ a formal symbol of zero class. We can obtain the following two propositions from the above definitions.

Proposition 2.4. $\widehat{S}(K)$ is a commutative ring under the sum and the product as formal power series in t_1 and t_2. (ref. [12])

Proposition 2.5. $\widehat{R}(K)$ is an ideal in $\widehat{S}(K)$. (ref. [12])

And we can define the equivalent class of a formal symbol of product type as follows.

Definition 2.6. We call an element in the ring $\widehat{S}(K)/\widehat{R}(K)$ a pseudodifferential operator of product type on K. We write $: \sum P_{j,k} :$ for the associated pseudodifferential operator of product type on K using an element $\sum P_{j,k}$ in $\widehat{S}(K)$.

Let $\Lambda_1(t),\ \Lambda_2(t^*)$ be infra-linear functions of t , t^*, respectively and put

$$\tilde{\Lambda}(\xi,\eta) = \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}.$$

Then for studying exponential calculus, we need the definition of a formal symbol of minimum type as a role of exponent.

Definition 2.7. A formal series $\displaystyle\sum_{j,j'=0}^{\infty} {t_1}^j {t_2}^{j'} p_{j,j'}(z,\xi,w,\eta)$ is called a formal symbol of minimum type defined on K if there exist $\tilde{\Lambda}$, some positive constants $d,\ 0 < A < 1$, and some open set $U \supset K$ such that the following hold:

(1) $p_{j,j'}$ is holomorphic in $\gamma^{-1}(U;(j+1)d,(j'+1)d)$ for each $j,j' \ge 0$.
(2) The inequality

$$|p_{j,j'}(z,\xi,w,\eta)| \le A^{j+j'} \tilde{\Lambda}(\xi,\eta) \tag{3}$$

holds on $\gamma^{-1}(U;(j+1)d,(j'+1)d)$ for each $j,j' \ge 0$.

Theorem 2.1. *If* $p = \sum_{j,j'=0}^{\infty} {t_1}^j {t_2}^{j'} p_{j,j'}(z,\xi,w,\eta)$ *is a formal symbol of minimum type defined on* K, *the formal power series representation for* e^p *is a formal symbol of product type on* K. *(ref.* [16]*)*

Theorem 2.2. *If* $p = \sum_{j,j'=0}^{\infty} {t_1}^j {t_2}^{j'} p_{j,j'}(z,\xi,w,\eta)$ *is a formal symbol of minimum type and of zero class defined on* K, $e^p - 1$ *is a formal symbol of zero class on* K.

Proof. $e^p \in \widehat{S}(K)$ by the theorem 2.1. To prove that $e^p - 1$ belongs to $\widehat{R}(K)$ we express e^p as a formal power series with indeterminates t_1 and t_2.
If we set
$\sum_{j,j'=0}^{\infty} {t_1}^j {t_2}^{j'} e_{j,j'}(z,\xi,w,\eta) := e^p,$
then, we can obtain the following coefficients.

$$e_{0,0} = e^{p_{0,0}} \tag{4}$$

$$e_{j,j'} = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\substack{j_1+\cdots+j_k=j \\ j'_1+\cdots+j'_k=j'}} \prod_{\nu=1}^{k} p_{j_\nu, j'_\nu}, \quad (j,j') \neq (0,0). \tag{5}$$

Then the following estimations hold on $\gamma^{-1}(U;(s+1)d,(t+1)d)$ for each $s,t \geq 0$, where we interpret as zero each second sigma notation in the last three terms appearing after the inequality sign in the case that the notation is meaningless for some s and t :

$$\begin{aligned}
\left| \sum_{\substack{0\leq j\leq s \\ 0\leq j'\leq t}} e_{j,j'}(z,\xi,w,\eta) - 1 \right| &= \left| \sum_{\substack{0\leq j\leq s \\ 0\leq j'\leq t}} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\substack{j_1+\cdots+j_k=j \\ j'_1+\cdots+j'_k=j'}} \prod_{\nu=1}^{k} p_{j_\nu, j'_\nu} \right| \\
&= \left| \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\substack{0\leq j\leq s \\ 0\leq j'\leq t}} \sum_{\substack{j_1+\cdots+j_k=j \\ j'_1+\cdots+j'_k=j'}} \prod_{\nu=1}^{k} p_{j_\nu, j'_\nu} \right| \\
&\leq \left(\left| \sum_{\substack{0\leq j\leq s \\ 0\leq j'\leq t}} p_{j,j'} \right| + \left| \sum_{k=2}^{\infty} \frac{1}{k!} \Big(\sum_{\substack{0\leq j\leq s \\ 0\leq j'\leq t}} p_{j,j'} \Big)^k \right| \right)
\end{aligned}$$

$$+\left|\sum_{k=2}^{\infty}\frac{1}{k!}\sum_{\substack{0\le j\le s\\ t+1\le j'\le tk}}\ \sum_{\substack{j_1+\cdots+j_k=j\\ j'_1+\cdots+j'_k=j'\\ 0\le j_1,\cdots,j_k\\ 0\le j'_1,\cdots,j'_k\le t}}\prod_{\nu=1}^{k}p_{j_\nu,j'_\nu}\right|$$

$$+\left|\sum_{k=2}^{\infty}\frac{1}{k!}\sum_{\substack{s+1\le j\le sk\\ 0\le j'\le t}}\ \sum_{\substack{j_1+\cdots+j_k=j\\ j'_1+\cdots+j'_k=j'\\ 0\le j_1,\cdots,j_k\le s\\ 0\le j'_1,\cdots,j'_k}}\prod_{\nu=1}^{k}p_{j_\nu,j'_\nu}\right|$$

$$+\left|\sum_{k=2}^{\infty}\frac{1}{k!}\sum_{\substack{s+1\le j\le sk\\ t+1\le j'\le tk}}\ \sum_{\substack{j_1+\cdots+j_k=j\\ j'_1+\cdots+j'_k=j'\\ 0\le j_1,\cdots,j_k\le s\\ 0\le j'_1,\cdots,j'_k\le t}}\prod_{\nu=1}^{k}p_{j_\nu,j'_\nu}\right|.$$

We can estimate the first of the above four terms as follows:

$$\begin{aligned}
\text{The first term } &\le \left|\sum_{\substack{0\le j\le s\\ 0\le j'\le t}}p_{j,j'}\right|\left|\sum_{k=1}^{\infty}\frac{1}{k!}\right|\left|\Big(\sum_{\substack{0\le j\le s\\ 0\le j'\le t}}p_{j,j'}\Big)^{k-1}\right|\\
&\le A^{\min\{s,t\}}e^{\Lambda(|\xi|+|\eta|)}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{\substack{0\le j\le s\\ 0\le j'\le t}}|\ p_{j,j'}\ |\right)^{k}\\
&\le A^{\min\{s,t\}}e^{\Lambda(|\xi|+|\eta|)}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{\substack{0\le j\le s\\ 0\le j'\le t}}A^{j+j'}\ \tilde\Lambda(\xi,\eta)\right)^{k}\\
&= A^{\min\{s,t\}}e^{\Lambda(|\xi|+|\eta|)}\sum_{k=0}^{\infty}\frac{1}{k!}(\tilde\Lambda(\xi,\eta))^{k}\left(\sum_{\substack{0\le j\le s\\ 0\le j'\le t}}A^{j+j'}\right)^{k}
\end{aligned}$$

$$
\begin{aligned}
&\le A^{\min\{s,t\}} e^{\Lambda(|\xi|+|\eta|)} \sum_{k=0}^{\infty} \frac{1}{k!} (\tilde{\Lambda}(\xi,\eta))^k ((1+A+\cdots)(1+A+\cdots))^k \\
&= A^{\min\{s,t\}} e^{\Lambda(|\xi|+|\eta|)} \sum_{k=0}^{\infty} \frac{1}{k!} (\tilde{\Lambda}(\xi,\eta))^k (\frac{1}{1-A})^{2k} \\
&= A^{\min\{s,t\}} e^{\Lambda(|\xi|+|\eta|)} e^{(1-A)^{-2}\tilde{\Lambda}(\xi,\eta)} \\
&\le A^{\min\{s,t\}} e^{\Lambda(|\xi|+|\eta|)} e^{(1-A)^{-2}(\Lambda_1(|\xi|+|\eta|)+\Lambda_2(|\xi|+|\eta|))}.
\end{aligned}
$$

If we let $B = A(2-A)$ and $C = 1/(2-A)$, then we see that $A = BC, A < B < 1$, and $A < C < 1$.
We thus can estimate the second, third, fourth term as follows:

The second term

$$
\begin{aligned}
&\le \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{\substack{0\le j\le s \\ t+1\le j'\le tk}} \sum_{\substack{j_1+\cdots+j_k=j \\ j'_1+\cdots+j'_k=j' \\ 0\le j_1,\cdots,j_k \\ 0\le j'_1,\cdots,j'_k\le t}} \prod_{\nu=1}^{k} (A^{j_\nu+j'_\nu}\ \tilde{\Lambda}(\xi,\eta)) \\
&= \sum_{k=2}^{\infty} \frac{1}{k!} (\tilde{\Lambda}(\xi,\eta))^k \sum_{\substack{0\le j\le s \\ t+1\le j'\le tk}} \sum_{\substack{j_1+\cdots+j_k=j \\ j'_1+\cdots+j'_k=j' \\ 0\le j_1,\cdots,j_k \\ 0\le j'_1,\cdots,j'_k\le t}} A^{j_1+j_2+\cdots+j_k+j'_1+\cdots+j'_k} \\
&\le \sum_{k=2}^{\infty} \frac{1}{k!} (\tilde{\Lambda}(\xi,\eta))^k \left(\sum_{0\le j\le s} \sum_{j_1+\cdots+j_k=j} A^j \right) \left(\sum_{t+1\le j'\le tk} \sum_{j'_1+\cdots+j'_k=j'} A^{j'} \right) \\
&\le \sum_{k=2}^{\infty} \frac{1}{k!} (\tilde{\Lambda}(\xi,\eta))^k (1+A+A^2+\cdots)^k \left(\sum_{t+1\le j'\le tk} \sum_{j'_1+\cdots+j'_k=j'} C^{j'} B^{j'} \right) \\
&\le C^{t+1} \sum_{k=2}^{\infty} \frac{1}{k!} (\tilde{\Lambda}(\xi,\eta))^k (\frac{1}{1-A})^k \left(\sum_{t+1\le j'\le tk} \sum_{j'_1+\cdots+j'_k=j'} B^{j'} \right) \\
&\le C^{t+1} \sum_{k=2}^{\infty} \frac{1}{k!} (\tilde{\Lambda}(\xi,\eta))^k \left(\frac{1}{1-A}\right)^k \left(\frac{1}{1-B}\right)^k \\
&\le C^{t+1} e^{(1-A)^{-1}(1-B)^{-1}\tilde{\Lambda}(\xi,\eta)} \\
&\le C \cdot C^{\min\{s,t\}} e^{(1-A)^{-1}(1-B)^{-1}\tilde{\Lambda}(\xi,\eta)} \\
&\le C^{\min\{s,t\}} e^{(1-A)^{-1}(1-B)^{-1}(\Lambda_1(|\xi|+|\eta|)+\Lambda_2(|\xi|+|\eta|))}.
\end{aligned}
$$

In like manners, we can estimate the third term as follows:

$$
\begin{aligned}
\text{The third term } &\le \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{\substack{s+1\le j\le sk \\ 0\le j'\le t}} \sum_{\substack{j_1+\cdots+j_k=j \\ j_1'+\cdots+j_k'=j' \\ 0\le j_1,\cdots,j_k\le s \\ 0\le j_1',\cdots,j_k'}} \prod_{\nu=1}^{k} (A^{j_\nu+j_\nu'}\tilde{\Lambda}(\xi,\eta)) \\
&\le C^{s+1} e^{(1-A)^{-1}(1-B)^{-1}\tilde{\Lambda}(\xi,\eta)} \\
&\le C\cdot C^{\min\{s,t\}} e^{(1-A)^{-1}(1-B)^{-1}\tilde{\Lambda}(\xi,\eta)} \\
&\le C^{\min\{s,t\}} e^{(1-A)^{-1}(1-B)^{-1}(\Lambda_1(|\xi|+|\eta|)+\Lambda_2(|\xi|+|\eta|))}.
\end{aligned}
$$

Finally we obtain the estimations of the fourth term:

The fourth term

$$
\begin{aligned}
&\le \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{\substack{s+1\le j\le sk \\ t+1\le j'\le tk}} \sum_{\substack{j_1+\cdots+j_k=j \\ j_1'+\cdots+j_k'=j' \\ 0\le j_1,\cdots,j_k\le s \\ 0\le j_1',\cdots,j_k'\le t}} \prod_{\nu=1}^{k} (A^{j_\nu+j_\nu'}\tilde{\Lambda}(\xi,\eta)) \\
&= \sum_{k=2}^{\infty} \frac{1}{k!} (\tilde{\Lambda}(\xi,\eta))^k \sum_{\substack{s+1\le j\le sk \\ t+1\le j'\le tk}} \sum_{\substack{j_1+\cdots+j_k=j \\ j_1'+\cdots+j_k'=j' \\ 0\le j_1,\cdots,j_k\le s \\ 0\le j_1',\cdots,j_k'\le t}} A^{j_1+j_2+\cdots+j_k+j_1'+\cdots+j_k'} \\
&\le \sum_{k=2}^{\infty} \frac{1}{k!} (\tilde{\Lambda}(\xi,\eta))^k \left(\sum_{s+1\le j\le sk} \sum_{j_1+\cdots+j_k=j} A^j \right) \left(\sum_{t+1\le j'\le tk} \sum_{j_1'+\cdots+j_k'=j'} A^{j'} \right) \\
&= \sum_{k=2}^{\infty} \frac{1}{k!} (\tilde{\Lambda}(\xi,\eta))^k \left(\sum_{s+1\le j\le sk} \sum_{j_1+\cdots+j_k=j} C^j B^j \right) \left(\sum_{t+1\le j'\le tk} \sum_{j_1'+\cdots+j_k'=j'} C^{j'} B^{j'} \right) \\
&\le C^{s+1}\cdot C^{t+1} \sum_{k=2}^{\infty} \frac{1}{k!} (\tilde{\Lambda}(\xi,\eta))^k \left(\frac{1}{1-B}\right)^k \left(\frac{1}{1-B}\right)^k \\
&\le C^{\min\{s,t\}} e^{(1-B)^{-2}\tilde{\Lambda}(\xi,\eta)} \\
&\le C^{\min\{s,t\}} e^{(1-B)^{-2}(\Lambda_1(|\xi|+|\eta|)+\Lambda_2(|\xi|+|\eta|))}.
\end{aligned}
$$

Consequently, the following inequality holds on $\gamma^{-1}(U;(s+1)d,(t+1)d)$ for each $s,t\geq 0$:

$$\left|\sum_{\substack{0\leq j\leq s\\0\leq j'\leq t}} e_{j,j'}(z,\xi,w,\eta)-1\right|\leq C^{\min\{s,t\}}e^{\Lambda^*(|\xi|+|\eta|)},$$

where $\Lambda^*=2+\Lambda+(1-B)^{-2}\Lambda_1+(1-B)^{-2}\Lambda_2$ and $\Lambda^*$is infra-linear.

This completes the proof of the Theorem 2.2. □

We then obtain the main result in this article.

Theorem 2.3. *Suppose that* $p(t_1,t_2;z,\xi,w,\eta)=\sum_{j,j'=0}^{\infty} {t_1}^j {t_2}^{j'} p_{j,j'}(z,\xi,w,\eta)$ *and* $q(t_1,t_2;z,\xi,w,\eta)=\sum_{j,j'=0}^{\infty} {t_1}^j {t_2}^{j'} q_{j,j'}(z,\xi,w,\eta)$ *are formal symbols of minimum type defined on* K. *If* $:p(t_1,t_2;z,\xi,w,\eta):=:q(t_1,t_2;z,\xi,w,\eta):$, *then*

$$:e^{p(t_1,t_2;z,\xi,w,\eta)}:=:e^{q(t_1,t_2;z,\xi,w,\eta)}:.$$

Proof. Since $p(t_1,t_2;z,\xi,w,\eta)-q(t_1,t_2;z,\xi,w,\eta)$ is of zero class on K,

$$e^{p(t_1,t_2;z,\xi,w,\eta)-q(t_1,t_2;z,\xi,w,\eta)}-1\in\widehat{R}(K)$$

by the theorem 2.2. Therefore, we can conclude that the following holds by the proposition 2.5 and the theorem 2.1.

$$e^{p(t_1,t_2;z,\xi,w,\eta)}-e^{q(t_1,t_2;z,\xi,w,\eta)}\in\widehat{R}(K)$$

This means that

$$:e^{p(t_1,t_2;z,\xi,w,\eta)}:=:e^{q(t_1,t_2;z,\xi,w,\eta)}:.$$

□

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