East Asian Math. J.
Vol. 34 (2018), No. 1, pp. 039–046
http://dx.doi.org/10.7858/eamj.2018.005

JACOBSON RADICAL AND NILPOTENT ELEMENTS

Chan Huh* †, Jeoung Soo Cheon, and Sun Hye Nam

Abstract. In this article we consider rings whose Jacobson radical contains all the nilpotent elements, and call such a ring an NJ-ring. The class of NJ-rings contains NI-rings and one-sided quasi-duo rings. We also prove that the Koethe conjecture holds if and only if the polynomial ring $R[x]$ is NJ for every NI-ring R.

1. Introduction

Throughout R denotes an associative ring with identity unless otherwise stated. An element $a \in R$ is *nilpotent* if $a^n = 0$ for some integer $n \geq 1$, and an (one-sided) ideal is nil if all the elements are nilpotent. R is *reduced* if it has no nonzero nilpotent elements. For a ring R, $Nil(R)$, $N(R)$, and $J(R)$ denote the set of all the nilpotent elements, the nil radical, and the Jacobson radical of R, respectively. Note that $N(R) \subseteq Nil(R)$ and $N(R) \subseteq J(R)$. Due to Marks [14], R is called an *NI-ring* if $Nil(R) \subseteq N(R)$ (or equilvalently $N(R) = Nil(R)$). Thus R is NI if and only if $Nil(R)$ forms an ideal if and only if the factor ring $R/N(R)$ is reduced. Hong et al [8, corollary 13] proved that R is NI if and only if every minimal strongly prime ideal of R is completely prime. Since $N(R) \subseteq J(R)$, it is natural to consider the rings in which $J(R)$ contains $Nil(R)$. We call R an *NJ-ring* if $Nil(R) \subseteq J(R)$. Note that an element $a \in R$ is *left quasi-regular* if $1 - a$ is left invertible, and a one-sided ideal I is left quasi-regular if every element of I is left quasi-regular. Since $J(R)$ is the (unique) largest left quasi-regular left ideal and contains all the left quasi-regular left ideals [7, Theorem 1.2.3], a ring R is NJ if and only if ra is left quasi-regular for every $a \in Nil(R)$ and $r \in R$.

Remark 1.1. If R is an NJ ring and $ab = 1$ for $a, b \in R$, then $ba = 1$. To see this, first note that ba and $1 - ba$ are idempotents and $(1 - ba)b = b - bab = b - b(ab) = 0$. So $[b(1 - ba)]^2 = 0$, and $b(1 - ba) \in J(R)$. Now $1 - ba =$

Received November 21, 2017; Accepted January 24, 2018.
2010 *Mathematics Subject Classification.* Primary 16D, 16S, 16W.
Key words and phrases. Jacobson Radical and Nilpotent Elements.
† This work was supported by a 2-year Research Grant of Pusan National University.
∗ Corresponding author.

$ab(1-ba) = a(b-(1-ba)) \in J(R)$. This means that the idempotent $1-ba$ is left quasi-regular, so must be zero, thus $ba = 1$.

A ring R is called *left(resp. right) quasi-duo* [17] if every maximal left(resp. right) ideal is two sided, thus R is left(resp. right) quasi-duo if and only if every left(resp. right) primitive factor ring of R is a division ring. Since $J(R)$ is the intersection of all the (left) primitive ideals, every one-sided quasi-duo rings is NJ. By Remark 1.1, if R is a one-sided quasi-duo ring and $ab = 1$ for $a, b \in R$ then $ba = 1$.

A ring R is called *semicommutative* if $ab = 0$ for $a, b \in R$ implies that $aRb = 0$. If R is semicommutative and $a^2 = 0$ for $a \in R$, then $aRa = 0$. This means that $(Ra)^2 = 0$, hence $a \in Ra \subseteq J(R)$. So R is NJ.

2. Examples and Properties of NJness

In this section, we investigate properties of NJ-rings and construct several examples related to the rings.

NJ-rings need not be NI or one-sided quasi-duo, by the following examples. For a ring R and an integer $n \geq 1$, $Mat_n(R)$ denotes the $n \times n$ matrix ring over R.

Example 2.1. (1) Let F be any field of characteristic 0 and
$A = M_2(F), B = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a, b \in F \right\}$ and $C = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \;\middle|\; b \in F \right\}$.
Then B is a subring of A and C is a nilpotent ideal of B. Let $R = B + A[[x]]x$, then $J(R) = C + A[[x]]x$. This means that R is NJ. On the other hand, consider $f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$ and $g(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x \in N(R)$. Then $(f+g)^k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^k x^k \neq 0$ for each $k \geq 1$. Therefore, R is NJ which is not NI.

(2) If a field F has characteristic 0, then the first Weyl algebra $W(F)$ over F is a simple domain but not a division ring, So $W(F)$ is NJ, but not left quasi-duo.

For an ideal I of R and an idempotent $e = e^2 \in R$, $J(I) = I \cap J(R)$ and $J(eRe) = eRe \cap J(R)$. Hence the followings hold.

Proposition 2.2. *(1) R is NJ if and only if so is any ideal I of R (as a ring which may not have identity).*
(2) R is NJ if and only if so is eRe for any idempotent $0 \neq e$ in R.

Theorem 2.3. *Let $\{R_\lambda | \lambda \in \Lambda\}$ be a class of NJ rings. Then we have the following :*
(1) $\prod_{\lambda \in \Lambda} R_\lambda$ is NJ.

(2) If $\{R_\lambda | \lambda \in \Lambda\}$ is a directed system, then the direct limit of $\{R_\lambda | \lambda \in \Lambda\}$ is NJ.
(3) If the index set Λ is finite, then the subdirect product of R_λ is NJ.

Proof. (1) holds by the fact that

$$J(\prod_{\lambda \in \Lambda} R_\lambda) = \prod_{\lambda \in \Lambda} J(R_\lambda).$$

(2) is trivial by the definition.
For (3), it satisfies to show that the subdirect product R of two NJ rings R_1 and R_2 is also NJ. By the property of subdirect products, there are two ideals A_1 and A_2 of R such that $A_1 \cap A_2 = 0$ and $R_i \cong R/A_i$ for any $i = 1, 2$. Let $r \in R$ and $x \in Nil(R)$. Since R_1 and R_2 are NJ, we have the following : $b_1(1-rx) = 1+a_1$ and $b_2(1 - rx) = 1 + a_2$, for some $b_1, b_2 \in R$, $a_1 \in A$, $a_2 \in A_2$. This implies that $[b_1 + b_2 - b_1(1 - rx)b_2](1 - rx) = 1 - a_1a_2 = 1$, since $a_1a_2 \in A_1 \cap A_2 = 0$. Therefore, $1 - rx$ is left invertible, entailing that R is NJ. □

Remark 2.4. For a ring R,
(1) if R is one-sided quasi-duo, then $\frac{R}{J(R)}$ is reduced.
(2) $\frac{R}{J(R)}$ is reduced if and only if $\frac{R}{J(R)}$ is NJ.
(3) if $\frac{R}{J(R)}$ is NJ, then R is NJ.

Proof. (1) If $a^2 \in J(R)$, then $a^2 \in P$ for every left primitive ideal P of R. Hence $a \in P$ since $\frac{R}{P}$ is a division ring, and so $a \in \bigcap\{P \mid P$ a left primitive ideal of $R\} = J(R)$. Thus $\frac{R}{J(R)}$ is reduced.
(2) Since reduced rings are NJ, it suffices to prove that the sufficient condition. Suppose $\frac{R}{J(R)}$ is NJ and $a^2 \in J(R)$; then $\bar{a} = a + J(R)$ is nilpotent, so $\bar{a} = a + J(R) \in J(\frac{R}{J(R)}) = (0)$. Thus $\frac{R}{J(R)}$ is reduced.
(3) If $a \in R$ and $a^2 = 0$, then $\bar{a}^2 = 0$ in $\frac{R}{J(R)}$ and so $\bar{a} = 0$. This shows that $a \in J(R)$. □

By Remark 2.4 (3) if $\frac{R}{J(R)}$ is NJ then R is NJ, but the converse is not true in general by the following example.

Example 2.5. Let $A = \{\frac{b}{a} \mid a, b \in \mathbb{Z},\ 3 \nmid a\}$ be the localization of $\mathbb{Z}$ at 3 and let $B = \{\frac{3b}{a} \mid a, b \in \mathbb{Z},\ 3 \nmid a\}$ the unique maximal ideal of A. Then $B = J(A)$ and $A/B \cong \mathbb{Z}_3$. Let $R = A + Ai + Aj + Ak$ be a subring of the Hamilton quaternions and $M = B + Bi + Bj + Bk = 3R$. Trivially R is NJ.

Now we claim that $M = J(R)$. First we prove that M is left quasi-regular. Indeed, let $0 \neq \alpha \in M$ then $\alpha = \frac{3}{s}(a + bi + cj + dk)$, where $a, b, c, d \in \mathbb{Z}$ and $3 \nmid s$ and so that $1 - \alpha = \frac{s-3a}{s} + \frac{-3b}{s}i + \frac{-3c}{s}j + \frac{-3d}{s}k$. Put $t = s - 3a$, then $u = s/t \in A$ is a central unit in A and $u(1 - \alpha) = 1 + \frac{-3b}{t}i + \frac{-3c}{t}j + \frac{-3d}{t}k$. Let $\beta = 1 + \frac{3b}{t}i + \frac{3c}{t}j + \frac{3d}{t}k$, then $v\beta u(1-\alpha) = 1$ where $v = \frac{t^2}{t^2+9b^2+9c^2+9d^2} \in A$. Thus α is a left quasi-regualr element and M is a left quasi-regular ideal. In order to show that M is a maximal ideal of R, let $\alpha = \frac{1}{s}(a + bi + cj + dk) \in R$, $\alpha \notin M$, say $3 \nmid b$. Let $\beta = k\alpha - \alpha k$. Then $\beta = \frac{2}{s}(bj - ci) \notin M$ and $i\beta - \beta i = \frac{4}{s}bk$. Since $\frac{4}{s}b$ is a unit in A, we have $1 \in R\alpha R$. Thus $M = 3R$ is maximal. Hence $M = J(R)$. Now let $a = (i + j + k) + J(R)$, then $a \neq 0$ and $a^2 = 0$ in $\frac{R}{J(R)}$. Thus $\frac{R}{J(R)}$ is not reduced, hence is not NJ by Remark 2.4.

3. Related Rings to NJness

In this last section, we investigate related rings to NJ-rings and prove some equivalent conditions to the Koethe conjecture. Note that a subring S of R is *unital* if $1_R \in S$. The following results are given by Rowen [16].

Lemma 3.1. ([16, Proposition 2.5.17]). *Let S be a unital subring of a ring R.*
(1) If every elements of S which is invertible in R is already invertible in S, then $S \cap J(R) \subseteq J(S)$.
(2) If S is left Artinian then $S \cap J(R) \subseteq J(S)$ is nilpotent.

Note that $Nil(S) = S \cap Nil(R) \subseteq S \cap J(R)$ for any subring S of an NJ ring R.

Proposition 3.2. *For each case of Lemma 3.1, if R is NJ, then so is S.*

For any ring R, we have $J(R[[x;\theta]]) = J(R) + J[[x;\theta]]x$, where θ is an endomorphism of R. An endomorphism θ of R is *locally finite order* if for any $r \in R$ there is an integer $n \geq 1$, depending on r, such that $\theta^n(r) = r$. Thus, we conclude the following theorem.

Theorem 3.3. *Let θ ba an endomorphism of a ring R. Then R is NJ if and only if $R[[x;\theta]]$ is NJ.*

Due to Bedi and Ram [3, Theorem 3.1] for any automorphism θ of a ring R, we have the following:
(1) $J(R[x;\theta] = I \cap J(R) + I[x;\theta]x$;
(2) $J(R[x,x^-1;\theta]) = K[x,x^-1;\theta] \subseteq J(R)[x,x^-1;\theta]$ and $J(R[x,x^-1;\theta]) \cap R[x;\theta] \subseteq J(R[x;\theta])$, where $I = \{r \in R \mid rx \in J(R[x;\theta])\}$ and $K = J(R[x,x^-1;\theta]) \cap R$. In addition, if θ is of locally finite order then I and K are nil ideals, and so $J(R[x;\theta]) = I[x;\theta]$.

Lemma 3.4. *Let θ be an automorphism of locally finite order and $R[x;\theta]$ is NJ. Then R is NI and $J(R[x;\theta]) = N(R)[x;\theta]$.*

Proof. By [3, Theorem 3.1] $J(R[x;\theta]) = N[x;\theta]$ for some nil ideal N of R. Since $R[x]$ is NJ, $Nil(R) \subseteq J(R[x;\theta]) \subset N(R)[x;\theta]$. Thus $Nil(R) = N(R)$, and hence R is NI, $J(R[x;\theta]) = N(R)[x;\theta]$. Hence R is NI and $J(R[x;\theta]) = N(R)[x;\theta]$. □

Corollary 3.5. *If $R[x]$ is NJ, then R is NI and $J(R[x]) = N(R)[x]$.*

Remark 3.6. By Lemma 3.4 and [13, Theorem 4.1], if an automorphism θ of a ring R is of locally finite order, then $R[x;\theta]$ is NJ if and only if it is one-sided quasi-duo. Therefore, in this case, one-sided quasi-duo condition is left right symmetric.

Proposition 3.7. *For a ring R, the following conditions are equivalent:*
(1) $R[x]$ is NJ.
(2) R is NI and $J(R[x]) = N(R)[x]$.
(3) $\frac{R[x]}{J(R[x])}$ is reduced.
In particular, if $R[x]$ is NJ then so is $\frac{R[x]}{J(R[x])}$.

Proof. (1) $\Rightarrow$ (2) is by Corollary 3.5.
(2) $\Rightarrow$ (3) Suppose R is NI and $J(R[x]) = N(R)[x]$. Then $\frac{R[x]}{J(R[x])} = \frac{R[x]}{N(R)[x]} \cong (\frac{R}{N(R)})[x]$ is a reduced ring, since $\frac{R}{N(R)}$ is reduced.
(3) $\Rightarrow$ (1) is trivial. □

In 1930, G.Koethe raised the following question which is known as the Koethe conjecture:

"Does a ring R with nonzero one-sided nil ideal have
a nonzero two-sided ideal?"

In spite of great effort of many reserchers, it remains still open. However many equivalent properties have been found. Below we list some of them.

Remark 3.8. The followings are equivalent:
(1) The Koethe conjecture holds.
(2) For any ring R if A and B are left nil ideals then $A + B$ is nil.
(3) $J(R[x]) = N(R)[x]$ for any ring R.
(4) $N(Mat_2(R)) = Mat_2(N(R))$ for any ring R
(5) $N(Mat_n(R)) = Mat_n(N(R))$ for any ring R and integer $n \geq 1$

Proof. See [5, 6, 10, 11, 15]. □

Note that if R is NI, then $Nil(R)$ forms a subring, and if $Nil(R)$ forms a subring, then the sum of any two nil left ideals is nil. So the Koethe conjecture holds for this kind of rings.

There is an example of NJ-ring R in which $Nil(R)$ is not a subring, and an example of non NJ-ring in which $Nil(R)$ is a subring.

Example 3.9. (1) Let R be the ring in Example 2.1(1). Then R is NJ, but $Nil(R)$ is not a subring of R as can be seen by the nilpotent elements

$f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$ and $g(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x$.
(2) Let K be a field and $A = K\{a, b\}$ be the free algebra generated by the noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^2 and set $R = A/I$. Let $\bar{c} = c+I$ be the image of $c \in A$ in R. Then $Nil(R)$ forms a subring of R by [2, Corollary 3.3 and Example 4.8]. Note that $\bar{b} \in Nil(R)$. Assume $\bar{b} \in J(R)$, then there exists $\bar{c} \in R$ such that $1 = (1 - \bar{c})(1 - \bar{a}\bar{b}) = (1 - \bar{a}\bar{b})(1 - \bar{c})$. Then $\bar{c}\bar{a}\bar{b} = \bar{a}\bar{b}\bar{c} = \bar{c} + \bar{a}\bar{b}$. However, note that A is F-graded and I is a homogeneous ideal, so R is F-graded. Therefore (by comparing the degrees of the homogeneous components) the equalities $\bar{c}\bar{a}\bar{b} = \bar{c} + \bar{a}\bar{b}$ is impossible, since $\bar{a}\bar{b}\bar{a}\bar{b} \neq 0$, hence R is not NJ.

The converse of Corollary 3.5 is equivalent to the Koethe's conjecture, and the following is a main result of this article.

Theorem 3.10. *The following are equivalent.*
(1) The Koethe conjecture holds.
(2) For any ring R, $J(R[x]) = N(R)[x]$.
(3) For any NI-ring R, $R[x]$ is an NJ-ring.

Proof. (1) $\Leftrightarrow$ (2) is obtained by [10, Theorem 22].
(2) $\Rightarrow$ (3) Suppose R is NI, then by the condition(2) $J(R[x]) = N(R)[x] = Nil(R)[x]$. Thus $\frac{R[x]}{J(R[x])} = \frac{R[x]}{N(R)[x]} = (\frac{R}{N(R)})[x]$ is reduced, hence $R[x]$ is NJ.
(3) $\Rightarrow$ (2) Let R be a ring and $S = \{(m, a) \mid m \in \mathbb{Z}, a \in N(R)\}$.
Define addition and multiplication in S by

$$(m, a) + (m, b) = (m + n, a + b),\ (m, a)(m, b) = (mn, mb + na + ab)$$

for $m, n \in \mathbb{Z}$, $a, b \in N(R)$. Then S is an NI ring with identity $1_S = (1, 0)$ and $Nil(S) = \{\ (0, a) \mid a \in N(R)\} = N(S)$. So by condition (3), $S[x]$ is NJ and so $J(S[x]) = N(S)[x] \cong N(R)[x]$. Thus $N(R)[x](\cong J(S[x]))$ is a left quasi-regular ideal of $R[x]$, and hence $N(R)[x] \subseteq J(R[x])$. Therefore we have $J(R[x]) = N(R)[x]$ by a Theorem of Amitsur [1, Theorem 2.5.23]. $\square$

Using the above theorem, we can construct an example that NJ condition is not closed under subrings.

Example 3.11. Let R be the ring in Example 2.1(1). Then R is NJ but not NI. Thus, $S = R[[x]]$ is also NJ by the Theorem 3.3. However its subring $T = R[x]$ is not NJ by Lemma 3.4.

Due to Lam and Leory [12, 13] a subring S of a ring R is called a *(right) corner ring* of R if there exists and additive subgroup C of R such that $R = S \oplus C, CS \subseteq C$. The subgroup C is called a *complement* of S.

A corner ring S of a ring R is called *Peirce corner* if there is an idempotent $e = e^2 \in R$ such that $S = eRe$. Lam [12] showed that every corner ring of a ring R is a unital corner of some Peirce corner of R and is also a Peirce corner of some unital corner of R.

We have the following result on corner rings.

Theorem 3.12. *A ring R is NJ if and only if so is every (right) corner ring of R.*

Proof. If we choose $e = 1_R \in R$ then $R = eRe$ is a corner ring of itself. Thus, we only prove that the necessary condition of this theorem. Let R be an NJ ring. By Propositon 2.2(2) every Peirce corner ring of R is NJ. Now consider the case of right unital corner ring of R. Let S be a right unital corner ring of R. Let $a \in N(S)$ and $s \in S$. Since R is NJ, there is an element $1 - r \in R$ such that $(1 - r)(1 - sa) = 1$. By the definition $r = t + c$ with $t \in S,\ c \in C$. Now $c(1 - sa) = 1 - (1 - t)(1 - sa) = sa + t - tsa \in C \cap S = (0)$. Thus $1 = (1 - r)(1 - sa) = (1 - t)(1 - sa) - c(1 - sa) = (1 - t)(1 - sa)$, and hence sa is left quasi-regular in S. Therefore S is NJ.

□

Since R is a corner of $R[x; \theta, \delta]$ and also is a corner of upper triangular matrix rings of itself, we have the following corollary for any endomorphism θ and a θ-derivation δ of R.

Corollary 3.13. *(1) If $R[x; \theta, \delta]$ is NJ, then so is R.*
(2) R is NJ if and only if the $n \times n$ upper triangular matrix ring over R is NJ for any $n \geq 1$.

References

[1] S.Amitsur *Radicals of polynomial rings*, Canadian J.Math., 8(1956), 355-361

[2] R.Antoine *Nilpotent elements and Armendariz rings*, J.Algebra, 319(8), (1974), 3128-3140

[3] S.S.Bedi, J.Ram *Jacobson radical of skew polynomial rings and skew group rings*, Israel.J.Math. 35(1980), 327-338

[4] G.F.Birkenmeier, H.E.Heatherly, E.K.Lee *Completely prime ideals and assciated radicals*, Proc. Biennial Ohio State-Denison Conference 1992, edited by S.K.Jain and S.T.Rizvi, World Scientific, Singapore-New Jersey-London-Hong Kong (1993), 102-129

[5] J.Fisher and K.Krempa *'R^G is nil implies R is nil' is equivalent to the Koethe conjecture*, Huston Journal Mathematics, 9 No. 2(1983), 177-180

[6] M.Frrero and E.Poczylowski *On rings which are sums of two subrings*, Arch.Math, 53(1989), 4-10

[7] I.N.Herstein *Noncommutative Rings*, The Carus Math. Monographs. 15(1968)

[8] C.Y.Hong, T.K.Kwak *On minimal strongly prime ideals*, Comm. Algebra. 28(2000), 4867-4878

[9] S.U.Hwang, Y.C.Jeon, Y.Lee *Structure and topological conditions of NI-rings*, J. Algebra. 302(2006), 186-199

[10] J.Krempa *Logical connections between some open problems concerning nil rings*, Fund. Math. 76(1972), 121-130

[11] J.Krempa *On radical properties of polynomial rings*, Bull.Acad.Polon.Sci.Ser.Math. Astronom.Phys. 29(1972), 545-548

[12] T.Y.Lam *Corner ring theory : a generalization of Peirce decompositions*, I.Algebras, rings and their representations, World Sci.Publ., Hackensack, NJ, (2006), 153-182

[13] A.Leory, J.Matczuk, E.R.Puczylowski *Quasi-duo skew polynomial rings*, J.Pure Appl.Algebra 212(2008), 1951-1959

[14] G.Marks *On 2-primal Ore extensions*, Comm.Algebra 29(2001), 2113-2123

[15] E.Poczylowski *Question Related to Koethe's Nil Ideal Problem*, Preprint, (2000)

[16] L.H.Rowen *Ring theory*, Academic Press,Inc., San Diego (1991), 2113-2123

[17] H.P.Yu *On quasi-duo rings*, Glasgow Math.J., 37(1995), 21-31

Chan Huh
Department of Mathematics
Pusan National University
Busan 46241, Korea
E-mail address: chuh@pusan.ac.kr

Jeoung Soo Cheon
Department of Mathematics
Pusan National University
Busan 46241, Korea
E-mail address: jeoungsoo@pusan.ac.kr

Sun Hye Nam
Department of Mathematics
Pusan National University
Busan 46241, Korea
E-mail address: icandoitsunhye@naver.com