EXPONENTIAL DECAY FOR THE SOLUTION OF THE VISCOELASTIC KIRCHHOFF TYPE EQUATION WITH MEMORY CONDITION AT THE BOUNDARY

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Abstract. In this paper, we study the viscoelastic Kirchhoff type equation with a nonlinear source for each independent kernels \( h \) and \( g \) with respect to Volterra terms. Under the smallness condition with respect to Kirchhoff coefficient and the relaxation function and other assumptions, we prove the uniform decay rate of the Kirchhoff type energy.

1. Introduction

In the present work, we are concerned with the following problem:

\[
\begin{align*}
    u_{tt}(x,t) &- M(x,t,\|\nabla u(t)\|^2)\Delta u(x,t) \\
    &+ \int_0^t h(t-\tau)\text{div}[a(x)\nabla u(\tau)]d\tau + |u|^\gamma u = 0 \quad \text{in} \quad \Omega \times (0,T), \\
    u(x,t) & = 0 \quad \text{on} \quad \Gamma_0 \times (0,T), \\
    u(x,t) + \int_0^t g(t-\tau)M(x,\tau,\|\nabla u(\tau)\|^2)\frac{\partial u}{\partial \nu}(\tau)d\tau = 0 \quad \text{on} \quad \Gamma_1 \times (0,T), \\
    [a(x)\nabla u(\tau)] \cdot \nu &= 0 \quad \text{on} \quad \Gamma_2 \times (0,t), \\
    u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega,
\end{align*}
\]

where \( \Omega \) be a bounded open set of \( \mathbb{R}^N (N \geq 1) \) with a smooth boundary \( \Gamma \) of class \( C^2 \), \( \gamma > 0 \), and other conditions such as \( M, h, a \) be in next section. Indeed, \( t < T \) in (4). We consider \( \Gamma_0, \Gamma_1, \Gamma_2 \) having positive Lebesque measures and \( \Gamma_0 \cap \Gamma_1 \cap \Gamma_2 = \phi \). Let \( \nu \) be the outward normal to \( \Gamma \) and \( T > 0 \) be a real number. In fact, \( u_0, u_1 \) are initially given functions and \( u(x,t) \) is the transversal displacement of the strip at spatial coordinate \( x \) and time \( t \) in the real world application.

Our system works independently with respect to kernels for Volterra terms and spatial part for the Kirchhoff term under internal space or not. Physically,
first, in the space $\Omega$, the Volterra energy is only acted on $h$. Second, in the space $\Gamma_0$, there is no Volterra energy. Third, the Volterra energy is only acted on $g$ in the space $\Gamma_1$. And also, the main system has a difference when it comes to the Kirchhoff type term under internal space or not in this work. More precisely, the Kirchhoff type term is not affected only by spatial part on the boundaries. That is, not only $M(x,t,\|\nabla u(t)\|^2) = 0$ on the space $\Gamma_0$, but also $M(x,t,\|\nabla u(t)\|^2) = M(t,\|\nabla u(t)\|^2)$ on the space $\Gamma_1$ in this work. So we let you know the follows again:

$$M(x,t,\|\nabla u(t)\|^2) := M(t,\|\nabla u(t)\|^2) \quad \text{on} \quad \Gamma_1. \quad (6)$$

This problem has its origin in the mathematical description of system in real world from the mathematical modeling for axially moving viscoelastic materials. It is well known that viscoelastic materials exhibit natural damping, which is due to the special property of these materials to retain a memory of their past history. From the mathematical point of view, these damping effects are modeled by integro-differential operators. For these reasons, there are not exist weak or strong damping term in our problem (1)-(5). Recently, problems with Timoshenko or basic hyperbolic type for viscoelastic materials have been considered by many authors (See [1, 2]). Besides, many engineering devices involve the transverse vibration of axially moving strings. Axially moving string is a typical model that is widely used, especially when the subject is long and narrow enough and has a negligible flexural rigidity, to represent threads, wires, magnetic tapes, belts, band saws, and cables. Various mathematical models and simulations have been established for a better understanding with linear or nonlinear dynamic behavior of these moving continua [3, 4, 5, 6, 7, 8, 9]. The mathematical model for axially moving strings was first introduced by Kirchhoff [10] (and see Carrier [3]), and the original equation is given in the form of

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left( p_0 + \frac{E h}{2L} \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 dx \right) \frac{\partial^2 u}{\partial x^2}$$

for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate $x$ and time $t$; $E$, the young’s modulus; $\rho$, the mass density; $h$, the cross section area; $L$, the length; and $p_0$, the initial axial tension. Recently, problems with the extended Kirchhoff type equation which is concerning axially moving heterogeneous or non heterogeneous materials (nonlinear vibrations of beams, strings, plates, and membranes) have been considered by many authors (See [11, 12, 13, 14]).

In this paper, we will mainly concern on an aspect of decay rate of the Kirchhoff type energy of the system. Our purpose is focused on not only main equation but also boundary condition which are involved in memory effects for the problem otherwise the previous result [15, 16, 17]. We get its proof by using the smallness condition functions with respect to Kirchhoff coefficient and the relaxation function. In fact, the difference of the energy consist in Kirchhoff type potential energy.
This paper is organized as follows. In Section 2, we will present some notations, material needed (assumptions, lemmas and so on) for our work and state a global existence and energy decay rate theorem (main result). Section 3 contains the proof of our main result.

2. Preliminaries and main results

We first introduce the elementary bracket pairing in $\Omega \subset \mathbb{R}^N$

$$\langle \varphi, \psi \rangle \equiv \int_{\Omega} (\varphi, \psi) dx,$$

provided that $(\varphi, \psi) \in L^1(\Omega)$. And we set the norms as follows.

$$\|u\|_{L^p(\Omega)} = \left( \int_{\Omega} |u|^p dx \right)^{\frac{1}{p}}.$$

To simplify the notations, we denote $\|u\|_{L^2(\Omega)}$, $\|u\|_{L^1(0, +\infty)}$, $\|v\|_{L^\infty(0, +\infty)}$ by $\|u\|, \|v\|_{L^1}, \|v\|_{L^\infty}$ respectively.

In the sequel we state the general hypotheses.

(A1) $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded $C^1$ function satisfying $h(0) > 0$, and there exists positive constant $t_0, \zeta_1, \zeta_2, \zeta_3$ such that

$$-\zeta_1 \leq h'(t) \leq -\zeta_2 h(t), \quad \forall t > t_0,$$

$$0 \leq h''(t) \leq \zeta_3 h(t), \quad \forall t > t_0.$$

(A2) $a : \Omega \to \mathbb{R}^+$ is a nonnegative bounded function and $a(x) \geq a_0 > 0$ on $\Omega$ with

$$\frac{m_0}{a_0} \geq 1 - \|a\|_\infty \int_0^\infty h(s) ds = l > 0,$$

where $m_0$ is in (B2). And also, the following smallness condition satisfy

$$\epsilon_7 < a_0^2 \int_0^t h(s) ds.$$

(A3) $\gamma$ satisfies

$$0 \leq \gamma \leq \frac{2}{n - 2}, \quad n \geq 3,$$

$$\gamma \geq 0, \quad n = 1, 2.$$

(B1) $M(x, t, \lambda)$ is a real-valued function of class $C^2$ on $x \in \overline{\Omega}$, $t \geq 0$, $\lambda \leq 0$.

(B2) $0 < m_0 \leq M(x, t, \lambda) \leq C_0 f(\lambda)$ with $M(x, t, \lambda) = M_1(x, t) + M_2(x, t, \lambda)$.

And also, the following smallness condition satisfy

$$f(\lambda) < \sqrt{\frac{a_0 h(t)}{2} - C_p \tilde{C}_1 + \epsilon_2 (m_0 - \frac{1}{2})}.$$

(B3) $\frac{\partial M_1}{\partial t} \leq 0$, $|\frac{\partial M_2}{\partial t}| \leq C_1 g_1(\lambda)$, $|\frac{\partial M}{\partial x}| \leq C_2 g_2(\lambda)$, $0 < m_1 \leq M(x, t, \lambda)$.

(B4) $f, g_1, g_2 \in C^1([0, +\infty) ; \mathbb{R}^+)$ are strictly increasing.

Furthermore, $C_i$ ($i = 0, 1, 2$) is a positive constant.
Next, we assume that the kernel $g$ is positive and $k$ satisfies:

\[ 0 < k(t) \leq b_0 e^{-s_0 t}, \]
\[-b_1 k(t) \leq k'(t) \leq -b_2 k(t), \]
\[-b_3 k'(t) \leq k''(t) \leq -b_4 k'(t) \]

for some positive constants $b_i$, $i = 0, 1, 2, 3, 4$ and $s_0$. To facilitate our analysis, we introduce the following binary operators:

\[(g \Box u)(t) = \int_0^t g(t - \tau) |u(t) - u(\tau)|^2 d\tau,\]
\[(g * u)(t) = \int_0^t g(t - \tau) u(\tau) d\tau\]

where * is the convolution product. Differentiating (3) we arrive at the Volterra equation:

\[ M(t, \|\nabla u(t)\|^2) \frac{\partial u}{\partial \nu} + \frac{1}{g(0)} g' M(t, \|\nabla u(t)\|^2) \frac{\partial u}{\partial \nu} = -\frac{1}{g(0)} u_t. \]

Using the Volterra inverse operator, we get

\[ M(t, \|\nabla u(t)\|^2) \frac{\partial u}{\partial \nu} = -\frac{1}{g(0)} u_t + k * u_t \]

where the resolvent kernel satisfy

\[ k + \frac{1}{g(0)} g' * k = -\frac{1}{g(0)} g'. \]

With $\varsigma = \frac{1}{g(0)}$ and using the above identity, we obtain

\[ M(t, \|\nabla u(t)\|^2) \frac{\partial u}{\partial \nu} = -\varsigma \{ u_t + k(0)u - k(t)u_0 + k' * u \}. \]  

(8)

In the following, we give a lemma which will be useful in this paper.

**Lemma 2.1.** For $g, \Psi \in C^1([0, \infty) : \mathbb{R})$. Then we have

\[ (g * \psi)_t = -\frac{1}{2} g(t) |\psi(t)|^2 + \frac{1}{2} g' \Box \psi - \frac{1}{2} \frac{d}{dt} \left[ g \Box \psi - \left( \int_0^t g(s) ds \right) |\psi|^2 \right] \] 

Proof. The proof of this lemma follows by differentiating the term $g \Box \psi$. \qed

**Lemma 2.2.** Denote $(h \diamond u)(t) = \int_0^t h(t - \tau) \|\sqrt{a(x)}(u(t) - u(\tau))\|^2 d\tau$. Then we have

\[ \int_0^t h(t - \tau) (a(x) \nabla u(\tau), \nabla u'(t)) d\tau = -\frac{1}{2} \frac{d}{dt} \left[ (h \diamond u)(t) \right] + \frac{1}{2} (h' \diamond u)(t) \]

\[ + \frac{1}{2} \frac{d}{dt} \left[ \|\sqrt{a(x)} \nabla u(t)\|^2 \int_0^t h(s) ds \right] \]

\[ - \frac{1}{2} h(t) \|\sqrt{a(x)} \nabla u(t)\|^2. \]  

(10)
Proof. A direct computation shows that
\[
\int_0^t h(t - \tau) (a(x) \nabla u(\tau), \nabla u'(t)) d\tau = \int_0^t h(t - \tau) (a(x) \nabla u(\tau) - a(x) \nabla u(t), \nabla u'(t)) d\tau \\
+ \int_0^t h(t - \tau) (a(x) \nabla u(t), \nabla u'(t)) d\tau \\
= -\frac{1}{2} \int_0^t h(t - \tau) \left[ \frac{d}{dt} \|a(x)(\nabla u(\tau) - \nabla u(t))\|^2 \right] d\tau \\
+ \frac{1}{2} \int_0^t h(t - \tau) \left[ \frac{d}{dt} \|a(x)\nabla u(t)\|^2 \right] d\tau \\
= -\frac{1}{2} \frac{d}{dt} \left[ \int_0^t h(t - \tau) \|a(x)(\nabla u(\tau) - \nabla u(t))\|^2 d\tau \right] \\
+ \frac{1}{2} \int_0^t h'(t - \tau) \|a(x)(\nabla u(\tau) - \nabla u(t))\|^2 d\tau \\
+ \frac{1}{2} \frac{d}{dt} \int_0^t h(t - \tau) \|a(x)\nabla u(t)\|^2 d\tau \\
- \frac{1}{2} h(t) \|a(x)\nabla u(t)\|^2.
\]

Lemma 2.3. (General Poincaré Inequality).
Denote \(H^1_{\Gamma_0}(\Omega) = \{u | u \in H^1(\Omega), u|_{\Gamma_0} = 0\}\) and \(\text{meas}(\Gamma_0) > 0\). Then there exists a positive constant \(B\) such that \(\|u\|_{L^2(\Gamma)} \leq B\|\nabla u\|_{L^2(\Omega)}\), for all \(u \in H^1_{\Gamma_0}(\Omega)\).

Proof. The proof can be found in [18].

Then, we can state our result as follows.

Theorem 2.4. Let the assumptions \((A_1), (A_3)\) and \((B_1)-(B_4)\) and the relating conditions (7) and (8) to the volterra term on boundary hold and the sobolev space \(V\) is \(\{u | u \in H^1_{\Gamma_0}(\Omega), u = 0 \text{ on } \Gamma_0\}\). If \((u_0, u_1) \in (H^2(\Omega) \cap V) \times V\) and satisfy the compatibility condition
\[
M(0, \|\nabla u_0\|^2) \frac{\partial u_0}{\partial \nu} = -\varsigma u_1 \text{ on } \Gamma_1.
\]

Then there exists a unique solution \(u\) of the problem (1)-(5) satisfying
\(u \in L^\infty(0, T; H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega))\), \(u' \in L^\infty(0, T; H^1_{\Gamma_0}(\Omega))\), \(u'' \in L^\infty(0, T; L^2(\Omega))\), and \(u(x, t) \to u_0(x) \text{ in } V \cap H^2(\Omega); \ u'(x, t) \to u_1(x) \text{ in } V\), as \(t \to 0\)

Proof. By using Galerkin’s approximation and a routine procedure similar to that of cite [12, 1], we can the global existence result for the solution subject to (1)-(5)
under the assumptions (A1)-(A3) and (B1)-(B4) and the relating conditions (7) and (8) to the volterra term on boundary.

**Theorem 2.5.** Let \( u \) be the global solution of the problem (1)-(5) with the above all conditions. We define the Kirchhoff type energy functional \( E(t) \) as
\[
E(t) = \frac{1}{2} \int_{\Omega} |u'|^2 dx + \frac{1}{2} \int_{\Omega} M(x, t, \|\nabla u(t)\|^2)|\nabla u(x, t)|^2 dx \\
+ \frac{1}{\gamma+2} \|u'(t)\|_{\gamma+2}^{\gamma+2} + \frac{\kappa}{2} (M(t, \|\nabla u(t)\|^2)) \int_{\Gamma_1} |u(x, t)|^2 d\Gamma_1 \\
- \int_{\Gamma_1} M'(t, \|\nabla u(t)\|^2) \Box u(t) d\Gamma_1.
\]

Then the energy functional decays exponentially to zero as the time goes to infinity, that is,
\[
E(t) \leq \kappa e^{-\vartheta t}, \quad \forall t \geq 0
\]
where \( \kappa, \vartheta \) are positive constants.

### 3. Proof of Theorem 2.5 (Energy decay)

**Proof.** Multiplying \( u' \) on both sides of Eq.(1), integrating the resulting equations over \( \Omega \), and using the Green formula, (6) and (3), we have
\[
\langle u''(t), u'(t) \rangle + \langle M(x, t, \|\nabla u(t)\|^2)\nabla u(t), \nabla u'(t) \rangle \\
+ \langle M_x(x, t, \|\nabla u(t)\|^2)\nabla u(t), u'(t) \rangle \\
+ \left\langle M(t, \|\nabla u(t)\|^2) \frac{\partial u}{\partial \nu}(t), u'(t) \right\rangle_{\Gamma_1} \\
- \int_0^t h(t - \tau) \langle a(x)\nabla u(\tau), \nabla u'(\tau) \rangle d\tau + \langle |\gamma u, u'| \rangle = 0,
\]
that is
\[
\frac{d}{dt} E(t) = \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} M_1(x, t)|\nabla u(x, t)|^2 dx \\
+ \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} M_2(x, t, \|\nabla u(t)\|^2)|\nabla u(x, t)|^2 dx \\
+ \left[ \int_{\Omega} \frac{\partial}{\partial \lambda} M_2(x, t, \|\nabla u(t)\|^2)|\nabla u(x, t)|^2 dx \right] \langle \nabla u'(t), \nabla u(t) \rangle \\
- \langle M_x(x, t, \|\nabla u(t)\|^2)\nabla u(t), u'(t) \rangle \\
- \int_0^t h(t - \tau) \langle a(x)\nabla u(\tau), \nabla u'(\tau) \rangle d\tau \\
+ \left\langle M(t, \|\nabla u(t)\|^2) \frac{\partial u}{\partial \nu}(t), u'(t) \right\rangle_{\Gamma_1} \\
+ \frac{s}{2} \frac{d}{dt} \left[ M(t, \|\nabla u(t)\|^2) \int_{\Gamma_1} |u(x, t)|^2 d\Gamma_1 \right] \\
- \frac{s}{2} \frac{d}{dt} \int_{\Gamma_1} M'(t, \|\nabla u(t)\|^2) \Box u(t) d\Gamma_1,
\]
Therefore, we have

\[
E(t) = \frac{1}{2} \int_{\Omega} |u'(t)|^2 \, dx + \frac{1}{2} \int_{\Omega} M(x, t, \|\nabla u(t)\|^2)|\nabla u(x, t)|^2 \, dx
+ \frac{1}{\gamma + 2} |u'(t)|^{\gamma + 2} + \frac{\varsigma}{2} M(t, \|\nabla u(t)\|^2) \int_{\Gamma_1} |u(x, t)|^2 \, d\Gamma_1
- \frac{\varsigma}{2} \int_{\Gamma_1} M'(t, \|\nabla u(t)\|^2)\Box u(t) \, d\Gamma_1.
\]

(14)

Therefore, we have

\[
\frac{d}{dt} E(t) = \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} M_1(x, t) |\nabla u(x, t)|^2 \, dx
+ \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} M_2(x, t, \|\nabla u(t)\|^2)|\nabla u(x, t)|^2 \, dx
+ \left[ \int_{\Omega} \frac{\partial}{\partial \lambda} M_2(x, t, \|\nabla u(t)\|^2) \right]|\nabla u'(t), \nabla u(t)\rangle
- (M_x(x, t, \|\nabla u(t)\|^2)|\nabla u(t), u'(t)\rangle
- \int_0^t h(t - \tau)\langle a(x)\nabla u(\tau), \nabla u'(\tau)\rangle \, d\tau
- \varsigma \int_{\Gamma_1} |u'(t)|^2 \, d\Gamma_1 - \frac{\varsigma}{2} k(0) \frac{d}{dt} \int_{\Gamma_1} |u|^2 \, d\Gamma_1
+ \varsigma \int_{\Gamma_1} k(t)u_0u_t \, d\Gamma_1 - \varsigma \int_{\Gamma_1} (k' * u)u_t \, d\Gamma_1
+ \frac{\varsigma}{2} \frac{d}{dt} \left[ M(t, \|\nabla u(t)\|^2) \int_{\Gamma_1} |u(x, t)|^2 \, d\Gamma_1 \right]
- \frac{\varsigma}{2} \frac{d}{dt} \int_{\Gamma_1} M'(t, \|\nabla u(t)\|^2)\Box u(t) \, d\Gamma_1.
\]

(15)

From (B3), (7), Lemma 2.1 and Hölder inequality, we obtain

\[
E'(t) \leq \|u(t)\|^2 \left\{ \frac{C_1}{2} g_1(\|\nabla u(t)\|^2) + C_2 g_2(\|\nabla u(t)\|^2)|\nabla u'(t)|\|u(t)\| \right\}
- (M_x(x, t, \|\nabla u(t)\|^2)|\nabla u(t), u'(t)\rangle
- \int_0^t h(t - \tau)\langle a(x)\nabla u(\tau), \nabla u'(\tau)\rangle \, d\tau
- \frac{\varsigma}{2} \int_{\Gamma_1} |u'(t)|^2 \, d\Gamma_1 + \frac{\varsigma}{2} k^2(t) \int_{\Gamma_1} |u_0|^2 \, d\Gamma_1
+ \frac{\varsigma}{2} k'(t) \int_{\Gamma_1} |u(t)|^2 \, d\Gamma_1 - \frac{\varsigma}{2} \int_{\Gamma_1} k'' \Box u \, d\Gamma_1
- \frac{\varsigma}{2} \frac{d}{dt} \left[ \left( \int_0^t k'(\tau) \, d\tau \right) \|u\|^2 \right] \, d\Gamma_1.
\]

(16)
\[ \leq \|u(t)\|^2 \left\{ \frac{C_1}{2} g_1(\|\nabla u(t)\|^2) + C_2 g_2(\|\nabla u(t)\|^2) \|\nabla u'(t)\|\|u(t)\| \right\} \]

\[ - \langle M_x(x, t, \|\nabla u(t)\|^2) \nabla u(t), u'(t) \rangle \]

\[ - \int_0^t h(t - \tau) \langle a(x) \nabla u(\tau), \nabla u'(t) \rangle d\tau \]

\[ - \frac{\varsigma}{2} \int_{\Gamma_1} |u'(t)|^2 d\Gamma_1 + \frac{\varsigma}{2} k^2(t) \int_{\Gamma_1} |u_0|^2 d\Gamma_1 \]

\[ + \frac{\varsigma}{2} k'(t) \int_{\Gamma_1} |u(t)|^2 d\Gamma_1 - \frac{\varsigma}{2} \int_{\Gamma_1} k''\Box u d\Gamma_1. \]

By \((B_3), (10)\) and Young’s inequality, we have

\[
E'(t) \leq \|u(t)\|^2 \widetilde{C}_1 + \epsilon_1 m_1 \|\nabla u(t)\|^2 + \frac{m_1}{4\epsilon_1} \|u'(t)\|^2
\]

\[
- \frac{1}{2} \frac{d}{dt} [\|h \circ u(t)\|] + \frac{1}{2} \langle h' \circ \nabla u(t) \rangle
\]

\[
+ \frac{1}{2} \frac{d}{dt} \left[ \|\sqrt{a(x)} \nabla u(t)\|^2 \int_0^t h(s) ds \right]
\]

\[
- \frac{1}{2} h(t)\|\sqrt{a(x)} \nabla u(t)\|^2
\]

\[
- \frac{\varsigma}{2} \int_{\Gamma_1} |u'(t)|^2 d\Gamma_1 + \frac{\varsigma}{2} k^2(t) \int_{\Gamma_1} |u_0|^2 d\Gamma_1
\]

\[
+ \frac{\varsigma}{2} k'(t) \int_{\Gamma_1} |u(t)|^2 d\Gamma_1 - \frac{\varsigma}{2} \int_{\Gamma_1} k''\Box u d\Gamma_1,
\]

where

\[
\widetilde{C}_1 = \frac{C_1}{2} g_1(\|\nabla u(t)\|^2) + C_2 g_2(\|\nabla u(t)\|^2) \|\nabla u'(t)\|\|u(t)\|
\]

is a positive constant. And \(\epsilon_1\) is also a positive constant.

In the boundary \(\Gamma_1\), note that

\[
-k(0)u(t) - k' * u(t) = - \int_0^t k'(t - \tau)[u(\tau) - u(t)]d\tau - k(t)u(t)
\]

\[
\leq \left( \int_0^t |k'(\tau)|d\tau \right)^\frac{1}{2} \left[ [k' \Box u(x, t)]^\frac{1}{2} + k(t)|u(t)| \right]
\]

\[
\leq |k(t) - k(0)|^\frac{1}{2} [ [k' \Box u(t)]^\frac{1}{2} + k(t)|u(t)|. \]
Using (8) and (19), follows that
\[
\int_{\Gamma_1} M(t, \|\nabla u(t)\|^2) \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma_1 
\]
(20)
\[
\leq C_{b_1} \int_{\Gamma_1} \{|u_t(t)|^2 + k^2(t)|u_0|^2 + k(0)|k'|\Box u(t) + k(0)k(t)|u(t)|^2\} d\Gamma_1.
\]
where $C_{b_1}$ is a positive constant.

Here we use (7) to conclude the following estimates for the corresponding two terms appearing in (22).
\[
-\frac{\varsigma}{2} \int_{\Gamma_1} k''\Box u(t) d\Gamma_1 \leq C_{c_1} \int_{\Gamma_1} k'\Box u(t) d\Gamma_1
\]
(21)
\[
-\frac{\varsigma}{2} \int_{\Gamma_1} k'|u(t)|^2 d\Gamma_1 \leq -C_{c_1} \int_{\Gamma_1} k|u(t)|^2 d\Gamma_1.
\]
where $C_{c_1}$ is a positive constant.

By using (20) and (21) in (22), we conclude
\[
E'(t) + \int_{\Gamma_1} M(t, \|\nabla u(t)\|^2) \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma_1
\leq \|u(t)\|^2 \widetilde{C}_1 + \epsilon_1 m_1 \|\nabla u(t)\|^2 + \frac{m_1}{4\epsilon_1} \|u'(t)\|^2
\]
\[+ \frac{1}{2} \frac{d}{dt} [(h \circ u)(t)] + \frac{1}{2} (h' \circ \nabla u)(t)
\]
\[+ \frac{1}{2} \frac{d}{dt} \left[ \|\sqrt{a(x)}\nabla u(t)\|^2 \int_0^t h(s) ds \right]
\]
\[+ \frac{1}{2} \int_{\Gamma_1} \{\|u_t(t)|^2 + k^2(t)|u_0|^2 + k(0)|k'|\Box u(t) + k(0)k(t)|u(t)|^2\} d\Gamma_1
\]
(22)
where $C_{c_2}$ is a positive constant.

Define the new energy functional $E_1(t)$ as follows
\[
E_1(t) = E(t) + \frac{1}{2} (h \circ \nabla u)(t) - \frac{1}{2} \|\sqrt{a(x)}\nabla u(t)\|^2 \int_0^t h(s) ds + \int_{\Gamma_1} M(t, \|\nabla u(t)\|^2) \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma_1.
\]
(23)
Then from (A1), (B2), (22) and Lemma (2.3), we have
\[
E_1'(t) \leq \|u(t)\|^2 \widetilde{C}_1 + (\epsilon_1 m_1 + C_B) \|\nabla u(t)\|^2 + \frac{m_1}{4\epsilon_1} \|u'(t)\|^2
\]
\[+ \frac{\varsigma_2}{2} (h \circ \nabla u)(t) - \frac{1}{2} a_0 h(t) \|\nabla u(t)\|^2,
\]
(24)
where $C_B$ is a positive constant relating Poincaré constant $B$ and also, by (A2), the energy $E_1(t)$ is a positive functional. Applying Poincaré inequality to (24),
we deduce
\[
E'(t) \leq \left( C_p C_1 + \epsilon_1 m_1 - \frac{1}{2} a_0 h(t) \right) \| \nabla u(t) \|^2 \\
+ \frac{m_1}{4\epsilon_1} \| u'(t) \|^2 - \frac{\zeta_2}{2} (h \cdot \nabla u)(t),
\]
(25)
where \( C_p \) is the Poincaré coefficient. Meanwhile, we note from (A1) and (A2) that
\[
E_1(t) \geq \frac{1}{2} \| u(t) \|^2 + \frac{1}{2} \int_\Omega M(x, t, \| \nabla u(t) \|^2) |\nabla u(x, t)|^2 dx \\
+ \frac{1}{2} \left( 1 + a \| u \|_\infty \right) \int_0^t h(s) ds \| \nabla u(t) \|^2 + \frac{1}{2} (h \cdot u)(t) + \frac{1}{\gamma + 2} \| u(t) \|_{\gamma + 2}^2 \\
\geq \frac{l}{2} \| u'(t) \|^2 + \frac{1}{2} \int_\Omega M(x, t, \| \nabla u(t) \|^2) |\nabla u(x, t)|^2 dx + \frac{1}{\gamma + 2} \| u(t) \|_{\gamma + 2}^2.
\]
(26)
So, we deduce the relation \( 0 \leq E(t) \leq l^{-1} E_1(t) \). Therefore, the uniform decay of \( E(t) \) is a result of the decay of \( E_1(t) \). For positive constants \( \epsilon_2 \) and \( \epsilon_3 \), let us define the perturbed modified energy by
\[
F(t) = E_1(t) + \epsilon_2 \varphi(t) + \epsilon_3 \psi(t),
\]
(27)
where
\[
\varphi(t) = \langle u'(t), u(t) \rangle.
\]
(28)
and
\[
\psi(t) = -\int_0^t h(t - \tau) \langle a(x) u'(t), u(t) - u(\tau) \rangle d\tau.
\]
(29)
By using the Cauchy’s inequality, Hölder inequality and Poincaré inequality, there exist positive constants \( \alpha_1, \alpha_2 \) such that for each \( t > 0 \)
\[
\alpha_1 F(t) \leq E_1(t) \leq \alpha_2 F(t).
\]
(30)

**Proposition 3.1. (Energy equivalence)**
\[
\alpha_1 F(t) \leq E_1(t) \leq \alpha_2 F(t) \quad \text{for all } t \geq 0,
\]
where
\[
\alpha_1 = \frac{1}{\max \{ 1, \epsilon_4 + \epsilon_5, (\epsilon_4 + \epsilon_5) C_p \} } > 0
\]
and
\[
\alpha_2 = \frac{1}{\min \{ 1 - (\epsilon_4 + \epsilon_5), m_0 - \epsilon_4 C_p, 1 - \epsilon_5 C_p(1 - l) \} } > 0.
\]
**Proof.** By the Cauchy inequality and Hölder inequality, we get
\[
F(t) \leq E_1(t) + \frac{\epsilon_4}{2} \| u'(t) \|^2 + \frac{\epsilon_4}{2} \| u(t) \|^2 \\
+ \frac{\epsilon_5}{2} \left\| \int_0^t h(t - \tau) a(x)(u(t) - u(\tau)) d\tau \right\| + \frac{\epsilon_5}{2} \| u'(t) \|,
\]
(31)
where $\epsilon_4, \epsilon_5$ are positive constants.

By using the Poincarè inequality, we have

\[
F(t) \leq E_1(t) + \frac{\epsilon_4 + \epsilon_5}{2} \|u'(t)\|^2 + \frac{\epsilon_4}{2} \|u(t)\|^2
\]
\[+ \frac{\epsilon_5}{2} \|a\|_{\infty} \int_{0}^{\infty} h(s)ds \int_{0}^{t} h(t - \tau) \left\|\sqrt{a(x)}(u(t) - u(\tau))\right\|^2 d\tau \]
\[\leq E_1(t) + \frac{\epsilon_4 + \epsilon_5}{2} \|u'(t)\|^2 + \frac{\epsilon_4}{2} \|u(t)\|^2
\]
\[+ \frac{\epsilon_5}{2} C_p (1 - l) \int_{0}^{\infty} h(s)ds \int_{0}^{t} h(t - \tau) \left\|\sqrt{a(x)}(u(t) - u(\tau))\right\|^2 d\tau \]
\[\leq E_1(t) + \frac{\epsilon_4 + \epsilon_5}{2} \|u'(t)\|^2 + \frac{\epsilon_4}{2} C_p \|\nabla u(t)\|^2
\]
\[+ \frac{\epsilon_5}{2} C_p (1 - l) (h \circ \nabla u)(t) \]
\[\leq \max\{1, \epsilon_4 + \epsilon_5, (\epsilon_4 + \epsilon_5) C_p\} E_1(t),
\]

where $C_p$ is the Poincarè coefficient. Besides, choosing $\epsilon_4, \epsilon_5$ small enough, we have

\[
F(t) \geq E_1(t) - \frac{\epsilon_4 + \epsilon_5}{2} \|u'(t)\|^2 - \frac{\epsilon_4}{2} C_p \|\nabla u(t)\|^2
\]
\[- \frac{\epsilon_5}{2} C_p (1 - l) (h \circ \nabla u)(t) \]
\[\geq \frac{1 - (\epsilon_4 + \epsilon_5)}{2} \|u'(t)\|^2 + \frac{m_0 - \epsilon_4 C_p}{2} \|\nabla u(t)\|^2
\]
\[- \frac{1}{2} \left\|\sqrt{a(x)} \nabla u(t)\right\|^2 \int_{0}^{t} h(s)ds + \frac{1}{\gamma + 2} \|u(t)\|_{\gamma + 2}^2
\]
\[+ \frac{1 - \epsilon_5 C_p (1 - l)}{2} (h \circ \nabla u)(t) \]
\[\geq \min\{1 - (\epsilon_4 + \epsilon_5), m_0 - \epsilon_4 C_p, 1 - \epsilon_5 C_p (1 - l)\} E_1(t).
\]

\[\square\]

In fact, using (1), we have

\[
\varphi'(t) = \langle u''(t), u(t) \rangle + \|u'(t)\|^2
\]
\[= \|u'(t)\|^2 + \langle u(t), M(x, t, \|\nabla u(t)\|^2) \Delta u(x, t)
\]
\[- \int_{0}^{t} h(t - \tau) \text{div}[a(x)\nabla u(\tau)]d\tau - |u(t)|^\gamma u(t) \rangle \]
\[= \|u'(t)\|^2 - \int_{\Omega} M(x, t, \|\nabla u(t)\|^2) |\nabla u(t)|^2 dx
\]
\[+ \int_{0}^{t} h(t - \tau) \langle a(x)\nabla u(\tau), \nabla u(t) \rangle d\tau - |u(t)|^\gamma u(t).
\]

(32)
By Cauchy inequality and Young’s inequality, we have

\[
\left| \int_0^t h(t - \tau) \langle a(x) \nabla u(\tau), \nabla u(t) \rangle d\tau \right| \\
\leq \frac{1}{2} \| \nabla u(t) \|^2 + \frac{1}{2} \left\| \int_0^t h(t - \tau) (a(x) |\nabla u(\tau) - \nabla u(t)| + a(x) |\nabla u(t)|) d\tau \right\|^2 \\
\leq \frac{1}{2} \| \nabla u(t) \|^2 + \left( \frac{1}{2} + \frac{1}{8\epsilon_6} \right) \left\| \int_0^t h(t - \tau) a(x) |\nabla u(\tau) - \nabla u(t)| d\tau \right\|^2 \\
+ \left( \frac{1}{2} + \frac{\epsilon_6}{2} \right) \left\| \int_0^t h(t - \tau) a(x) |\nabla u(t)| d\tau \right\|^2,
\]

where \( \epsilon_6 \) with respect to Young’s inequality is a positive constant. Using the assumption \( (A_2) \) and (33), we get

\[
\left| \int_0^t h(t - \tau) \langle a(x) \nabla u(\tau), \nabla u(t) \rangle d\tau \right| \\
\leq \left( \frac{1}{2} + \frac{1}{8\epsilon_6} \right) \| a \|_\infty \int_0^t h(s) ds \int_0^t h(t - \tau) \left\| \sqrt{a(x)} |\nabla u(\tau) - \nabla u(t)| \right\|^2 d\tau \\
+ \left( \frac{1}{2} + \frac{\epsilon_6}{2} \right) \| \nabla u(t) \|^2 \left( \| a \|_\infty \int_0^t h(s) a(x) ds \right)^2 + \frac{1}{2} \| \nabla u(t) \|^2 \\
\leq \frac{1}{2} (1 + (1 + \epsilon_6)(1 - l)^2) \| \nabla u(t) \|^2 + \frac{(4\epsilon_6 + 1)(1 - l)}{8\epsilon_6} (h \circ \nabla u)(t).
\]

By combining (32) and (34), we conclude

\[
\varphi'(t) \leq \| u'(t) \|^2 + \frac{1}{2} (1 - 2m_0 + (1 + \epsilon_6)(1 - l)^2) \| \nabla u(t) \|^2 \\
+ \frac{(4\epsilon_6 + 1)(1 - l)}{8\epsilon_6} (h \circ \nabla u)(t) - \| u(t) \|_{\gamma+2}^\gamma.
\]
Next, we estimate $\psi'(t)$ as follows. In fact, using (1), we have

\[
\psi'(t) = -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t) - u(\tau) \rangle d\tau.
\]

\[
- \int_0^t h(t-\tau)\langle a(x)u''(t), u(t) - u(\tau) \rangle d\tau - \|\sqrt{a(x)}u(t)\|^2 \int_0^t h(s)ds
\]

\[
= -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t) - u(\tau) \rangle d\tau.
\]

\[
- \int_0^t h(t-\tau)\langle M(x,t,\|\nabla u(t)\|^2)a(x)\nabla u(t), \nabla u(t) - \nabla u(\tau) \rangle d\tau
\]

\[
- \left( \int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau, \int_0^t h(t-\tau)a(x)(\nabla u(t) - \nabla u(\tau))d\tau \right)
\]

\[
+ \int_0^t h(t-\tau)|a(x)|u^\gamma u, u(t) - u(\tau) d\tau
\]

\[
- \|\sqrt{a(x)}u'(t)\|^2 \int_0^t h(s)ds.
\]

Using Cauchy inequality, Poincaré inequality and (A1), we have

\[
\left| -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t) - u(\tau) \rangle d\tau \right|
\]

\[
\leq \epsilon_7\|\nabla u(t)\|^2 + \frac{\zeta_1}{4\epsilon_7}\left\| \int_0^t h(t-\tau)a(x)|u(t) - u(\tau)|d\tau \right\|^2
\]

\[
\leq \epsilon_7\|\nabla u(t)\|^2 + \frac{\zeta_1}{4\epsilon_7}(1 - l)C_p^2(h \circ \nabla u)(t),
\]

where $\epsilon_7$ is a positive constant with respect to Cauchy inequality and $C_p$ is the Poincaré coefficient. Similarly, using Cauchy inequality and (B2), we get

\[
\left| -\int_0^t h(t-\tau)\langle M(x,t,\|\nabla u(t)\|^2)a(x)\nabla u(t), \nabla u(t) - \nabla u(\tau) \rangle d\tau \right|
\]

\[
\leq \epsilon_8 f^2(\|\nabla u(t)\|^2)\|u'(t)\|^2 + \frac{C_0(1 - l)}{4\epsilon_8}(h \circ \nabla u)(t)
\]
and

\(| - \left( \int_0^t h(t - \tau)a(x)\nabla u(\tau) d\tau, \int_0^t h(t - \tau)a(x)(\nabla u(t) - \nabla u(\tau)) d\tau \right) | \leq \varepsilon_9 \left| \int_0^t h(t - \tau)a(x)|\nabla u(t) - \nabla u(\tau)| + a(x)|\nabla u(t)| d\tau \right|^2 \]

\[ + \frac{1}{4\varepsilon_9} \left( \|a\|_\infty \int_0^t h(s)ds \right) \int_0^t h(t - \tau) \|\sqrt{a(x)(\nabla u(t) - \nabla u(\tau))}\|^2 d\tau \]

\[ \leq 2\varepsilon_9 \left( \left| \int_0^t h(t - \tau)a(x)|\nabla u(t) - \nabla u(\tau)| d\tau \right|^2 + \left| \int_0^t h(t - \tau)a(x)|\nabla u(t)| d\tau \right|^2 \right) \]

\[ + \frac{1 - l}{4\varepsilon_9} (h \circ \nabla u)(t) \]

\[ \leq \left( 2\varepsilon_9 + \frac{1}{4\varepsilon_9} \right) (1 - l)(h \circ \nabla u)(t) + 2\varepsilon_9(1 - l)^2 \|\nabla u(t)\|^2, \]

where \( \varepsilon_8, \varepsilon_9 \) are positive constants with respect to Cauchy inequality. And also, using Cauchy inequality and Poincarè inequality, we have

\[ \left| \int_0^t h(t - \tau)\langle a(x)|u(t)|^\gamma u, u(t) - u(\tau) \rangle d\tau \right| \leq \varepsilon_{10} \|u(t)\|^{2(\gamma + 1)} \frac{C_p(1 - l)}{4\varepsilon_{10}} (h \circ \nabla u)(t), \]

where \( \varepsilon_{10} \) is a positive constant with respect to Cauchy inequality and \( C_p \) is the Poincarè coefficient. Noting \( H^1(\Omega) \hookrightarrow L^{2(\gamma + 1)}(\Omega) \) and using Poincarè inequality, (23), (24) and (40), we get

\[ \left| \int_0^t h(t - \tau)\langle a(x)|u(t)|^\gamma u, u(t) - u(\tau) \rangle d\tau \right| \leq \varepsilon_{10} C_p^{2(\gamma + 1)} \left( \frac{2E_1(0)}{l} \right) \gamma \|\nabla u(t)\|^2 + \frac{C_p(1 - l)}{4\varepsilon_{10}} (h \circ \nabla u)(t), \]

where \( C_p \) is the Poincarè coefficient. Combining (34)-(39) and (41) and also using (A2), we deduce

\[ \psi'(t) \leq \left( \varepsilon_7 - a_0^2 \int_0^t h(s)ds \right) \|u'(t)\|^2 \]

\[ + \left( \varepsilon_8 f^2(\|\nabla u(t)\|^2) + 2\varepsilon_9(1 - l)^2 + \varepsilon_{10} C_p^{2(\gamma + 1)} \left( \frac{2E_1(0)}{l} \right) \gamma \right) \|\nabla u(t)\|^2 \]

\[ + \left( \frac{\zeta_1}{4\varepsilon_7} C_p^2 + \frac{C_0}{4\varepsilon_8} + 2\varepsilon_9 + \frac{1}{4\varepsilon_9} + \frac{C_p}{4\varepsilon_{10}} \right) (1 - l)(h \circ \nabla u)(t). \]
Combining (25), (27), (35) and (42), we deduce

\begin{equation}
F(t) = E_1(t) + \epsilon_2 \varphi(t) + \epsilon_3 \psi(t)
\end{equation}

\begin{equation}
\leq w_1 \|u'(t)\|^2 + w_2 \int_{\Omega} M(x, t, \|\nabla u(t)\|^2)\nabla u(x, t)^2 \,dx + w_3 (h \circ \nabla u(t)) - \|u(t)\|_{\gamma+2}^2
\end{equation}

where

\begin{equation}
w_1 = \frac{m_1}{4\epsilon_1} + \epsilon_2 + \epsilon_3 \left( \epsilon_7 - a_0^2 \int_0^t h(s) \,ds \right),
\end{equation}

\begin{equation}
w_2 = f(\|\nabla u(t)\|^2)C_0 \left[ C_p \bar{C}_1 + \epsilon_1 m_1 + C_B - \frac{1}{2}a_0 h(t) \right]
+ \frac{\epsilon_2 f(\|\nabla u(t)\|^2)C_0}{2} (1 - 2m_0 + (1 + \epsilon_6)(1 - l)^2)
+ \epsilon_3 f(\|\nabla u(t)\|^2)C_0 \left( \epsilon_8 f^2(\|\nabla u(t)\|^2) + 2\epsilon_9 (1 - l)^2 + \epsilon_10 C_p^{2(\gamma+1)} \left( \frac{2E_1(0)}{l} \right)^\gamma \right),
\end{equation}

\begin{equation}
w_3 = -\frac{\zeta_2}{2} + \left[ \frac{\epsilon_2 (4\epsilon_6 + 1)}{8\epsilon_6} \right] + \epsilon_3 \left( \frac{\zeta_1}{4\epsilon_7} C_p^2 + \frac{C_0}{4\epsilon_8} + 2\epsilon_9 + \frac{1}{4\epsilon_9} + \frac{C_p}{4\epsilon_10} \right) (1 - l),
\end{equation}

By using the smallness condition in (A_2) and (B_2), for the fixed \(\epsilon_i, i = 1, 4, \cdots, 10\), we choose \(\epsilon_j > 0, j = 2, 3\) such that \(w_k < 0, k = 1, 2, 3\). According to (23) and (43), there exist a positive constant \(s\) such that

\begin{equation}
F(t) \leq -sE_1(t)
\end{equation}

for all \(t\) which is larger than the fixed time \(T_0\). We conclude from (30) and (44) that

\begin{equation}
F(t) \leq -s\alpha_1 F(t)
\end{equation}

for all \(t\) which is larger than the fixed time \(T_0\). That is, for all \(t\) which is larger than the fixed time \(T_0\),

\begin{equation}
F(t) \leq F(T_0)e^{\alpha_1 T_0}e^{-s\alpha_1 t}.
\end{equation}

Therefore, we deduce from (30), (26) and (45) that there are positive constants \(\kappa\) and \(\vartheta\) such that

\(E(t) \leq \kappa \exp\{-\vartheta t\} \quad \text{for all } t \geq 0 \text{ and as } t \to +\infty.\)

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\(\square\)
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References


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