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EXPONENTIAL DECAY FOR THE SOLUTION OF THE VISCOELASTIC KIRCHHOFF TYPE EQUATION WITH MEMORY CONDITION AT THE BOUNDARY

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Abstract. In this paper, we study the viscoelastic Kirchhoff type equation with a nonlinear source for each independent kernels h and g with respect to Volterra terms. Under the smallness condition with respect to Kirchhoff coefficient and the relaxation function and other assumptions, we prove the uniform decay rate of the Kirchhoff type energy.

1. Introduction

In the present work, we are concerned with the following problem:

$$\begin{aligned}
&u_{tt}(x,t) - M(x,t,\|\nabla u(t)\|^2)\Delta u(x,t) \qquad\qquad (1)\\
&\qquad + \int_0^t h(t-\tau)div[a(x)\nabla u(\tau)]d\tau + |u|^\gamma u = 0 \quad \text{in} \quad \Omega\times(0,T),\\
&u(x,t) = 0 \quad \text{on} \quad \Gamma_0\times(0,T), \qquad\qquad (2)\\
&u(x,t) + \int_0^t g(t-\tau)M(x,\tau,\|\nabla u(\tau)\|^2)\frac{\partial u}{\partial\nu}(\tau)d\tau = 0 \quad \text{on} \quad \Gamma_1\times(0,T), \qquad (3)\\
&[a(x)\nabla u(\tau)]\cdot\nu = 0 \quad \text{on} \quad \Gamma_2\times(0,t), \qquad\qquad (4)\\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \qquad \text{in} \quad \Omega, \qquad\qquad (5)
\end{aligned}$$

where Ω be a bounded open set of $\mathbb{R}^N(N \geq 1)$ with a smooth boundary Γ of class C^2, $\gamma > 0$, and other conditions such as M, h, a be in next section. Indeed, $t < T$ in (4). We consider $\Gamma_0, \Gamma_1, \Gamma_2$ having positive Lebesque measures and $\overline{\Gamma_0}\cap\overline{\Gamma_1}\cap\overline{\Gamma_2} = \phi$. Let ν be the outward normal to Γ and $T > 0$ be a real number. In fact, u_0, u_1 are initially given functions and $u(x,t)$ is the transversal displacement of the strip at spatial coordinate x and time t in the real world application.

Our system works independently with respect to kernels for Volterra terms and spatial part for the Kirchhoff term under internal space or not. Physically,

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first, in the space Ω, the Volterra energy is only acted on h. Second, in the space Γ_0, There is no Volterra energy. Third, the Volterra energy is only acted on g in the space Γ_1 And also, the main system has a difference when it comes to the Kirchhoff type term under internal space or not in this work. More precisely, the Kirchhoff type term is not affected only by spatial part on the boundaries That is, not only $M(x,t,\|\nabla u(t)\|^2) = 0$ on the space Γ_0 but also $M(x,t,\|\nabla u(t)\|^2) = M(t,\|\nabla u(t)\|^2)$ on the space Γ_1 in this work. So we let you know the follows again:

$$M(x,t,\|\nabla u(t)\|^2) := M(t,\|\nabla u(t)\|^2) \quad \text{on} \quad \Gamma_1. \tag{6}$$

This problem has its origin in the mathematical description of system in real world from the mathematical modeling for axially moving viscoelastic materials. It is well known that viscoelastic materials exhibit natural damping, which is due to the special property of these materials to retain a memory of their past history. From the mathematical point of view, these damping effects are modeled by integro-differential operators. For these reasons, there are not exist weak or strong damping term in our problem (1)-(5). Recently, problems with Timoshenko or basic hyperbolic type for viscoelastic materials have been considered by many authors (See [1, 2]). Besides, many engineering devices involve the transverse vibration of axially moving strings. Axially moving string is a typical model that is widely used, especially when the subject is long and narrow enough and has a negligible flexural rigidity, to represent threads, wires, magnetic tapes, belts, band saws, and cables. Various mathematical models and simulations have been established for a better understanding with linear or nonlinear dynamic behavior of these moving continua [3, 4, 5, 6, 7, 8, 9]. The mathematical model for axially moving strings was first introduced by Kirchhoff [10] (and see Carrier [3]), and the original equation is given in the form of

$$\rho h \frac{\partial^2 u}{\partial t^2} = \Big(p_0 \ + \ \frac{Eh}{2L} \int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2 dx \Big) \frac{\partial^2 u}{\partial x^2}$$

for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate x and time t; E, the young's modulus; ρ, the mass density; h, the cross section area; L, the length; and p_0, the initial axial tension. Recently, problems with the extended Kirchhoff type equation which is concerning axially moving heterogeneous or non heterogeneous materials (nonlinear vibrations of beams, strings, plates, and membranes) have been considered by many authors (See [11, 12, 13, 14]).

In this paper, we will mainly concern on an aspect of decay rate of the Kirchhoff type energy of the system. Our purpose is focused on not only main equation but also boundary condition which are involved in memory effects for the problem otherwise the previous result [15, 16, 17]. We get its proof by using the smallness condition functions with respect to Kirchhoff coefficient and the relaxation function. In fact, the difference of the energy consist in Kirchhoff type potential energy.

This paper organized as follows. In Section 2, we will present some notations, material needed (assumptions, lemmas and so on) for our work and state a global existence and energy decay rate theorem (main result). Section 3 contains the proof of our main result.

2. Preliminaries and main results

We first introduce the elementary bracket pairing in $\Omega \subset \mathbb{R}^N$

$$\langle \varphi, \psi \rangle \equiv \int_\Omega (\varphi, \psi) dx,$$

provided that $(\varphi, \psi) \in L^1(\Omega)$. And we set the norms as follows.

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}.$$

To simplify the notations, we denote $\|u\|_{L^2(\Omega)}, \|u\|_{L^1(0,+\infty)}, \ \|v\|_{L^\infty(0,+\infty)}$ by $\|u\|, \|v\|_{L^1}, \|v\|_{L^\infty}$ respectively.

In the sequel we state the general hypotheses.

(A_1) $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function satisfying $h(0) > 0$, and there exists positive constant $t_0, \zeta_1, \zeta_2, \zeta_3$ such that

$$-\zeta_1 \le h'(t) \le -\zeta_2 h(t), \quad \forall t > t_0,$$
$$0 \le h''(t) \le \zeta_3 h(t), \quad \forall t > t_0.$$

(A_2) $a : \Omega \to \mathbb{R}^+$ is a nonnegative bounded function and $a(x) \ge a_0 > 0$ on Ω with

$$\frac{m_0}{a_0} \ge 1 - \|a\|_\infty \int_0^\infty h(s) ds = l > 0,$$

where m_0 is in (B_2). And also, the following smallness condition satisfy

$$\epsilon_7 < a_0^2 \int_0^t h(s) ds.$$

(A_3) γ satisfies

$$0 \le \gamma \le \frac{2}{n-2}, \quad n \ge 3,$$
$$\gamma \ge 0, \quad n = 1, 2.$$

(B_1) $M(x,t,\lambda)$ is a real-valued function of class C^2 on $x \in \overline{\Omega}, \ t \ge 0, \ \lambda \le 0$.

(B_2) $0 < m_0 \le M(x,t,\lambda) \le C_0 f(\lambda)$ with $M(x,t,\lambda) = M_1(x,t) + M_2(x,t,\lambda)$. And also, the following smallness condition satisfy

$$f(\lambda) < \sqrt{\frac{\frac{a_0 h(t)}{2} - C_p \widetilde{C_1} + \epsilon_2 \left(m_0 - \frac{1}{2} \right)}{\epsilon_3 \epsilon_8}}.$$

(B_3) $\frac{\partial M_1}{\partial t} \le 0, \ \left| \frac{\partial M_2}{\partial t} \right| \le C_1 g_1(\lambda), \ \left| \frac{\partial M}{\partial \lambda} \right| \le C_2 g_2(\lambda), \ 0 < m_1 \le M_x(x,t,\lambda).$

(B_4) $f, g_1, g_2 \in C^1([0,+\infty); \mathbb{R}_+)$ are strictly increasing.
Furthermore, $C_i \ (i = 0,1,2)$ is a positive constant.

Next, we assume that the kernel g is positive and k satisfies:

$$\begin{aligned} 0 < k(t) \le b_0 e^{-s_0 t}, \\ -b_1 k(t) \le k'(t) \le -b_2 k(t), \\ -b_3 k'(t) \le k''(t) \le -b_4 k'(t) \end{aligned} \tag{7}$$

for some positive constants $b_i,\ i = 0, 1, 2, 3, 4$ and s_0. To facilitate our analysis, we introduce the following binary operators:

$$(g\Box u)(t) = \int_0^t g(t-\tau)|u(t) - u(\tau)|^2 d\tau,$$

$$(g * u)(t) = \int_0^t g(t-\tau)u(\tau) d\tau$$

where $*$ is the convolution product. Differentiating (3) we arrive at the Volterra equation:

$$M(t, \|\nabla u(t)\|^2)\frac{\partial u}{\partial \nu} + \frac{1}{g(0)} g' * M(t, \|\nabla u(t)\|^2)\frac{\partial u}{\partial \nu} = -\frac{1}{g(0)} u_t.$$

Using the Volterra inverse operator, we get

$$M(t, \|\nabla u(t)\|^2)\frac{\partial u}{\partial \nu} = -\frac{1}{g(0)} u_t + k * u_t$$

where the resolvent kernel satisfy

$$k + \frac{1}{g(0)} g' * k = -\frac{1}{g(0)} g'.$$

With $\varsigma = \frac{1}{g(0)}$ and using the above identity, we obtain

$$M(t, \|\nabla u(t)\|^2)\frac{\partial u}{\partial \nu} = -\varsigma\{u_t + k(0)u - k(t)u_0 + k' * u\}. \tag{8}$$

In the following, we give a lemma which will be useful in this paper.

Lemma 2.1. *For $g, \Psi \in C^1([0,\infty) : \mathbb{R})$. Then we have*

$$(g * \psi)\Psi_t = -\frac{1}{2} g(t)|\psi(t)|^2 + \frac{1}{2} g'\Box\psi - \frac{1}{2}\frac{d}{dt}\left[g\Box\psi - \left(\int_0^t g(s) ds\right)|\psi|^2\right] \tag{9}$$

Proof. The proof of this lemma follows by differentiating the term $g\Box\psi$. □

Lemma 2.2. *Denote $(h \diamond u)(t) = \int_0^t h(t-\tau)\|\sqrt{a(x)}(u(t) - u(\tau))\|^2 d\tau$. Then we have*

$$\begin{aligned} \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau = &- \frac{1}{2}\frac{d}{dt}[(h \diamond u)(t)] + \frac{1}{2}(h' \diamond u)(t) \\ &+ \frac{1}{2}\frac{d}{dt}\left[\|\sqrt{a(x)}\nabla u(t)\|^2 \int_0^t h(s) ds\right] \\ &- \frac{1}{2} h(t)\|\sqrt{a(x)}\nabla u(t)\|^2. \end{aligned} \tag{10}$$

Proof. A direct computation shows that

$$\begin{aligned}
\int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau = &\int_0^t h(t-\tau)\langle a(x)\nabla u(\tau) - a(x)\nabla u(t), \nabla u'(t)\rangle d\tau \\
&+ \int_0^t h(t-\tau)\langle a(x)\nabla u(t), \nabla u'(t)\rangle d\tau \\
= &-\frac{1}{2}\int_0^t h(t-\tau)\left[\frac{d}{dt}\|\sqrt{a(x)}(\nabla u(\tau) - \nabla u(t))\|^2\right] d\tau \\
&+ \frac{1}{2}\int_0^t h(t-\tau)\left[\frac{d}{dt}\|\sqrt{a(x)}\nabla u(t)\|^2\right] d\tau \\
= &-\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(t-\tau)\|\sqrt{a(x)}(\nabla u(\tau) - \nabla u(t))\|^2 d\tau\right] \\
&+ \frac{1}{2}\int_0^t h'(t-\tau)\|\sqrt{a(x)}(\nabla u(\tau) - \nabla u(t))\|^2 d\tau \\
&+ \frac{1}{2}\frac{d}{dt}\int_0^t h(t-\tau)\|\sqrt{a(x)}\nabla u(t)\|^2 d\tau \\
&- \frac{1}{2}h(t)\|\sqrt{a(x)}\nabla u(t)\|^2.
\end{aligned}$$

□

Lemma 2.3. (General Poincaré Inequality).
Denote $H^1_{\Gamma_0}(\Omega) = \{u | u \in H^1(\Omega), u|_{\Gamma_0} = 0\}$ *and* $meas(\Gamma_0) > 0$. *Then there exists a positive constant* B *such that* $\|u\|_{L^2(\Gamma)} \le B\|\nabla u\|_{L^2(\Omega)}$, *for all* $u \in H^1_{\Gamma_0}(\Omega)$.

Proof. The proof can be found in [18]. □

Then, we can state our result as follows.

Theorem 2.4. *Let the assumptions* $(A_1), (A_3)$ *and* (B_1)-(B_4) *and the relating conditions* (7) *and* (8) *to the volterra term on boundary hold and the sobolev space* V *is* $\{u \mid u \in H^1_0(\Omega),\ u = 0 \quad on \quad \Gamma_0\}$. *If* $(u_0, u_1) \in (H^2(\Omega) \cap V) \times V$ *and satisfy the compatibility condition*

$$M(0, \|\nabla u_0\|^2)\frac{\partial u_0}{\partial \nu} = -\varsigma u_1 \quad on\ \Gamma_1. \tag{11}$$

Then there exists a unique solution u *of the problem* (1)-(5) *satisfying*

$$u \in L^\infty(0,T; H^1_0(\Omega) \cap H^2(\Omega)),\ u' \in L^\infty(0,T; H^1_0(\Omega)),\ u'' \in L^\infty(0,T; L^2(\Omega)),$$

and

$$u(x,t) \to u_0(x)\ in\ V \cap H^2(\Omega); \qquad u'(x,t) \to u_1(x)\ in\ V,$$

as $t \to 0$

Proof. By using Galerkin's approximation and a routine procedure similar to that of cite [12, 1], we can the global existence result for the solution subject to (1)-(5)

under the assumptions (A_1)-(A_3) and (B_1)-(B_4) and the relating conditions (7) and (8) to the volterra term on boundary. $\square$

Theorem 2.5. *Let u be the global solution of the problem* (1)-(5) *with the above all conditions. We define the Kirchhoff type energy functional $E(t)$ as*

$$\begin{aligned} E(t) = {} & \tfrac{1}{2}\int_\Omega |u'(t)|^2 dx + \tfrac{1}{2}\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx \\ & + \tfrac{1}{\gamma+2}\|u'(t)\|_{\gamma+2}^{\gamma+2} + \tfrac{\varsigma}{2}(M(t,\|\nabla u(t)\|^2)\int_{\Gamma_1}|u(x,t)|^2 d\Gamma_1 \\ & - \int_{\Gamma_1} M'(t,\|\nabla u(t)\|^2)\square u(t) d\Gamma_1). \end{aligned}$$

Then the energy functional decays exponentially to zero as the time goes to infinity, that is,

$$E(t) \le \kappa e^{-\vartheta t}, \ \forall t \ge 0$$

where κ, ϑ are positive constants.

3. Proof of Theorem 2.5 (Energy decay)

Proof. Multiplying u' on both sides of Eq.(1), integrating the resulting equations over Ω, and using the Green formula, (6) and (3), we have

$$\begin{aligned} & \langle u''(t), u'(t)\rangle + \langle M(x,t,\|\nabla u(t)\|^2)\nabla u(t), \nabla u'(t)\rangle \\ & \quad + \langle M_x(x,t,\|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\ & \quad + \left\langle M(t,\|\nabla u(t)\|^2)\frac{\partial u}{\partial \nu}(t), u'(t)\right\rangle_{\Gamma_1} \\ & \quad - \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau + \langle |u|^\gamma u, u'\rangle = 0, \end{aligned} \tag{12}$$

that is

$$\begin{aligned} \frac{d}{dt}E(t) = {} & \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}M_1(x,t)|\nabla u(x,t)|^2 dx \\ & + \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}M_2(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx \\ & + \left[\int_\Omega \frac{\partial}{\partial \lambda}M_2(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx\right]\langle \nabla u'(t), \nabla u(t)\rangle \\ & - \langle M_x(x,t,\|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\ & - \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau \\ & + \left\langle M(t,\|\nabla u(t)\|^2)\frac{\partial u}{\partial \nu}(t), u'(t)\right\rangle_{\Gamma_1} \\ & + \frac{\varsigma}{2}\frac{d}{dt}\left[M(t,\|\nabla u(t)\|^2)\int_{\Gamma_1}|u(x,t)|^2 d\Gamma_1\right] \\ & - \frac{\varsigma}{2}\frac{d}{dt}\int_{\Gamma_1} M'(t,\|\nabla u(t)\|^2)\square u(t) d\Gamma_1, \end{aligned} \tag{13}$$

where

$$
\begin{aligned}
E(t) = {} & \frac{1}{2}\int_\Omega |u'(t)|^2 dx + \frac{1}{2}\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx \\
& + \frac{1}{\gamma+2}\|u'(t)\|_{\gamma+2}^{\gamma+2} + \frac{\varsigma}{2} M(t,\|\nabla u(t)\|^2)\int_{\Gamma_1} |u(x,t)|^2 d\Gamma_1 \\
& - \frac{\varsigma}{2}\int_{\Gamma_1} M'(t,\|\nabla u(t)\|^2)\Box u(t) d\Gamma_1.
\end{aligned} \tag{14}
$$

Therefore, we have

$$
\begin{aligned}
\frac{d}{dt}E(t) = {} & \frac{1}{2}\int_\Omega \frac{\partial}{\partial t} M_1(x,t)|\nabla u(x,t)|^2 dx \\
& + \frac{1}{2}\int_\Omega \frac{\partial}{\partial t} M_2(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx \\
& + \left[\int_\Omega \frac{\partial}{\partial \lambda} M_2(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx\right]\langle \nabla u'(t), \nabla u(t)\rangle \\
& - \langle M_x(x,t,\|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\
& - \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau \\
& - \varsigma\int_{\Gamma_1} |u'(t)|^2 d\Gamma_1 - \frac{\varsigma}{2}k(0)\frac{d}{dt}\int_{\Gamma_1} |u|^2 d\Gamma_1 \\
& + \varsigma\int_{\Gamma_1} k(t)u_0 u_t d\Gamma_1 - \varsigma\int_{\Gamma_1} (k' * u)u_t d\Gamma_1 \\
& + \frac{\varsigma}{2}\frac{d}{dt}\left[M(t,\|\nabla u(t)\|^2)\int_{\Gamma_1} |u(x,t)|^2 d\Gamma_1\right] \\
& - \frac{\varsigma}{2}\frac{d}{dt}\int_{\Gamma_1} M'(t,\|\nabla u(t)\|^2)\Box u(t) d\Gamma_1.
\end{aligned} \tag{15}
$$

From (B_3), (7), Lemma 2.1 and Hölder inequality, we obtain

$$
\begin{aligned}
E'(t) \le {} & \|u(t)\|^2 \left\{\frac{C_1}{2} g_1(\|\nabla u(t)\|^2) + C_2 g_2(\|\nabla u(t)\|^2)\|\nabla u'(t)\|\|u(t)\|\right\} \\
& - \langle M_x(x,t,\|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\
& - \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau \\
& - \frac{\varsigma}{2}\int_{\Gamma_1} |u'(t)|^2 d\Gamma_1 + \frac{\varsigma}{2}k^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1 \\
& + \frac{\varsigma}{2}k'(t)\int_{\Gamma_1} |u(t)|^2 d\Gamma_1 - \frac{\varsigma}{2}\int_{\Gamma_1} k''\Box u d\Gamma_1 \\
& - \frac{\varsigma}{2}\frac{d}{dt}\left[\left(\int_0^t k'(\tau)d\tau\right)|u|^2\right] d\Gamma_1
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&\leq \|u(t)\|^2 \left\{ \frac{C_1}{2} g_1(\|\nabla u(t)\|^2) + C_2 g_2(\|\nabla u(t)\|^2)\|\nabla u'(t)\|\|u(t)\| \right\} \\
&- \langle M_x(x,t,\|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\
&- \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau \\
&- \frac{\varsigma}{2}\int_{\Gamma_1} |u'(t)|^2 d\Gamma_1 + \frac{\varsigma}{2}k^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1 \\
&+ \frac{\varsigma}{2}k'(t)\int_{\Gamma_1} |u(t)|^2 d\Gamma_1 - \frac{\varsigma}{2}\int_{\Gamma_1} k''\Box u d\Gamma_1.
\end{aligned}
$$

By (B_3), (10)and Young's inequality, we have

$$
\begin{aligned}
E'(t) \leq & \|u(t)\|^2 \widetilde{C_1} + \epsilon_1 m_1 \|\nabla u(t)\|^2 + \frac{m_1}{4\epsilon_1}\|u'(t)\|^2 \\
& - \frac{1}{2}\frac{d}{dt}\left[(h \diamond u)(t)\right] + \frac{1}{2}(h' \diamond \nabla u)(t) \\
& + \frac{1}{2}\frac{d}{dt}\left[\|\sqrt{a(x)}\nabla u(t)\|^2 \int_0^t h(s)ds\right] \\
& - \frac{1}{2}h(t)\|\sqrt{a(x)}\nabla u(t)\|^2 \\
& - \frac{\varsigma}{2}\int_{\Gamma_1} |u'(t)|^2 d\Gamma_1 + \frac{\varsigma}{2}k^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1 \\
& + \frac{\varsigma}{2}k'(t)\int_{\Gamma_1} |u(t)|^2 d\Gamma_1 - \frac{\varsigma}{2}\int_{\Gamma_1} k''\Box u d\Gamma_1,
\end{aligned} \tag{17}
$$

where

$$
\widetilde{C_1} = \frac{C_1}{2} g_1(\|\nabla u(t)\|^2) + C_2 g_2(\|\nabla u(t)\|^2)\|\nabla u'(t)\|\|u(t)\| \tag{18}
$$

is a positive constant. And ϵ_1 is also a positive constant.

In the boundary Γ_1, note that

$$
\begin{aligned}
-k(0)u(t) - k' * u(t) =& -\int_0^t k'(t-\tau)[u(\tau) - u(t)]d\tau - k(t)u(t) \\
& \leq \left(\int_0^t |k'(\tau)|d\tau\right)^{\frac{1}{2}} [|k'|\Box u(x,t)]^{\frac{1}{2}} + k(t)|u(t)| \\
& \leq |k(t) - k(0)|^{\frac{1}{2}} [|k'|\Box u(t)]^{\frac{1}{2}} + k(t)|u(t)|.
\end{aligned} \tag{19}
$$

Using (8) and (19), follows that

$$\begin{aligned}&\int_{\Gamma_1} M(t,\|\nabla u(t)\|^2)\left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma_1\\ \le& C_{b_1}\int_{\Gamma_1}\{|u_t(t)|^2+k^2(t)|u_0|^2+k(0)|k'|\Box u(t)+k(0)k(t)|u(t)|^2\}d\Gamma_1.\end{aligned}\tag{20}$$

where C_{b_1} is a positive constant.

Here we use (7) to conclude the following estimates for the corresponding two terms appearing in (22).

$$\begin{aligned}-\frac{\varsigma}{2}\int_{\Gamma_1} k''\Box u(t)d\Gamma_1 &\le C_{\varsigma_1}\int_{\Gamma_1} k'\Box u(t)\Gamma_1\\ -\frac{\varsigma}{2}\int_{\Gamma_1} k'|u(t)|^2 d\Gamma_1 &\le -C_{\varsigma_1}\int_{\Gamma_1} k|u(t)|^2\Gamma_1.\end{aligned}\tag{21}$$

where C_{ς_1} is a positive constant.

By using (20) and (21) in (22), we conclude

$$\begin{aligned}&E'(t)+\int_{\Gamma_1} M(t,\|\nabla u(t)\|^2)\left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma_1\\ \le&\|u(t)\|^2\widetilde{C_1}+\epsilon_1 m_1\|\nabla u(t)\|^2+\frac{m_1}{4\epsilon_1}\|u'(t)\|^2\\ &-\frac{1}{2}\frac{d}{dt}\left[(h\diamond u)(t)\right]+\frac{1}{2}(h'\diamond\nabla u)(t)\\ &+\frac{1}{2}\frac{d}{dt}\left[\|\sqrt{a(x)}\nabla u(t)\|^2\int_0^t h(s)ds\right]\\ &-\frac{1}{2}h(t)\|\sqrt{a(x)}\nabla u(t)\|^2\\ &C_{\varsigma_2}\int_{\Gamma_1}\{|u_t(t)|^2+k^2(t)|u_0|^2+k(0)|k'|\Box u(t)+k(0)k(t)|u(t)|^2\}d\Gamma_1\end{aligned}\tag{22}$$

where C_{ς_2} is a positive constant.

Define the new energy functional $E_1(t)$ as follows

$$E_1(t)=E(t)+\frac{1}{2}(h\diamond\nabla u)(t)-\frac{1}{2}\|\sqrt{a(x)}\nabla u(t)\|^2\int_0^t h(s)ds+\int_{\Gamma_1} M(t,\|\nabla u(t)\|^2)\left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma_1.\tag{23}$$

Then from (A_1), (B_2), (22) and Lemma (2.3), we have

$$\begin{aligned}E_1'(t)\le&\|u(t)\|^2\widetilde{C_1}+(\epsilon_1 m_1+C_B)\|\nabla u(t)\|^2+\frac{m_1}{4\epsilon_1}\|u'(t)\|^2\\ &-\frac{\zeta_2}{2}(h\diamond\nabla u)(t)-\frac{1}{2}a_0 h(t)\|\nabla u(t)\|^2,\end{aligned}\tag{24}$$

where C_B is a positive constant relating Poincaré constant B and also, by (A_2), the energy $E_1(t)$ is a positive functional. Applying Poincarè inequality to (24), we deduce

$$E'(t) \leq \left(C_p\widetilde{C_1} + \epsilon_1 m_1 - \frac{1}{2}a_0 h(t) \right) \|\nabla u(t)\|^2 + \frac{m_1}{4\epsilon_1}\|u'(t)\|^2 - \frac{\zeta_2}{2}(h \diamond \nabla u)(t), \tag{25}$$

where C_p is the Poincarè coefficient. Meanwhile, we note from (A_1) and (A_2) that

(26)

$$\begin{aligned} E_1(t) \geq & \frac{1}{2}\|u(t)\|^2 + \frac{1}{2}\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx \\ & + \frac{1}{2}\left(1 - \|a\|_\infty \int_0^t h(s)ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(h \diamond u)(t) + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2} \\ \geq & l\left[\frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2}\right]. \end{aligned}$$

So, we deduce the relation $0 \leq E(t) \leq l^{-1}E_1(t)$. Therefore, the uniform decay of $E(t)$ is a result of the decay of $E_1(t)$. For positive constants ϵ_2 and ϵ_3, let us define the perturbed modified energy by

$$F(t) = E_1(t) + \epsilon_2\varphi(t) + \epsilon_3\psi(t), \tag{27}$$

where

$$\varphi(t) = \langle u'(t), u(t)\rangle. \tag{28}$$

and

$$\psi(t) = -\int_0^t h(t-\tau)\langle a(x)u'(t), u(t) - u(\tau)\rangle d\tau. \tag{29}$$

By using the Cauchy's inequality, Hölder inequality and Poincarè inequality, there exist positive constants α_1, α_2 such that for each $t > 0$

$$\alpha_1 F(t) \leq E_1(t) \leq \alpha_2 F(t). \tag{30}$$

Proposition 3.1. (Energy equivalence)

$$\alpha_1 F(t) \leq E_1(t) \leq \alpha_2 F(t) \quad \textit{for all } t \geq 0,$$

where

$$\alpha_1 = \frac{1}{\max\{1, \epsilon_4 + \epsilon_5, (\epsilon_4 + \epsilon_5)C_p\}} > 0$$

and

$$\alpha_2 = \frac{1}{\min\{1 - (\epsilon_4 + \epsilon_5), m_0 - \epsilon_4 C_p, 1 - \epsilon_5 C_p(1-l)\}} > 0.$$

Proof. By the Cauchy inequality and Hölder inequality, we get

$$\begin{aligned} F(t) \leq & E_1(t) + \frac{\epsilon_4}{2}\|u'(t)\|^2 + \frac{\epsilon_4}{2}\|u(t)\|^2 \\ & + \frac{\epsilon_5}{2}\left\|\int_0^t h(t-\tau)a(x)(u(t) - u(\tau))d\tau\right\| + \frac{\epsilon_5}{2}\|u'(t)\|, \end{aligned} \tag{31}$$

where ϵ_4, ϵ_5 are positive constants.

By using the Poincarè inequality, we have

$$
\begin{aligned}
F(t) \leq & E_1(t) + \frac{\epsilon_4 + \epsilon_5}{2}\|u'(t)\|^2 + \frac{\epsilon_4}{2}\|u(t)\|^2 \\
& + \frac{\epsilon_5}{2}\|a\|_\infty \int_0^\infty h(s)ds \int_0^t h(t-\tau) \left\| \sqrt{a(x)}(u(t)-u(\tau)) \right\|^2 d\tau \\
\leq & E_1(t) + \frac{\epsilon_4 + \epsilon_5}{2}\|u'(t)\|^2 + \frac{\epsilon_4}{2}\|u(t)\|^2 \\
& + \frac{\epsilon_5}{2}C_p(1-l) \int_0^\infty h(s)ds \int_0^t h(t-\tau) \left\| \sqrt{a(x)}(u(t)-u(\tau)) \right\|^2 d\tau \\
\leq & E_1(t) + \frac{\epsilon_4 + \epsilon_5}{2}\|u'(t)\|^2 + \frac{\epsilon_4}{2}C_p\|\nabla u(t)\|^2 \\
& + \frac{\epsilon_5}{2}C_p(1-l)(h \diamond \nabla u)(t) \\
\leq & \max\{1, \epsilon_4 + \epsilon_5, (\epsilon_4 + \epsilon_5)C_p\} E_1(t),
\end{aligned}
$$

where C_p is the Poincarè coefficient. Besides, choosing ϵ_4, ϵ_5 small enough, we have

$$
\begin{aligned}
F(t) \geq & E_1(t) - \frac{\epsilon_4 + \epsilon_5}{2}\|u'(t)\|^2 - \frac{\epsilon_4}{2}C_p\|\nabla u(t)\|^2 \\
& - \frac{\epsilon_5}{2}C_p(1-l)(h \diamond \nabla u)(t) \\
\geq & \frac{1-(\epsilon_4+\epsilon_5)}{2}\|u'(t)\|^2 + \frac{m_0 - \epsilon_4 C_p}{2}\|\nabla u(t)\|^2 \\
& - \frac{1}{2}\|\sqrt{a(x)}\nabla u(t)\|^2 \int_0^t h(s)ds + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2} \\
& + \frac{1 - \epsilon_5 C_p(1-l)}{2}(h \diamond \nabla u)(t) \\
\geq & \min\{1-(\epsilon_4+\epsilon_5), m_0 - \epsilon_4 C_p, 1 - \epsilon_5 C_p(1-l)\} E_1(t).
\end{aligned}
$$

□

In fact, using (1), we have

(32)
$$
\begin{aligned}
\varphi'(t) = & \langle u''(t), u(t) \rangle + \|u'(t)\|^2. \\
= & \|u'(t)\|^2 + \Big\langle u(t), M(x,t,\|\nabla u(t)\|^2)\Delta u(x,t) \\
& - \int_0^t h(t-\tau) div[a(x)\nabla u(\tau)]d\tau - |u(t)|^\gamma u(t) \Big\rangle \\
= & \|u'(t)\|^2 - \int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(t)|^2 dx \\
& + \int_0^t h(t-\tau) \langle a(x)\nabla u(\tau), \nabla u(t) \rangle] d\tau - |u(t)|^\gamma u(t).
\end{aligned}
$$

By Cauchy inequality and Young's inequality, we have

$$
\begin{aligned}
&\left|\int_0^t h(t-\tau)\left\langle a(x)\nabla u(\tau), \nabla u(t)\right\rangle] d\tau\right| \\
\le&\frac{1}{2}\|\nabla u(t)\|^2+\frac{1}{2}\left\|\int_0^t h(t-\tau)(a(x)|\nabla u(\tau)-\nabla u(t)|+a(x)|\nabla u(t)|)d\tau\right\|^2 \\
\le&\frac{1}{2}\|\nabla u(t)\|^2+\left(\frac{1}{2}+\frac{1}{8\epsilon_6}\right)\left\|\int_0^t h(t-\tau)a(x)|\nabla u(\tau)-\nabla u(t)|d\tau\right\|^2 \\
&+\left(\frac{1}{2}+\frac{\epsilon_6}{2}\right)\left\|\int_0^t h(t-\tau)a(x)|\nabla u(t)|d\tau\right\|^2,
\end{aligned}
\tag{33}
$$

where ϵ_6 with respect to Young's inequality is a positive constant. Using the assumption (A_2) and (33), we get

$$
\begin{aligned}
&\left|\int_0^t h(t-\tau)\left\langle a(x)\nabla u(\tau), \nabla u(t)\right\rangle] d\tau\right| \\
\le&\left(\frac{1}{2}+\frac{1}{8\epsilon_6}\right)\|a\|_\infty\int_0^t h(s)ds\int_0^t h(t-\tau)\left\|\sqrt{a(x)}(\nabla u(\tau)-\nabla u(t))\right\|^2 d\tau \\
&+\left(\frac{1}{2}+\frac{\epsilon_6}{2}\right)\|\nabla u(t)\|^2\left(\|a\|_\infty\int_0^t h(s)a(x)ds\right)^2+\frac{1}{2}\|\nabla u(t)\|^2 \\
\le&\frac{1}{2}(1+(1+\epsilon_6)(1-l)^2)\|\nabla u(t)\|^2+\frac{(4\epsilon_6+1)(1-l)}{8\epsilon_6}(h\diamond\nabla u)(t).
\end{aligned}
\tag{34}
$$

By combining (32) and (34), we conclude

$$
\begin{aligned}
\varphi'(t)\le&\|u'(t)\|^2+\frac{1}{2}(1-2m_0+(1+\epsilon_6)(1-l)^2)\|\nabla u(t)\|^2 \\
&+\frac{(4\epsilon_6+1)(1-l)}{8\epsilon_6}(h\diamond\nabla u)(t)-\|u(t)\|_{\gamma+2}^{\gamma+2}.
\end{aligned}
\tag{35}
$$

Next, we estimate $\psi'(t)$ as follows. In fact, using (1), we have

(36)

$$\begin{aligned}
\psi'(t) =& -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t)-u(\tau)\rangle d\tau. \\
& -\int_0^t h(t-\tau)\langle a(x)u''(t), u(t)-u(\tau)\rangle d\tau - \|\sqrt{a(x)}u'(t)\|^2 \int_0^t h(s)ds \\
=& -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t)-u(\tau)\rangle d\tau. \\
& -\int_0^t h(t-\tau)\langle M(x,t,\|\nabla u(t)\|^2)a(x)\nabla u(t), \nabla u(t)-\nabla u(\tau)\rangle d\tau \\
& -\left\langle \int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau, \int_0^t h(t-\tau)a(x)(\nabla u(t)-\nabla u(\tau))d\tau \right\rangle \\
& +\int_0^t h(t-\tau)\langle a(x)|u|^\gamma u, u(t)-u(\tau)\rangle d\tau \\
& -\|\sqrt{a(x)}u'(t)\|^2 \int_0^t h(s)ds.
\end{aligned}$$

Using Cauchy inequality, Poincarè inequality and (A_1), we have

$$\begin{aligned}
&\left| -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t)-u(\tau)\rangle d\tau \right| \\
\le& \epsilon_7\|\nabla u(t)\|^2 + \frac{\zeta_1}{4\epsilon_7}\left\| \int_0^t h(t-\tau)a(x)|u(t)-u(\tau)|d\tau \right\|^2 \\
\le& \epsilon_7\|\nabla u(t)\|^2 + \frac{\zeta_1}{4\epsilon_7}(1-l)C_p^2(h\diamond\nabla u)(t),
\end{aligned} \tag{37}$$

where ϵ_7 is a positive constant with respect to Cauchy inequality and C_p is the Poincarè coefficient. Similarly, using Cauchy inequality and (B_2), we get

$$\begin{aligned}
&\left| -\int_0^t h(t-\tau)\langle M(x,t,\|\nabla u(t)\|^2)a(x)\nabla u(t), \nabla u(t)-\nabla u(\tau)\rangle d\tau \right| \\
\le& \epsilon_8 f^2(\|\nabla u(t)\|^2)\|u'(t)\|^2 + \frac{C_0(1-l)}{4\epsilon_8}(h\diamond\nabla u)(t)
\end{aligned} \tag{38}$$

and

(39)
$$\begin{aligned}
&\left|-\left\langle\int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau,\int_0^t h(t-\tau)a(x)(\nabla u(t)-\nabla u(\tau))d\tau\right\rangle\right|\\
\le&\epsilon_9\left\|\int_0^t h(t-\tau)(a(x)|\nabla u(t)-\nabla u(\tau)|+a(x)|\nabla u(t)|)d\tau\right\|^2\\
&+\frac{1}{4\epsilon_9}\left(\|a\|_\infty\int_0^t h(s)ds\right)\int_0^t h(t-\tau)\|\sqrt{a(x)}(\nabla u(t)-\nabla u(\tau))\|^2d\tau\\
\le&2\epsilon_9\left(\left\|\int_0^t h(t-\tau)a(x)|\nabla u(t)-\nabla u(\tau)|d\tau\right\|^2+\left\|\int_0^t h(t-\tau)a(x)|\nabla u(t)|d\tau\right\|^2\right)\\
&+\frac{1-l}{4\epsilon_9}(h\diamond\nabla u)(t)\\
\le&\left(2\epsilon_9+\frac{1}{4\epsilon_9}\right)(1-l)(h\diamond\nabla u)(t)+2\epsilon_9(1-l)^2\|\nabla u(t)\|^2,
\end{aligned}$$

where ϵ_8,ϵ_9 are positive constants with respect to Cauchy inequality. And also, using Cauchy inequality and Poincarè inequality, we have

$$\begin{aligned}
&\left|\int_0^t h(t-\tau)\langle a(x)|u(t)|^\gamma u,u(t)-u(\tau)\rangle d\tau\right|\\
\le&\epsilon_{10}\|u(t)\|_{2(\gamma+1)}^{2(\gamma+1)}+\frac{C_p(1-l)}{4\epsilon_{10}}(h\diamond\nabla u)(t),
\end{aligned}\tag{40}$$

where ϵ_{10} is a positive constant with respect to Cauchy inequality and C_p is the Poincarè coefficient. Noting $H^1(\Omega)\hookrightarrow L^{2(\gamma+1)}(\Omega)$ and using Poincarè inequality, (23), (24) and (40), we get

$$\begin{aligned}
&\left|\int_0^t h(t-\tau)\langle a(x)|u(t)|^\gamma u,u(t)-u(\tau)\rangle d\tau\right|\\
\le&\epsilon_{10}C_p^{2(\gamma+1)}\left(\frac{2E_1(0)}{l}\right)^\gamma\|\nabla u(t)\|^2+\frac{C_p(1-l)}{4\epsilon_{10}}(h\diamond\nabla u)(t),
\end{aligned}\tag{41}$$

where C_p is the Poincarè coefficient. Combining (34)-(39) and (41) and also using (A_2), we deduce

(42)
$$\begin{aligned}
\psi'(t)\le&\left(\epsilon_7-a_0^2\int_0^t h(s)ds\right)\|u'(t)\|^2\\
&+\left(\epsilon_8f^2(\|\nabla u(t)\|^2)+2\epsilon_9(1-l)^2+\epsilon_{10}C_p^{2(\gamma+1)}\left(\frac{2E_1(0)}{l}\right)^\gamma\right)\|\nabla u(t)\|^2\\
&+\left(\frac{\zeta_1}{4\epsilon_7}C_p^2+\frac{C_0}{4\epsilon_8}+2\epsilon_9+\frac{1}{4\epsilon_9}+\frac{C_p}{4\epsilon_{10}}\right)(1-l)(h\diamond\nabla u)(t).
\end{aligned}$$

Combining (25), (27), (35) and (42), we deduce

(43)

$$
\begin{aligned}
&F(t) = E_1(t) + \epsilon_2\varphi(t) + \epsilon_3\psi(t)\\
\leq& w_1\|u'(t)\|^2 + w_2\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2dx + w_3(h\diamond\nabla u(t)) - \|u(t)\|_{\gamma+2}^{\gamma+2},
\end{aligned}
$$

where

$$
w_1 = \frac{m_1}{4\epsilon_1} + \epsilon_2 + \epsilon_3\left(\epsilon_7 - a_0^2\int_0^t h(s)ds\right),
$$

$$
\begin{aligned}
w_2 =& f(\|\nabla u(t)\|^2)C_0\left[C_p\widetilde{C_1} + \epsilon_1 m_1 + C_B - \frac{1}{2}a_0h(t)\right]\\
&+\frac{\epsilon_2 f(\|\nabla u(t)\|^2)C_0}{2}(1-2m_0+(1+\epsilon_6)(1-l)^2)\\
&+\epsilon_3 f(\|\nabla u(t)\|^2)C_0\left(\epsilon_8 f^2(\|\nabla u(t)\|^2) + 2\epsilon_9(1-l)^2 + \epsilon_{10}C_p^{2(\gamma+1)}\left(\frac{2E_1(0)}{l}\right)^\gamma\right),\\
w_3 =& -\frac{\zeta_2}{2} + \left[\frac{\epsilon_2(4\epsilon_6+1)}{8\epsilon_6} + \epsilon_3\left(\frac{\zeta_1}{4\epsilon_7}C_p^2 + \frac{C_0}{4\epsilon_8} + 2\epsilon_9 + \frac{1}{4\epsilon_9} + \frac{C_p}{4\epsilon_{10}}\right)\right](1-l),
\end{aligned}
$$

By using the smallness condition in (A_2) and (B_2), for the fixed $\epsilon_i, i = 1, 4, \cdots, 10$, we choose $\epsilon_j > 0, j = 2, 3$ such that $w_k < 0, k = 1, 2, 3$. According to (23) and (43), there exist a positive constant s such that

$$
F(t) \leq -sE_1(t) \tag{44}
$$

for all t which is larger than the fixed time T_0. We conclude from (30) and (44) that

$$
F(t) \leq -s\alpha_1 F(t)
$$

for all t which is larger than the fixed time T_0. That is, for all t which is larger than the fixed time T_0,

$$
F(t) \leq F(T_0)e^{s\alpha_1 T_0}e^{-s\alpha_1 t}. \tag{45}
$$

Therefore, we deduce from (30), (26) and (45) that there are positive constants κ and ϑ such that

$$
E(t) \leq \kappa\exp\{-\vartheta t\} \quad for\ all\ t \geq 0\ and\ as\ t \to +\infty.
$$

□

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