**L^p-ESTIMATES FOR THE \( \bar{\partial} \)-EQUATION WITH EXACT SUPPORT ON \( q \)-CONVEX INTERSECTIONS**

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**Abstract.** We construct bounded linear integral operators that giving solutions to the \( \bar{\partial} \)-equation in \( L^p \)-spaces and with compact supports on a \( q \)-convex intersection (\( q \geq 1 \)) with \( C^3 \) boundary in Kähler manifolds, and we apply it to obtain a Hartogs-like extension theorems for \( \bar{\partial} \)-closed forms for some bidegree.

**1. Introduction**

The problem of solving \( \bar{\partial} \) with exact support was initiated for differential forms by Andreotti and Hill in [5] and [6]. Later, it has been widely investigated (cf. [11], [12], [13] and [18]). More precisely, let \( \Omega \subset \subset X \) be a relatively compact domain in a complex manifold \( X \) of complex dimension \( n \) and \( E \) be a holomorphic Hermitian vector bundle over \( X \). If \( \Omega \) is a bounded pseudoconvex domain in \( C^n \), Chen and Shaw proved in [11, Chapter 9] that for any \( \bar{\partial} \)-closed (\( r,s \))-form \( f \) with \( L^2 \) (or \( C^\infty \)) coefficients in \( C^n \) and compactly supported in \( \Omega \), there exists a form \( u \) in \( L^2_{r,s-1}(C^n) \) (or in \( C^\infty_{r,s-1}(C^n) \)) such that \( u \) is compactly supported in \( \Omega \) and \( \bar{\partial}u = f \) in \( C^n \), for \( 0 \leq r \leq n \) and \( 1 \leq s \leq n-1 \). When \( \Omega \) is a bounded domain with Lipschitz boundary and satisfies a convexity condition called log \( \delta \)-pseudoconvex in an \( n \)-dimensional Kähler manifold \( X \), Brinkshulte proved in [9] that if \( f \) is a \( \bar{\partial} \)-closed \( E \)-valued (\( r,s \))-form with \( C^\infty \)-coefficients in \( X \) and with compact support in \( \Omega \), then there exists a (\( r,s-1 \))-form \( u \) with \( C^\infty \)-coefficients in \( X \) and with compact support in \( \Omega \) such that \( \bar{\partial}u = f \) in \( X \), for \( 0 \leq r \leq n \) and \( 1 \leq s \leq n-1 \). Moreover, she proved that the range of the \( \bar{\partial} \)-operator acting on the subspace of (\( r,n-1 \))-forms of class \( C^\infty \) and compactly supported in \( \Omega \) is closed.

Analogous results to those of [9] have been obtained by Sambou in [23] for \( C \)-valued (\( r,s \))-forms with compact support in \( \Omega \) when \( \Omega \) is a completely strictly \( q \)-convex domain (\( 0 \leq q \leq n-1 \)) with smooth boundary in an \( n \)-dimensional complex manifold \( X \) for all \( 1 \leq s \leq q \). This domain is defined in the sense

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of Henkin by $\Omega = \{ z \in U | \rho(z) < 0 \}$ where $\rho$ is a smooth function defined on an open neighborhood $U$ of $\Omega$ whose Levi form has at least $q + 1$ positive eigenvalues everywhere (see e.g. [16]). In addition, he showed that the range of the $\partial$-operator acting on the subspace of $C^\infty (r, s - 1)$-forms with compact support in $\Omega$ is closed for $1 \leq s \leq q + 1$. Further, he proved that the $\partial$-equation is solvable on such domain for extensible currents of bidegree $(n, n - s)$ for all $s$ with $1 \leq n - q \leq s \leq n$. Furthermore, the case for strictly $q$-concave domains is dealt by the author in [24].

It worth also to mention that solving the $\bar{\partial}$-equation with prescribed support enables one to obtain $\bar{\partial}$-closed extensions of forms from boundaries of bounded domains (see e.g. [9], [10], [11] and [14]).

Solving $\bar{\partial}$ with compact support in $L^p$-spaces (or the so called the weak $L^p$ $\bar{\partial}$-Cauchy problem) is formulated by giving a $\bar{\partial}$-closed form $f$ in $L^p_{r,s}(X,E)$ with compact support in $\Omega$, $0 \leq r \leq n$, $1 \leq s \leq n$ and $p \geq 1$, the problem is then to find a form $u$ in $L^p_{r,s-1}(X,E)$ such that

\begin{equation}
\begin{cases}
\bar{\partial}u = f \text{ in the weak sense in } X, \\
supp u \subset \Omega.
\end{cases}
\end{equation}

This problem was solved by Amar and Mongodi in [4] for $C$-valued forms on Stein open domains of the form $D^n \setminus Z$ in $\mathbb{C}^n$, where $D^n$ is a polydisc and $Z$ is the zero locus of some holomorphic function, and by Amar in [3] for weakly $p$-regular domains in Stein manifolds. In [19], Laurent-Thiébaut gave some general cohomological and geometric conditions on $X$ and $\Omega$ under which (1) can be solved. In particular, she solved (1) for $C$-valued $(r,s)$-forms on completely $q$-convex domains in complex manifolds for all $1 \leq s \leq n - q$ (See Corollary 2.23 in [19]).

The plane of the paper is as follows. We first extend some complex analysis results of Amar [3] to $E$-valued currents on relatively compact domains in Kähler manifolds. Next, via a partition of unity, we globalize the local compact integral homotopy operators constructed in [20] for the $\partial$-equation to get global ones for $E$-valued forms on a $C^3$ $q$-convex intersection $\Omega$ in an $n$-dimensional Kähler manifold $X$. We moreover show that the $L^p$-$\bar{\partial}$-cohomology group $H^*_r(\Omega, E)$ is finite dimensional and the space $\bar{\partial}(L^p_{r,s-1}(\Omega, E))$ is closed subspace of $L^p_{r,s}(\Omega, E)$ for all $q \leq s \leq n - 1$. By using these integral homotopy formulas, we then solve the $\bar{\partial}$-equation with global $L^p$-estimates for $\bar{\partial}$-closed $E$-valued forms of type $(n, s)$ with $1 \leq s \leq \min\{n - q, n - m\}$ and $1 \leq q, m \leq n - 1$ (respectively of type $(0, s)$ with $m \leq s \leq n - q$ and $1 \leq q, m \leq n - 1$) if $E$ is Nakano semi-positive (respectively Nakano semi-negative) of type $m$ on $\Omega$ (see Theorem 3.4 below). This result generalizes some results of [17] to $E$-valued forms for some bidegree. Combining these results together with some arguments inspired from Laurent-Thiébaut [19], we have the following main theorem.
Theorem 1.1. Let $\Omega \subset X$ be a $C^3$ $q$-convex intersection ($1 \leq q \leq n-1$) in an $n$-dimensional Kähler manifold $X$ with $n \geq 2$ and $E$ be a holomorphic Hermitian vector bundle over $X$ such that $X \setminus \Omega$ is connected. Then the following assertions hold true.

(i) If $E$ is Nakano semi-positive of type $m$ ($1 \leq m \leq n-1$) on $\overline{\Omega}$, then for any $\bar{\partial}$-closed form $f$ in $L^p_{0,s}(X,E^*)$ such that $f$ is supported in $\overline{\Omega}$, there exists a form $u$ in $L^p_{0,s-1}(X,E^*)$ supported in $\overline{\Omega}$ such that $\bar{\partial}u = f$ in $X$, for $1 \leq s \leq \min\{n-q,n-m\}$.

(ii) If $E$ is Nakano semi-negative of type $m$ on $\overline{\Omega}$, then for any $\bar{\partial}$-closed form $f$ in $L^p_{n,s}(X,E^*)$ with compact support in $\overline{\Omega}$, there exists a form $u$ in $L^p_{n,s-1}(X,E^*)$ with compact support in $\overline{\Omega}$ such that $\bar{\partial}u = f$ in $X$, for $m \leq s \leq n-q$.

This result generalizes Corollary 2.23 of Laurent-Thiébaut [19] (which was obtained for $C$-valued $(r,s)$-forms, $0 \leq r \leq n$, $1 \leq s \leq n-q$, on completely $q$-convex domains ($q \geq 1$) in complex manifolds) to more general $q$-convex domains and to $E^*$-valued forms for some bidegree.

Remark 1.2. We note that $\max\{q,m\} = 1$ means that $m = q = 1$. The case $m = 1$ implies that $E$ is Nakano-positive on $\overline{\Omega}$, then there is a Kähler metric on $\overline{\Omega}$, so the Kählerity assumption is automatically satisfied (see e.g. [1]) and $q = 1$ means that $\Omega$ is a strictly pseudoconvex domain with piecewise $C^3$-boundary (see e.g. [21]). Therefore the assertion (i) in Theorem 1.1 for this case still valid if $X$ is replaced by any complex manifold of complex dimension $n \geq 2$.

Actually, the Kählerity property of $X$ and the positivity assumptions on $E$ play a crucial role in the proof of our results, these conditions ensure the $L^2$-existence theorem for the $\bar{\partial}$-equation in our setting (see [2]).

In addition, by using these results, we prove a Hartogs-like theorem for $\bar{\partial}$-closed forms in $L^p$-setting with $p \geq 1$ (see Theorem 4.1 below). This result generalizes the result of Theorem 5 in [10] which was obtained for $C$-valued forms with coefficients in the usual $L^2$-Sobolev spaces $W^k$, $k \geq 0$, on bounded pseudoconvex domains with Lipschitz boundaries in Stein manifolds. As a results, we finally solve the $\bar{\partial}$-equation for forms with $W^{1,p}$-coefficients on an annulus type domain between two strictly $q$-convex domains with smooth boundaries in a Kähler manifold. This result was also proved in [10] for forms with $W^k$-coefficients on an annulus domain between two pseudoconvex domains in a Stein manifold.

2. Preliminaries

In this section, we fix notations, definitions, and auxiliary results that will be used throughout the paper. Let $X$ be a complex manifold of complex dimension $n$ and $E$ be a holomorphic Hermitian vector bundle of rank $N$ over $X$. Let $\{U_j\}; j \in I$, be an open covering of $X$ consisting of coordinates neighborhoods.
If \( N \) of rank \( D \) associated cohomologies groups are denoted by \( H \) subspace of \( \Lambda_{r,s}(2) \) (see e.g. [2])

**Definition 2.1** (see e.g. [2]). Let \( E \) be a holomorphic Hermitian vector bundle of rank \( N \) over a complex manifold \( X \) of complex dimension \( n \).

(a) \( E \) is said to be Nakano m-positive (respectively \( m \)-negative), at \( x \in U_j \), if there exists an \((n-\ell+1)\)-dimensional subspace \( S_x \) of the holomorphic tangent \( T_x(X) \) such that the Hermitian form

\[
\sum H_{j\rho\beta,\nu\alpha}(x) \zeta_\alpha^\rho \zeta_\beta^\nu
\]

is positive (respectively negative) definite for any \( \zeta = (\zeta_\alpha^\rho) \in S_x \otimes E_x; \zeta \neq 0 \).

(b) \( E \) is said to be Nakano semi-positive (respectively semi-negative), at \( x \in U_j \), if the Hermitian form (2) is positive (respectively negative) semi-definite for any \( \zeta = (\zeta_\alpha^\rho) \in T_x(X) \otimes E_x \).

(c) \( E \) is said to be Nakano semi-positive (respectively semi-negative) of type \( m \) if \( E \) is both Nakano semi-positive and Nakano \( m \)-positive (respectively Nakano semi-negative and Nakano \( m \)-negative) at \( x \).

For all \( 0 \leq r, s \leq n \), we denote by \( \Lambda^{r,s}(X,E) \) the space of \( E \)-valued forms of bidegree \((r,s)\) and of class \( C^\infty \) on \( X \) with the topology of uniform convergence of forms and all their derivatives on compact subsets of \( X \) and by \( D^{r,s}(X,E) \) the subspace of \( \Lambda^{r,s}(X,E) \) consisting of forms with compact supports in \( X \). The associated cohomologies groups are denoted by \( H^{r,s}(X,E) \) and \( H^{r,s}_c(X,E) \) respectively. Let \( K \) be a compact subset of \( X \) and \( D^{r,s}_K(X,E) \) the closed subspace of \( \Lambda^{r,s}(X,E) \) of forms with supports in \( K \) endowed with the induced topology. Let \( \{K_i\}_{i \in \mathbb{N}} \) be an increasing sequence of compact subsets of \( X \) such that \( K_i \subset K_{i+1} \) and \( \bigcup \{K_i\} = X \). Then \( D^{r,s}_K(X,E) = \bigcup_{i \in \mathbb{N}} D^{r,s}_K(X,E) \). We put on \( D^{r,s}_K(X,E) \) the strict inductive limit topology defined by the spaces \( D^{r,s}_K(X,E) \). If \( \varphi \in \Lambda^{r,s}(X,E) \), then \( \partial \varphi = \{\partial \varphi_j\} \), where \( \partial \varphi_j = (\partial \varphi_1,j, \partial \varphi_2,j, \ldots, \partial \varphi_N,j) \). Let \( ds^2 \) be a Kähler metric on \( X \) defined by

\[
ds^2 = \sum_{\alpha,\beta=1}^{n} g_{j\alpha\beta} dz^\alpha_j d\bar{z}^\beta_j,
\]
where $g_{j_0,j_1}$ is a $C^\infty$-section of $T^*(X) \otimes \bar{T}^*(X)$ on $U_j$. For $\varphi, \psi \in \Lambda^{r,s}(M, E)$, we define a local inner product, at $z \in U_j$, by

$$
\sum_{\nu, \mu=1}^{N} h_{j_{\nu},j_{\mu}} \bar{\varphi}^j(z) \wedge \star \psi^j(z) = a(\varphi(z), \psi(z))dv,
$$

where the Hodge star operator $\star$ and the volume element $dv$ are defined by $ds^2$ and $a(\varphi, \psi)$ is a function on $X$ independent of $j$. For $\varphi, \psi \in \mathcal{D}^{r,s}(X, E)$, a global inner product is then defined by $(\varphi, \psi) = \int_X a(\varphi, \psi)dv$. Let $L_{r,s}^2(X, E)$ be the Hilbert space obtained by completing the space $\mathcal{D}^{r,s}(X, E)$ under the norm $\|\varphi\|^2 = (\varphi, \varphi)$.

We now extend the complex analysis results obtained in [3] to domains in complex manifolds. Let $\Omega \subset X$ be a bounded domain with smooth boundary in a Kähler manifold $X$ of complex dimension $n$, $E$ be a holomorphic Hermitian vector bundle of rank $N$ over $X$, and $E^*$ be the dual vector bundle of $E$. The space of $E^*$-valued currents of bidegree $(n-r, n-s)$ (or bidimension $(r,s)$) denoted by $\mathcal{D}^{n-r,n-s}_{\text{cur}}(\Omega, E^*)$ is the topological dual to the space $\mathcal{D}^{r,s}(\Omega, E)$. The $\bar{\partial}$-operator is defined from $\mathcal{D}^{n-r,n-s}_{\text{cur}}(\Omega, E^*)$ into $\mathcal{D}^{n-r,n-s+1}_{\text{cur}}(\Omega, E^*)$ as the transpose of the original $\bar{\partial}$-operator from $\mathcal{D}^{r,s}(\Omega, E)$ into $\mathcal{D}^{r,s+1}(\Omega, E)$. The topological dual to the space $\Lambda^{r,s}(\Omega, E)$ denoted by $\Lambda^{n-r,n-s}_{\text{cur}}(\Omega, E^*)$ is the space of $E^*$-valued currents of of bidegree $(n-r, n-s)$ with compact supports in $\Omega$. The restriction of the $\bar{\partial}$-operator to $\Lambda^{n-r,n-s}_{\text{cur}}(\Omega, E^*)$ gives unbounded operator $\bar{\partial} : \Lambda^{n-r,n-s}_{\text{cur}}(\Omega, E^*) \to \Lambda^{n-r,n-s+1}_{\text{cur}}(\Omega, E^*)$. For further details on duality for complexes of topological vector spaces, we refer to [7] and the references therein.

Let $A_{r,s}(\Omega, E)$ be a topological space of $E$-valued $(r,s)$-forms on $\Omega$ and $A'_{r,s}(\Omega, E)$ be its dual. Assume that the injections

$$
\mathcal{D}^{r,s}(\Omega, E) \hookrightarrow A_{r,s}(\Omega, E) \hookrightarrow \mathcal{D}^{n-r,n-s}_{\text{cur}}(\Omega, E^*)
$$

being continuous. Then $B_{n-r,n-s}(\Omega, E^*) = A'_{r,s}(\Omega, E)$ still a space of currents and asking that the duality pairing $\langle \phi, \psi \rangle = \int_\Omega \phi \wedge \psi$ be $\bar{\partial}$ compatible with currents, i.e., $\forall \phi \in \mathcal{D}^{r,s}_{\text{cur}}(\Omega, E)$ and $\psi \in \mathcal{D}^{n-r,n-s+1}(\Omega, E^*)$,

$$
\langle \bar{\partial} \phi, \psi \rangle = (-1)^{r+s+1} \langle \phi, \bar{\partial} \psi \rangle.
$$

For $0 \leq r \leq n, 1 \leq s \leq n$, we recall that the equation $\bar{\partial}g = f$ is solvable in $A_{r,s}(\Omega, E)$ if for any $\bar{\partial}$-closed form $f$ in $A_{r,s}(\Omega, E)$ there exists a form $g$ in $A_{r,s-1}(\Omega, E)$ such that $\bar{\partial}g = f$ in $\Omega$. Suppose now that the $\bar{\partial}$-equation is solvable in $A_{r,s}(\Omega, E)$ and $A_{r,s+1}(\Omega, E)$ for all $1 \leq s \leq n-1$. Let $u$ be a $\bar{\partial}$-closed form in $B_{n-r,n-s}(\Omega, E^*)$ and consider the form

$$
\mathcal{L}_u(\eta) = \langle g, u \rangle \quad \forall \eta \in A_{r,s+1}(\Omega, E), \quad \bar{\partial}\eta = 0,
$$

with $\bar{\partial}g = \eta$, which exists by hypothesis. Denote by $\mathcal{H}_r(\Omega, E)$, the space of $E$-valued $\bar{\partial}$-closed $(r,0)$-forms on $\Omega$. Then we have:
Theorem 2.3. Let $g, h \in A_{r,s}(\Omega, E)$ be such that $\partial g = \partial h = \eta$, then the difference $g - h$ is a $\partial$-closed form in $A_{r,s}(\Omega, E)$. By hypothesis, there exists $\phi \in A_{r,s-1}(\Omega, E)$ such that $\partial \phi = g - h$. Hence,

$$\langle g - h, u \rangle = \langle \partial \phi, u \rangle = (-1)^{r+s+1} \langle \phi, \partial u \rangle = 0.$$  

Thus $L_u$ is also well defined for all $s \geq 1$.

For $s = 0$, we have $\partial u = 0$ (because $u$ is an $(n-r,n)$-form). Again, let $g, h \in A_{r,0}(\Omega, E)$ with $\partial g = \partial h$, hence $g - h$ is a $\partial$-closed $(r,0)$-form, i.e., $g - h \in H_r(\Omega, E)$. Since, by hypothesis, $u \perp H_r(\Omega, E)$, we have $\langle g - h, u \rangle = 0$.

Then $L_u$ is also well defined in this case.

Next, we show that the form $L_u$ is linear. Let $\eta_1$ and $\eta_2$ be in $A_{r,s+1}(\Omega, E)$ such that $\partial \eta_1 = \partial \eta_2 = 0$ and put $\eta = \eta_1 + \eta_2$, then $\partial \eta = 0$ and so there are $g_1$ and $g_2$ in $A_{r,s}(\Omega, E)$ such that $\partial g = \eta$, $\partial g_1 = \eta_1$, and $\partial g_2 = \eta_2$. Thus $\partial (g - g_1 - g_2) = 0$ and hence there is a form $h$ in $A_{r,s-1}(\Omega, E)$ such that $g = g_1 + g_2 + \partial h$, therefore

$$L_u(\eta) = \langle g, u \rangle = \langle g_1 + g_2 + \partial h, u \rangle = \langle g_1, u \rangle + \langle g_2, u \rangle + \langle \partial h, u \rangle = L_u(\eta_1) + L_u(\eta_2),$$

where $\langle \partial h, u \rangle = (-1)^{r+s+1} \langle h, \partial u \rangle = 0$. Similarly for $\lambda \eta; \lambda \in \mathbb{C}$. The proof is complete.

Following [3], the equation $\partial u = \eta$ is continuously solvable in $A_{r,s+1}(\Omega, E)$ if it is solvable in $A_{r,s}(\Omega, E)$ and $A_{r,s+1}(\Omega, E)$ and, moreover, if the form $L_u(\eta)$ is continuously linear on the subspace of all $\partial$-closed forms $\eta$ in $A_{r,s+1}(\Omega, E)$.

Theorem 2.3. If the $\partial$-equation is continuously solvable in $A_{r,s+1}(\Omega, E)$, then it is solvable in $B_{n-r,n-s}(\Omega, E^*)$, that is, for any $f \in B_{n-r,n-s}(\Omega, E^*)$ with $\partial f = 0$ if $1 \leq s \leq n - 1$ and $\langle f, g \rangle = 0$ for all $g \in H_r(\Omega, E)$ if $s = 0$, there exists $\omega \in B_{n-r,n-s-1}(\Omega, E^*)$ such that $\partial \omega = f$.

Proof. Let $f \in B_{n-r,n-s}(\Omega, E^*)$ with $\partial f = 0$ if $1 \leq s \leq n - 1$ and $\langle f, g \rangle = 0$ for $s = 0$ and $g \in H_r(\Omega, E)$. Consider the form $L_f$ on the subspace of all $\eta$ in $A_{r,s+1}(\Omega, E)$ with $\partial \eta = 0$, which exists by hypothesis on $A_{r,s+1}(\Omega, E)$ and is continuous by assumption.

By the Hahn-Banach extension theorem, it can be extended to the whole $A_{r,s+1}(\Omega, E)$. By duality, the extended form can be represented by a current $\omega \in B_{n-r,n-s-1}(\Omega, E^*)$. Then we have

$$\langle \eta, \omega \rangle = \langle \partial g, \omega \rangle = L_f(g) = \langle g, f \rangle.$$  

But $\langle \partial g, \omega \rangle = (-1)^{r+s+1} \langle g, \partial \omega \rangle$ and hence $\langle g, f \rangle = (-1)^{r+s+1} \langle g, \partial \omega \rangle$ for all $g \in D'^{\ast}(\Omega, E)$. This means that $\partial \omega = f$. The proof is complete.
3. Solving $\bar{\partial}$ with exact support in $L^p$

Let $\Omega \subseteq X$ be a relatively compact domain with smooth boundary in a Kähler manifold $X$ of complex dimension $n$ and $E$ be holomorphic Hermitian vector bundle of rank $N$ over $X$. Let $\{U_{j_{\nu}}\}$ be a finite elements of the covering $\{U_j\}$ such that $\cup_{\nu} U_{j_{\nu}}$ cover $\overline{\Omega}$ and $\{\chi_{j_{\nu}}\}$ be a partition of unity subordinate to $\{U_{j_{\nu}}\}$. Then every $E$-valued $(r,s)$-form $f$ can be identified with a system $\{f_{j_{\nu}}\}$ of vectors $f_{j_{\nu}} = (f_{j_{\nu}}^1, f_{j_{\nu}}^2, \ldots, f_{j_{\nu}}^N)$ of differential forms $f_{j_{\nu}}^\mu$ on $U_{j_{\nu}} \cap \Omega$. For $1 \leq p \leq \infty$, we denote by $L^p_{\omega}(\Omega, E)$ the Banach space of $E$-valued forms $f$ of bidegree $(r,s)$ on $\Omega$ for which $\|f\|_{L^p_{\omega}(\Omega, E)} < \infty$. The norm $\|f\|_{L^p_{\omega}(\Omega, E)}$ is defined by means of a partition of unity in the following way: On each $\cup_{\nu} U_{j_{\nu}}$, we can choose an orthonormal basis $\omega^1, \ldots, \omega^N$ for the fibers $E_z$ for every $z \in U_{j_{\nu}}$. In such a basis, the $L^p(\Omega, E)$-norm is defined by $\|f\|_{L^p_{\omega}(\Omega, E)} = \sum_{\nu=1}^N \sum_{j_{\nu}} \|\chi_{j_{\nu}} f_{j_{\nu}}^\mu\|_{L^p(U_{j_{\nu}} \cap \Omega)}$, where $\|\chi_{j_{\nu}} f_{j_{\nu}}^\mu\|_{L^p(U_{j_{\nu}} \cap \Omega)} = \text{ess sup} |\chi_{j_{\nu}} f_{j_{\nu}}^\mu|$. This norm depends on the choice of the coverings and their local coordinates, however, as $\overline{\Omega}$ is compact, different choices give equivalent norms. The associated $\bar{\partial}$-cohomology group is denoted by $H^a_{\bar{\partial}}(\Omega, E)$.

For $p \geq 1$, we denote by $L^p_{\omega}(\Omega, E)$ the subspace of $\mathcal{D}^*_{\text{cusp}}(\Omega, E)$ consisting of $E$-valued $(r,s)$-currents with coefficients in $L^p(\Omega, E)$ and endowed with the topology of $L^p$-convergence on compact subsets of $\Omega$. Taking the restriction to $L^p_{f,s}(\Omega)$ of the $\bar{\partial}$-operator in the sense of distributions we get an unbounded operator whose domain of definition is the set of forms $f$ with $L^p_{\omega}(\Omega, E)$-coefficients such that $\bar{\partial} f$ has also $L^p_{\omega}(\Omega, E)$-coefficients, moreover, since $\bar{\partial} \circ \bar{\partial} = 0$, we get a complex of unbounded operators $(L^p_{f,s}(\Omega, E), \bar{\partial})$. The associated $\bar{\partial}$-cohomology group is denoted by $H^a_{f,s}(\Omega, E)$. By $L^p_{f,s}(\Omega, E)$, we denote the subspace of $L^p_{\omega}(\Omega, E)$ consisting of forms with compact supports in $\Omega$. We also consider the subcomplex $(L^p_{f,s}(\Omega, E), \bar{\partial})$ of the previous one consisting of forms with compact supports. For all $k \geq 1$ and $1 \leq p \leq \infty$, the $L^p$-Sobolev spaces $W^{k,p}_{f,s}(\Omega, E)$ and their norms are defined in similar manner. Finally, for $1 < p < \infty$ and $p'$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, $L^p_{f,s}(\Omega, E)^*$ is the dual space of $L^p_{f,s}(\Omega, E)$ with respect to the duality pairing $\langle f, g \rangle = \int_{\Omega} f \wedge g$. Using a partition of unity, as in [3], we have the following duality theorem.

**Theorem 3.1.** For any $p$ with $1 \leq p < \infty$ and $p'$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, $L^p_{\omega, f,s}(\Omega, E)^*$ is the dual space of $L^p_{\omega}(\Omega, E)$ with respect to the duality pairing $\langle f, g \rangle = \int_{\Omega} f \wedge g$.

For $0 \leq r \leq n$ and $1 \leq s \leq n - 1$, let $\bar{\partial}_r : L^p_{\omega}(\Omega, E) \to L^p_{r+s}(\Omega, E)$ be the minimal closed extension of $\bar{\partial}_r \mathcal{D}^r(\Omega, E)$. The domain of $\bar{\partial}_r$ denoted $\text{Dom}(\bar{\partial}_r)$ consists of those forms $f$ in $L^p_{\omega}(\Omega, E)$ for which there exist a sequence $\{f_i\}$ of elements $f_i$ in $\mathcal{D}^r(\Omega, E)$ and a form $g$ in $L^p_{r+s}(\Omega, E)$ such that $f_i \to f$ and $\bar{\partial}_r f_i \to g$ in the $L^p(\Omega, E)$-norm. We then set $\bar{\partial}_r f = g$. 
We consider also $\bar{\partial}_s : L^p_{r,s}(\Omega, E) \to L^p_{r,s+1}(\Omega, E)$ the minimal closed extension of $\bar{\partial}|_{A^{r,s}(X,E)|_{\Omega}}$, it is also a closed operator and $\text{Dom}(\bar{\partial}_s)$ consists of those forms $f$ in $L^p_{r,s}(\Omega, E)$ for which there exist a sequence $\{f_i\}$ of elements $f_i$ in $A^{r,s}(\Omega, E)$ and a form $g \in L^p_{r,s+1}(\Omega, E)$ such that $f_i \to f$ and $\bar{\partial}_s f_i \to g$ in the $L^p(\Omega, E)$-norm. We then set $\bar{\partial}_s f = g$.

The operator $\bar{\partial}$ extends to $L^p_{r,s}(\Omega, E)$, in the sense of distributions, so we can consider the operators $\hat{\partial}_s : L^p_{r,s}(\Omega, E) \to L^p_{r,s+1}(\Omega, E)$ and $\partial : L^p_{r,s}(\Omega, E) \to L^p_{r,s+1}(\Omega, E)$ which coincides with the original $\bar{\partial}$ such that

$$\text{Dom}(\hat{\partial}_s) = \{ f \in L^p_{r,s}(X, E), \text{supp } f \subset \Pi, \hat{\partial} f \in L^p_{r,s+1}(X, E) \},$$

and

$$\text{Dom}(\partial) = \{ f \in L^p_{r,s}(\Omega, E), \hat{\partial} f \in L^p_{r,s+1}(\Omega, E) \}.$$
Proof. Let $\Omega \subset \subset U \subset \subset \mathbb{C}^n$ be a $C^3$ $q$-convex intersection with the defining functions $\{\rho_i\}_{i=1}^b$ and $U$ as in Definition 3.2. Set

$$\Omega_I = \{z \in U \mid \rho_i(z) < 0, \ i \in I\} \quad \text{and} \quad S_I = \{z \in U \mid \rho_i(z) = 0, \ i \in I\}.$$

For each $\xi \in S_I$ there exists a smoothly bounded strictly pseudoconvex domain $D^*$ defined by $D^* = \{z \in U; \rho_s(z) < 0\}$ such that $\partial D^*$ intersects real transversely $\{z \in U; \rho_s(z) < 0\}, \ldots, \{z \in U; \rho_s(z) < 0\}$ and $\xi \in D^*$.

Denote by $I_s$ the multi-index $(i_1 \ldots, i_\ell, *)$, where $I = (i_1 \ldots, i_\ell)$, $1 \leq i_1 < \cdots < i_\ell < b$, and define

$$\Omega_{I_s} = \{z \in U; \rho_j(z) < 0, \ j \in I_s\}.$$

The domain $\Omega_{I_s}$ is still $q$-convex and is called a local $q$-convex intersection. Since $\Omega_I$ is $q$-convex intersection, for every $z \in \Omega_I$ there is then an $(n-q+1)$-linear vector subspace $T^I_z$ of $\mathbb{C}^n$ such that the Levi forms $L_{\rho_s}$ are positive definite on $T^I_z$ for all $i \in I$. Therefore, by means of generalized Berndtsson-Andersson formula with multiple weights, Lan Ma and Vassiliadou proved in [20] that if $f \in \mathcal{C}^1_{r,s}(\Omega_{I_s})$ with $\partial f \in \mathcal{C}^1_{r,s+1}(\Omega_{I_s})$, $0 \leq r \leq q$, $q \leq s \leq n-1$, there exist local kernels $K^c_s(z, \xi)$ of bidegree $(r, s)$ in $z$ and of bidegree $(n-r, n-s-1)$ in $\xi$ such that the map

$$f \mapsto \int_{\zeta \in \Omega_{I_s}} f(\zeta) \wedge K^c_{s-1}(\zeta, \xi)$$

defines a bounded linear operator $T^c_s : \mathcal{C}^1_{r,s}(\Omega_{I_s}) \rightarrow \mathcal{C}^1_{r,s-1}(\Omega_{I_s})$, the map

$$f \mapsto \int_{\zeta \in \partial \Omega_{I_s}} f(\zeta) \wedge K^c_s(\zeta, \xi)$$

defines a compact linear operator $K_s : \mathcal{C}^1_{r,s}(\Omega_{I_s}) \rightarrow \mathcal{C}^1_{r,s}(\Omega_{I_s})$ and the homotopy formula

$$f = \partial T^c_s f + T^c_{s+1} \partial f + K_s f$$

holds on $\Omega_{I_s}$ for every $f$ in $\mathcal{C}^1_{r,s}(\Omega_{I_s})$ with $\partial f$ in $\mathcal{C}^1_{r,s+1}(\Omega_{I_s})$.

Now we extend these operators to $E$-valued $(r, s)$-forms defined on $q$-convex intersections in complex manifolds. Let $\Omega \subset \subset X$ be a $C^3$ $q$-convex intersection $(q \geq 1)$ in an $n$-dimensional complex manifold $X$ and $E$ be a holomorphic Hermitian vector bundle over $X$. Cover $\Omega$ by a finite number of open sets $V_1, V_2, \ldots, V_m$ such that $\Omega \subset V_1 \cup \cdots \cup V_m$ and for every $1 \leq j \leq m$ the intersection $V_j \cap \Omega$ is a local $q$-convex intersection, moreover, we may assume that $E$ is trivial over some coordinates neighborhoods $z_j = (z_1^j, z_2^j, \ldots, z_n^j)$ of each $V_j \cap \Omega$. Then, for every $f \in \mathcal{C}^1_{r,s}(\Omega \cap V_j, E)$, $q \leq s \leq n-1$, with $\partial f = 0$, there exist bounded linear operators

$$T^c_j : \mathcal{C}^1_{r,s}(\Omega \cap V_j, E) \rightarrow \mathcal{C}^1_{r,s-1}(\Omega \cap V_j, E)$$

and compact operators

$$K^c_j : \mathcal{C}^1_{r,s}(\Omega \cap V_j) \rightarrow \mathcal{C}^1_{r,s}(\Omega \cap V_j)$$
such that the homotopy formulas

\[ f = \bar{\partial}T^j_s f + K^j_s f \]

hold on \( \Omega \cap V_j \) for all \( f \in C^1_r(\Omega \cap V_j) \cap \text{Ker}(\bar{\partial}) \).

Choose a \( C^\infty \) partition of unity \( \{\chi_j\} \) subordinate to the covering \( \{V_j\} \) and define

\[ \tilde{T}_s f = \sum_{j=1}^m \chi_j T^j_s f \quad \text{and} \quad \tilde{K}_s f = \sum_{j=1}^m \chi_j K^j_s f \]

for \( f \in C^1_r(\Omega, E) \cap \text{Ker}(\bar{\partial}), \ q \leq s \leq n - 1 \).

We then have

\[ f = \bar{\partial}\tilde{T}_s f + \tilde{K}_s f, \quad f \in C^1_r(\Omega, E) \cap \text{Ker}(\bar{\partial}), \ q \leq s \leq n - 1. \]

By using the \( L^p \)-estimates proved in [20] and the mollification method of Friedrichs (see e.g. [11]), the formula (5) extends to forms in \( L^p_r(\Omega, E) \cap \text{Ker}(\bar{\partial}) \) for all \( 1 \leq p \leq \infty \) and \( q \leq s \leq n - 1 \). This proves (4). As the operators \( \tilde{K}_s \) are compact operators from \( L^p_r(\Omega, E) \) into itself, the operator \( \text{Id} - \tilde{K}_s \) is a Fredholm operator maps \( L^p_r(\Omega, E) \cap \text{Ker}(\bar{\partial}) \) into itself whose range is contained in \( \bar{\partial}(L^p_{r,s-1}(\Omega, E)) \) by the formula (4) and hence the dimension of the cohomology group \( H^q_{\bar{\partial}}(\Omega, E) \) is smaller than the codimension of the range of \( \text{Id} - \tilde{K}_s \) which is finite. Therefore, the open mapping theorem implies that \( \bar{\partial}(L^p_{r,s-1}(\Omega, E)) \) is a closed subspace of \( L^p_r(\Omega, E) \). The proof is complete. \( \square \)

By using a partition of unity and the \( L^p \)-estimates obtained in [2, Theorem 0.1], we have the following \( L^p \)-existence theorem.

**Theorem 3.4.** Let \( \Omega \subset\subset X \) be a \( C^3 \) \( q \)-convex intersection (\( q \geq 1 \)) in a Kähler manifold \( X \) of complex dimension \( n \) and \( E \) be a holomorphic Hermitian vector bundle of rank \( N \) over \( X \). Then

(i) If \( E \) is Nakano semi-positive of type \( m \) on \( \Omega \), then for any \( \bar{\partial} \)-closed form \( f \) in \( L^1_r(\Omega, E) \) there exists a form \( g \) in \( L^1_{n,s}(\Omega, E) \) such that \( \bar{\partial}g = f \) for all \( s \) so that \( \max\{q, m\} \leq s \leq n - 1 \). Moreover, if \( f \) is in \( L^p_{r,s}(\Omega, E), \ 1 \leq p \leq \infty \), then \( g \) is in \( L^p_{n,s}(\Omega, E) \) and there is a constant \( C_s > 0 \) (independent of \( f \) and \( p \)) such that

\[ \|g\|_{L^p_{n,s}(\Omega, E)} \leq C_s \|f\|_{L^p_{r,s}(\Omega, E)}, \quad 1 \leq p \leq \infty. \]

(ii) If \( E \) is Nakano semi-negative of type \( m \) on \( \Omega \), the assertion (i) holds for \( E \)-valued \((0,s)\)-forms with \( L^p \)-coefficients, for all \( q \leq s \leq n - m, \ 1 \leq q, m \leq n - 1 \) and \( n \geq 2 \).

Since the \( q \)-convexity is stable with respect to small \( C^3 \) perturbations, we may assume that the defining functions \( \rho_i \) of \( \Omega \) are Morse functions (i.e., all critical points of \( \rho_i \) are non-degenerate and if \( \xi_1 \) and \( \xi_2 \) are two different critical points of \( \rho_i \), then \( \rho_i(\xi_1) = \rho_i(\xi_2) \)). Then we can approximate \( \Omega \) from inside
by a sequence of $C^3$ $q$-convex intersections $\{\Omega_k\}$ such that $\Omega_k \subset \subset \Omega_{k+1} \subset \subset \Omega$ and $\Omega = \bigcup_k \Omega_k$. This approach is known as Grauert’s bumping method where each $\Omega_{k+1}$ is obtained from $\Omega_k$ by an appropriate small bump (see e.g. [16] for the $q$-convex ($q$-concave) domains or [22] for $q$-convex intersections). Then, as in [19, Theorem 2.10], the next theorem follows immediately from Theorems 3.3 and 3.4.

**Theorem 3.5.** Let $\Omega$, $X$ and $E$ be given as in Theorem 3.4. Then we have the following assertions.

(i) If $E$ is Nakano semi-positive of type $m$ on $\overline{\Omega}$, then for all $s$ so that $\max\{q, m\} \leq s \leq n - 1$, we have

$$H^{n,s}_{L^p}(\Omega, E) \sim H^{n,s}_{L^p,\text{loc}}(\Omega, E).$$

(ii) If $E$ is Nakano semi-negative of type $m$ on $\overline{\Omega}$, then for all $s$ so that $q \leq s \leq n - m$, $1 \leq q, m \leq n - 1$, $n \geq 2$, we have

$$H^{0,s}_{L^p}(\Omega, E) \sim H^{0,s}_{L^p,\text{loc}}(\Omega, E).$$

We note that since every smooth domain in the complex plane is strictly pseudoconvex, the assertions (i) in Theorems 3.4 and 3.5 are still valid when $n = 1$ and $E$ is the trivial line bundle with the flat metric with $q = s = m = 1$.

Following [19], we recall that for any two real numbers $p$ and $p'$ so that $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$ and any $r \in \mathbb{N}$ with $0 \leq r \leq n$, the complexes $(L^p_{r,*}(\Omega, E), \partial)$ and $(L^p_{n-r,*}(\Omega, E^*, \bar{\partial}))$ are dual complexes. Moreover, we recall the following abstract result on duality.

**Proposition 3.6.** Let $(E^*, d)$ and $(E'_*, d')$ be two dual complexes of reflexive Banach spaces with densely defined unbounded operators. Assume that $H_s(E'_*)$ is Hausdorff and $H_{s+1}(E'_*) = 0$, then $H^{s+1}(E^*) = 0$.

Let $p, p' > 1$ be real numbers with $\frac{1}{p} + \frac{1}{p'} = 1$. It follows from Theorem 3.3 and Theorem 3.4(i) that the cohomology group $H^{n,s}_{L^p}(\Omega, E)$ is Hausdorff for all $s$ such that $q \leq s \leq n - 1$ and $H^{n,s}_{L^p}(\Omega, E) = 0$ for all $s$ such that $\max\{q, m\} \leq s \leq n - 1$. Moreover, by Theorem 3.3 and Theorem 3.4(ii), we get that $H^{0,s}_{L^p}(\Omega, E)$ is Hausdorff for all $q \leq s \leq n - 1$ and $H^{0,s}_{L^p}(\Omega, E) = 0$ for all $q \leq s \leq n - m$, where $1 \leq q, m \leq n - 1$ and $n \geq 2$.

**End proof of Theorem 1.1.** On applying Proposition 3.6 to the complex $(E^*, d)$ with, for fixed $r$ so that $0 \leq r \leq n$, $E^s = L^p_{r,*}(\Omega, E^*)$ if $0 \leq s \leq n$ and $E^s = \{0\}$ if $s < 0$ or $s > n$, and $d = \bar{\partial}$, we deduce that the cohomological hypotheses of Theorem 2.20 in [19] are satisfied in the current situations. This implies $L^p$-solvability for the $\bar{\partial}$-equation with exact support on a $q$-convex intersection in a complex manifold, and this completes the proof of Theorem 1.1. \[\square\]
4. $\bar{\partial}$-closed extensions of forms in $L^p$

As an application of Theorem 1.1, we obtain a Hartogs-like extension theorem for $\bar{\partial}$-closed forms.

**Theorem 4.1.** Let $\Omega \subset \subset X$ be a $C^3$ $q$-convex intersection $(q \geq 1)$ in a Kähler manifold $X$ of complex dimension $n \geq 3$ such that $X \setminus \Omega$ is connected. Let $E$ be a holomorphic Hermitian vector bundle of rank $N$ over $X$.

1. If $E$ is Nakano semi-positive of type $m$ on $\overline{\Omega}$, then for every $\bar{\partial}$-closed form $f$ in $W^{1,p}(\Omega \setminus \Omega, E^*)$, $1 \leq s \leq \min\{n-q, n-m\}$, $2 \leq q, m \leq n-1$, there exists a form $F$ in $L^{p,s}_{0}(X, E^*)$ such that $F|_{X \setminus \Omega} = f$ and $\bar{\partial}F = 0$ in $X$ in the distribution sense.

For $s = n - 1$, if we assume furthermore that the restriction of $f$ to $\partial \Omega$ satisfies the moment condition

$$\int_{\partial \Omega} f \wedge \phi = 0, \quad \forall \phi \in L^{p,s}_{n,0}(\Omega, E) \cap \text{Ker}(\bar{\partial}),$$

then the same statement holds.

2. If $E$ is Nakano semi-negative of type $m$ on $\overline{\Omega}$, then statement (1) holds for all $\bar{\partial}$-closed form $f$ in $W^{1,p}(X \setminus \Omega, E^*)$ for $m \leq s \leq n - q$ and $2 \leq q, m \leq n - 1$.

For $s = n - 1$, the same statement true if we assume furthermore that $f$ satisfies the moment condition

$$\int_{\partial \Omega} f \wedge \phi = 0, \quad \forall \phi \in L^{p,s}_{n,0}(\Omega, E) \cap \text{Ker}(\bar{\partial}).$$

**Proof.** We consider the assertion in (1), i.e., the case when $E$ is Nakano semi-positive of type $m$ on $\overline{\Omega}$, as the defining functions $\rho_i$ of $\Omega$ are of class $C^3$, there is a bounded extension operator of $W^{k,\rho}(X \setminus \Omega, E^*)$ into $W^{k,\rho}(X, E^*)$ for all $k \geq 0$ and $1 \leq \rho < \infty$ (see e.g. [8, Theorem 9.7]). Let $\tilde{f} \in W^{k,\rho}(X, E^*)$ be the extension of $f$ such that $f|_{X \setminus \Omega} = \tilde{f}$, and $\bar{\partial}\tilde{f}$ is in $L^{p,s}_{0,+1}(X, E^*)$ and is compactly supported in $\overline{\Omega}$. In view of Theorem 1.1, there exists a form $g$ in $L^{p,s}_{0}(X, E^*)$ with compact support in $\overline{\Omega}$ such that $\bar{\partial}g = \bar{\partial}\tilde{f}$ in the distribution sense in $X$. Set $F = \tilde{f} - g$, we have $\bar{\partial}F = 0$ in $X$, $F|_{X \setminus \Omega} = f$ and $F$ is compactly supported in $\overline{\Omega}$. Thus the form $F \in L^{p,s}_{0}(X, E^*)$ is the desired $\bar{\partial}$-closed extension of $f$ to $X$. The assertion in (2) follows on using similar arguments. This completes the proof. \[\square\]

**Corollary 4.2.** Let $\Omega_1$ and $\Omega_2$ be two strictly $q$-convex and $q^*$-convex intersections with smooth $C^\infty$ boundaries in an $n \geq 3$-dimensional Kähler manifold $X$, respectively, such that $\overline{\Omega_2} \subset \subset \Omega_1 \subset \subset X$. Assume that $H^{p,s}_{\bar{\partial}}(\Omega) = 0$. Then for any $\bar{\partial}$-closed form $f$ in $W^{1,p}(\Omega_1 \setminus \overline{\Omega_2})$ there exists a form $u$ in
\[ W_{r,s}^1(\Omega_1 \setminus \overline{\Omega}_2) \cap W_{r,s}^1(\Omega_1 \setminus \overline{\Omega}_2) \text{ such that } \partial u = f \text{ in } \Omega_1 \setminus \overline{\Omega}_2, \text{ where } r \geq 0, q^* \leq s \leq n - q - 1. \]

**References**


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