CONSTANT CURVATURE FACTORABLE SURFACES IN
3-DIMENSIONAL ISOTROPIC SPACE

Muhittin Evren Aydin

Abstract. In the present paper, we study and classify factorable surfaces in a 3-dimensional isotropic space with constant isotropic Gaussian ($K$) and mean curvature ($H$). We provide a non-existence result relating to such surfaces satisfying $\frac{H}{K} = \text{const}$. Several examples are also illustrated.

1. Introduction

Let $E^3$ be a 3-dimensional Euclidean space and $(x, y, z)$ rectangular coordinates. A surface in $E^3$ is said to be factorable (so-called homothetical) if it is a graph of the form $z(x, y) = f(x)g(y)$, where $f$ and $g$ are smooth functions (see [4, 14]). Such surfaces in $E^3$ with constant Gaussian ($K$) and mean curvature ($H$) were obtained in [10, 14, 24].

As more general case, Zong et al. [25] defined that an affine factorable surface in $E^3$ is a graph of the form

$$z(x, y) = f(x)g(y + ax), \quad a \neq 0$$

and classified these ones with $K, H$ constants.

A surface in a 3-dimensional Minkowski space $E^3_1$ is said to be factorable if it can be expressed by one of the explicit forms ([15])

$$\Phi_1 : z(x, y) = f(x)g(y), \quad \Phi_2 : y(x, z) = f(x)g(z), \quad \Phi_3 : x(y, z) = f(y)g(z).$$

Up to the causal characters of the directions, six different classes of these surfaces in $E^3_1$ appear. The surfaces in $E^3_1$ with $K, H$ constants were described in [9, 15, 21].

In 3-dimensional context, the factorable surfaces are closely connected with translation surfaces, namely the surfaces generated by translating of two curves. For instance; in the homogeneous Riemannian space $H^2 \times \mathbb{R}$ that is a Lie group, up to its group operation, a translation surface of type 2 is a graph of the form...
$y(x,z) = f(x)g(z)$ \(\text{(see [22])}\). For more details, we refer to [7, 8], [11–13], [17,23].

Besides the Minkowskian space, a 3-dimensional isotropic space $\mathbb{I}^3$ provides two different types of the factorable surfaces. This special ambient space which is one of the real Cayley-Klein spaces is the product of the $xy$-plane and the isotropic $z$-direction with a degenerate parabolic distance metric (cf. [5]).

Due to the absolute figure of $\mathbb{I}^3$, the factorable surface $\Phi_1$ distinctly behaves from others. We call it \textit{factorable surface of type 1} \(\text{(see [1–3])}\). The surfaces $\Phi_2$, $\Phi_3$ in $\mathbb{I}^3$ are locally isometric and, up to a sign, have same second fundamental form. This means to have same isotropic Gaussian $K$ and, up to a sign, mean curvature $H$. These surfaces are said to be of type 2.

In this manner we are mainly interested in the factorable surfaces of type 2 in $\mathbb{I}^3$. We describe such surfaces in $\mathbb{I}^3$ with $K, H, H/K$ constants by the following results:

\textbf{Theorem 1.1.} A factorable surface of type 2 ($\Phi_3$) in $\mathbb{I}^3$ has constant isotropic mean curvature $H_0$ if and only if, up to suitable translations and constants, one of the following occurs:

(i) If $\Phi_3$ is isotropic minimal, i.e., $H_0 = 0$;
   (i.1) $\Phi_3$ is a non-isotropic plane,
   (i.2) $x(y,z) = y \tan (cz)$,
   (i.3) $x(y,z) = c_y^2$.

(ii) Otherwise ($H_0 \neq 0$), $x(y,z) = \pm \sqrt{\frac{z}{H_0}}$,

where $c$ is some nonzero constant.

\textbf{Theorem 1.2.} A factorable surface of type 2 ($\Phi_3$) in $\mathbb{I}^3$ has constant isotropic Gaussian curvature $K_0$ if and only if, up to suitable translations and constants, one of the following holds:

(i) If $\Phi_3$ is isotropic flat, i.e., $K_0 = 0$;
   (i.1) $x(y,z) = c_1 g(z), \frac{dg}{dz} \neq 0$,
   (i.2) $x(y,z) = c_1 e^{c_2y + c_3z}$,
   (i.3) $x(y,z) = c_1 y^2 z c_3, c_2 + c_3 = 1$.

(ii) Otherwise ($K_0 \neq 0$);
   (ii.1) $K_0$ is negative and $x(y,z) = \pm \frac{z}{\sqrt{-K_0}}$,
   (ii.2) $x(y,z) = c_0 g(z) \text{ for}$

$$z = \pm \int \left( c_0 g^{-1} - \frac{K_0}{c_1} \right)^{1/2} dg,$$

where $c_1, c_2, c_3$ are some nonzero constants.

\textbf{Theorem 1.3.} There does not exist a factorable surface of type 2 in $\mathbb{I}^3$ that satisfies $\frac{H}{K} = \text{const.} \neq 0$. 

We point out that the above results are also valid for the factorable surface $\Phi_2$ in $I^3$ by replacing $x$ with $y$ as well as taking $y = \pm \sqrt{z/H_0}$ in the last statement of Theorem 1.1.

2. Preliminaries

For detailed properties of isotropic spaces, see [6,16], [18–20].

Let $P(\mathbb{R}^3)$ be a real 3-dimensional projective space and $(x_0 : x_1 : x_2 : x_3)$ denote the projective coordinates in $P(\mathbb{R}^3)$. A 3-dimensional isotropic space $I^3$ is a Cayley-Klein space obtained from $P(\mathbb{R}^3)$ such that its absolute figure consists of a plane (absolute plane) $\omega$ and complex-conjugate straight lines (absolute lines) $l_1,l_2$ in $\omega$. In coordinate form, $\omega$ is given by $x_0 = 0$ and $l_1,l_2$ by $x_0 = x_1 \pm ix_2 = 0$. The absolute point, $(0 : 0 : 0 : 1)$, is the intersection of the absolute lines.

For $x_0 \neq 0$, we have the affine coordinates by $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}, z = \frac{x_3}{x_0}$. The group of motions of $I^3$ is given by

\[
\begin{pmatrix} x', y', z' \end{pmatrix} \mapsto \begin{pmatrix} x_0 & y_0 & z_0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \
1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 & y_0 & z_0 \end{pmatrix}^{-1},
\]

where $a_1, \ldots, a_5, \phi \in \mathbb{R}$. The isotropic metric that is an invariant of (2.1) is induced by the absolute figure, namely $ds^2 = dx^2 + dy^2$.

There are two types of the lines and the planes in $I^3$ arising from its absolute figure: The lines parallel (resp. non-parallel) to $z$-direction are said to be isotropic (resp. non-isotropic). A plane is said to be isotropic if it involves an isotropic line. Otherwise it is called non-isotropic plane or Euclidean plane. For example the equations $ax + by + cz = 0 \ (a,b,c \in \mathbb{R}, \ c \neq 0)$ and $ax + by = 0$ determine a non-isotropic plane and an isotropic plane, respectively.

We restrict our framework to regular surfaces whose the tangent plane at each point is non-isotropic, namely admissible surfaces.

Let $M$ be a regular admissible surface in $I^3$ locally parameterized by

\[
r(u,v) = (x(u,v), y(u,v), z(u,v))
\]

for a coordinate pair $(u,v)$. The components $E,F,G$ of the first fundamental form of $M$ in $I^3$ are computed by the induced metric from $I^3$. The unit normal vector of $M$ is the unit vector parallel to the $z$-direction. The components of the second fundamental form $II$ of $M$ are given by

\[
l = \frac{\det(r_{uu}, r_{u}, r_{v})}{\sqrt{EG - F^2}}, \quad m = \frac{\det(r_{uv}, r_{u}, r_{v})}{\sqrt{EG - F^2}}, \quad n = \frac{\det(r_{vv}, r_{u}, r_{v})}{\sqrt{EG - F^2}}.
\]

Accordingly, the isotropic Gaussian (or relative) and mean curvature of $M$ are respectively defined by

\[
K = \frac{ln - m^2}{EG - F^2}, \quad H = \frac{En - 2Fm + Gl}{2(EG - F^2)}.
\]
A surface in $\mathbb{I}^3$ is said to be isotropic minimal (resp. flat) if $H$ (resp. $K$) vanishes identically. Further, it is said to have constant isotropic mean (resp. Gaussian) curvature if $H$ (resp. $K$) is a constant function on whole surface.

3. Proof of Theorem 1.1

A factorable surface of type 2 in $\mathbb{I}^3$ can be locally expressed by either

$\Phi_2 : r(x,z) = (x, f(x)g(z), z)$ or $\Phi_3 : r(y,z) = (f(y)g(z), y, z)$.

All over this paper, all calculations shall be done for the surface $\Phi_3$. Its first fundamental form in $\mathbb{I}^3$ turns to

$$ds^2 = \left(1 + (f'g)^2\right)dy^2 + 2(fgf'g')dydz + (fgg')^2dz^2,$$

where $f' = \frac{df}{dy}, g' = \frac{dg}{dz}$. Note that $g'$ must be nonzero to obtain a regular admissible surface. By a calculation for the second fundamental form of $\Phi_3$ we have

$$II = \left(\frac{f''g}{fg}\right)dy^2 + 2\left(\frac{f'}{f}\right)dydz + \left(\frac{g''}{g'}\right)dz^2,$$

$g' \neq 0$.

Therefore, the isotropic mean curvature $H$ of $\Phi_3$ becomes

$$(3.1) \quad H = \frac{\left((f'g)^2 + 1\right)g'' + \left(ff'' - 2(f')^2\right)g(g')}{2f^2(g')^3}.$$

Let us assume that $H = H_0 = \text{const}$. First we distinguish the case in which $\Phi_3$ is isotropic minimal:

**Case A:** $H_0 = 0$. (3.1) reduces to

$$(3.2) \quad \left((f'g)^2 + 1\right)g'' + \left(ff'' - 2(f')^2\right)g(g') = 0.$$

We have three cases in order to solve (3.2):

**Case A.1.** $f = f_0 \neq 0 \in \mathbb{R}$, (3.2) immediately implies $g = c_1z + c_2, c_1, c_2 \in \mathbb{R}$, and thus we deduce that $\Phi_3$ is a non-isotropic plane. This gives the statement (i.1) of Theorem 1.1.

**Case A.2.** $f = c_1y + c_2, c_1, c_2 \in \mathbb{R}, c_1 \neq 0$. (3.2) turns to

$$g'' = \frac{2c_2gg'}{1 + (c_1g)^2}.$$

By solving this one, we obtain

$$g = \frac{1}{c_1}\tan(c_2z + c_3), \quad c_2, c_3 \in \mathbb{R}, \quad c_2 \neq 0,$$

which proves the statement (i.2) of Theorem 1.1.

**Case A.3.** $f'' \neq 0$. By dividing (3.2) with $g(g')^2$ one can be rewritten as

$$(3.3) \quad \left((f'g)^2 + 1\right)\frac{g''}{g(g')} + ff'' - 2(f')^2 = 0.$$
Taking partial derivative of (3.3) with respect to $z$ and after dividing with $(f')^2$, we get

$$2\frac{g''}{g'} + \left( \frac{1}{(f')^2} + g^2 \right) \left( \frac{g''}{g (g')^2} \right)' = 0. \quad (3.4)$$

By taking partial derivative of (3.4) with respect to $y$, we find $g'' = c_1 g (g')^2$, $c_1 \in \mathbb{R}$. We have two cases:

**Case A.3.1.** $c_1 = 0$. (3.3) reduces to

$$ff'' - 2 (f')^2 = 0. \quad (3.5)$$

By solving (3.5) we derive

$$f = -\frac{1}{c_2 y + c_3}, \ c_2, c_3 \in \mathbb{R}, \ c_2 \neq 0.$$  

This implies the statement (i.3) of Theorem 1.1.

**Case A.3.2.** $c_1 \neq 0$. (3.4) immediately leads to the contradiction $2c_1 gg' = 0$.

**Case B:** $H_0 \neq 0$. We have cases:

**Case B.1.** $f = f_0 \neq 0 \in \mathbb{R}$. Then (3.1) follows

$$2H_0 f_0^2 = \frac{g''}{(g')^3}. \quad (3.6)$$

Solving it gives $g(z) = \pm \frac{1}{2H_0 f_0^2} \sqrt{-4H_0 f_0^2 z + c_1 + c_2}, \ c_1, c_2 \in \mathbb{R}$.

This is the proof of the statement (ii) of Theorem 1.1.

**Case B.2.** $f = c_1 y + c_2$, $c_1, c_2 \in \mathbb{R}, \ c_1 \neq 0$. By considering this one into (3.1) we conclude

$$2(c_1 y + c_2)^2 H_0 = (1 + c_1^2 g^2) \frac{g''}{(g')^3} - 2c_1 \frac{g}{g'} \quad (3.7)$$

The left side in (3.7) is a function of $y$ while other side is either a constant or a function of $z$. This is not possible.

**Case B.3.** $f'' \neq 0$. By multiplying both side of (3.1) with $2f^2 \frac{g'}{g}$ one can be rearranged as

$$2H_0 f \frac{g'}{g} = \left( (f'g)' + 1 \right) \frac{g''}{g (g')^2} + ff'' - 2 (f')^2. \quad (3.8)$$

Taking partial derivative of (3.8) with respect to $z$ and thereafter dividing with $(f')^2$ yields

$$2H_0 \left( \frac{f}{f'} \right)^2 \left( \frac{g'}{g} \right)' = 2\frac{g''}{g'} + \left( \frac{1}{(f')^2} + g^2 \right) \left( \frac{g''}{g (g')^2} \right)' \quad (3.9)$$

It is obvious in (3.9) that $g'' \neq 0$. To solve (3.9) we have two cases:
Case B.3.1. $g'' = c_1 g'(g')^2$, $c_1 \in \mathbb{R}$, $c_1 \neq 0$. This implies that

\[(3.10) \quad g' = e^{\frac{c_1}{2}g^2+c_2}, \quad c_2 \in \mathbb{R} .\]

Substituting (3.10) into (3.9) gives an equation in the following form:

\[
\left( c_1 e^{\frac{c_1}{2}g^2} - c_2 \right) g^3 - \left( c_1 H_0 \left( \frac{f}{f'} \right)^2 \right) g^2 + H_0 \left( \frac{f}{f'} \right)^2 = 0,
\]

where all coefficients with respect to $g$ must be zero and this is a contradiction.

Case B.3.2. $\left( \frac{g''}{g'(g')^2} \right)^t \neq 0$. By dividing (3.9) with $\left( \frac{g''}{g'(g')^2} \right)^t$, it turns to the following form:

\[(3.11) \quad A_1(y) B_1(z) = A_2(y) + B_2(z),\]

where

\[
\begin{align*}
A_1(y) &= 2H_0 \left( \frac{f}{f'} \right)^2, \quad A_2(y) = \frac{1}{(f')^2}, \\
B_1(z) &= \left( \frac{g''}{g'(g')^2} \right), \quad B_2(z) = 2g'' + g^2.
\end{align*}
\]

The fact that all terms in (3.11) must be constant for every $y$ and $z$ yields $A_2(y) = \frac{1}{(f')^2} = \text{const.}$, which contradicts with the assumption of Case B.3.

4. Proof of Theorem 1.2

By a calculation for a factorable graph of type 2 in $\mathbb{R}^3$, the isotropic Gaussian curvature turns to

\[(4.1) \quad K = \frac{fgf''g' - (f'g')^2}{(fg')^3} .\]

Let us assume that $K = K_0 = \text{const}$. We have cases:

Case A: $K_0 = 0$. (4.1) reduces to

\[(4.2) \quad fgf''g' - (f'g')^2 = 0 .\]

$f$ or $g$ constants are solutions for (4.2) and by regularity we have the statement (i.1) of Theorem 1.2. Suppose that $f, g$ are non-constants. Then (4.2) yields $f''g' \neq 0$. Thereby (4.2) can be arranged as

\[(4.3) \quad \frac{ff''}{(f')^2} = \frac{(g')^2}{gg''} .\]
Both sides of (4.3) are equal to same nonzero constant, namely
\[(4.4) \quad ff'' - c_1(f')^2 = 0 \text{ and } gg'' - \frac{1}{c_1}(g')^2 = 0.\]

If \(c_1 = 1\) in (4.4), then by solving it we obtain
\[
f(y) = c_2 e^{c_3 y} \quad \text{and} \quad g(z) = c_4 e^{c_5 z}, \quad c_2, \ldots, c_5 \in \mathbb{R}.
\]

This gives the statement (i.2) of Theorem 1.2. Otherwise, i.e., \(c_1 \neq 1\) in (4.4), we derive
\[
f(y) = ((1 - c_1)(c_6 y + c_7))^{\frac{1}{c_1-1}} \quad \text{and} \quad g(z) = \left(\frac{c_1 - 1}{c_1}\right) (c_8 z + c_9) \cdot \frac{1}{c_1},
\]
where \(c_6, \ldots, c_9 \in \mathbb{R}\). This completes the proof of the statement (i) of Theorem 1.2.

**Case B**: \(K_0 \neq 0\). (4.1) can be rewritten as
\[(4.5) \quad K_0 (g')^2 = \frac{f''}{f^3} \left(\frac{gg''}{(g')^2}\right) - \left(\frac{f'}{f^2}\right)^2.
\]

Taking partial derivative of (4.5) with respect to \(z\) leads to
\[(4.6) \quad 2K_0 g'g'' = \frac{f''}{f^3} \left(\frac{gg''}{(g')^2}\right)'.
\]

We have two cases for (4.6):

**Case B.1.** The situation that \(g'' = 0, g(z) = c_1 z + c_2, c_1, c_2 \in \mathbb{R}\), is a solution for (4.6). Hence, from (4.5), we deduce
\[
K_0 (c_1)^2 = -\left(\frac{f'}{f^2}\right)^2,
\]
which implies that \(K_0\) is negative and
\[
f(y) = \frac{1}{\pm c_1 \sqrt{-K_0 y} + c_3}.
\]

This proves the statement (ii.1) of Theorem 1.2.

**Case B.2.** \(g'' \neq 0\). (4.6) immediately implies
\[(4.7) \quad f'' = c_1 f^3, \quad c_1 \in \mathbb{R}, \quad c_1 \neq 0.
\]

Considering (4.7) into (4.5) yields to
\[(4.8) \quad f' = c_2 f^2, \quad c_2 \in \mathbb{R}, \quad c_2 \neq 0.
\]

It follows from (4.7) and (4.8) that \(c_1 = 2c_2^2\) and
\[
f(y) = -\frac{1}{c_2 y + c_3}.
\]
for some constant $c_3$. Nevertheless, by substituting (4.7) and (4.8) into (4.5), we conclude

\[(4.9)\]
\[
\frac{K_0}{c_4^2}r^3 + r = 2g\dot{r},
\]

where $r = g'$ and $\dot{r} = \frac{dg}{d\gamma} = \frac{g''}{g'}$. After solving (4.9) we obtain

\[
r = \pm \left(\frac{c_4^2g^{-1} - K_0}{c_4^2}\right)^{1/2},
\]

or

\[
z = \pm \int \left(\frac{c_4^2g^{-1} - K_0}{c_4^2}\right)^{1/2} dg,
\]

which proves the statement (ii.2) of Theorem 1.2.

5. Proof of Theorem 1.3

Assume that a factorable surface of type 2 in $\mathbb{R}^3$ fulfills the condition $H + \lambda K = 0$, $\lambda HK \neq 0$, $\lambda \in \mathbb{R}$. Then (3.1) and (4.1) give rise to

\[(5.1)\]
\[
\left(1 + (f')^2\right)f^2g'g'' + \left(f f'' - 2(f')^2\right)f^2g(g')^3 + 2\lambda \left(ff''g'' - (f')^2g\right)^2 = 0.
\]

Due to $K \neq 0$, $f$ must be a non-constant function and therefore dividing (5.1) with $(ff')^2$ leads to

\[(5.2)\]
\[
\left(\frac{1}{(f')^2} + g^2\right)g'g'' + \left(\frac{ff''}{(f')^2} - 2\right)g(g')^3 + 2\lambda \left(\frac{f''}{f(f')^2g} - \frac{(g')^2}{f^2}\right) = 0.
\]

If $g'' = 0$, namely $g = c_1z + c_2$, $c_1, c_2 \in \mathbb{R}, c_1 \neq 0$, then (5.2) reduces to the following polynomial equation on $z$:

\[(5.3)\]
\[
c_1^2 \left(\frac{ff''}{(f')^2} - 2\right)z + c_1c_2 \left(\frac{ff''}{(f')^2} - 2\right) - \frac{2\lambda}{f} = 0.
\]

All coefficients in (5.3) must be zero and this fact yields the contradiction $\lambda = 0$. Then $g'' \neq 0$ and, by dividing (5.2) with the product $g'g''$, we get

\[(5.4)\]
\[
\frac{1}{(f')^2} + g^2 - 2\left(\frac{g'}{g''}\right)^2 + \left(\frac{ff''}{(f')^2}\right)g(g')^2 + 2\lambda \left(\left(\frac{f''}{f(f')^2g}\right)\frac{g}{g'} - \frac{1}{f^2}\right)\frac{g'}{g''} = 0.
\]

Putting $p = f'$, $\dot{p} = \frac{dp}{d\gamma} = \frac{f''}{f'}$ and $r = g'$, $\dot{r} = \frac{dr}{d\gamma} = \frac{g''}{g'}$, (5.4) turns to

\[(5.5)\]
\[
\frac{1}{p^2} + g^2 - 2\frac{gr}{\dot{r}} + \left(\frac{\dot{p}}{p}\right)\frac{gr}{\dot{r}} + 2\lambda \left[\left(\frac{\dot{p}}{fp}\right)\frac{g}{\dot{r}} - \left(\frac{1}{f^2}\right)\frac{1}{\dot{r}}\right] = 0.
\]
Taking partial derivatives of (5.5) with respect to $f$ and $g$ implies an equation in the following form:

\[ A_1 (f) B_1 (g) + 2\lambda (A_2 (f) B_2 (g) - A_3 (f) B_3 (g)) = 0, \]

where

\[
\begin{align*}
A_1 (f) &= \frac{d}{df} \left( \frac{f\dot{p}}{p} \right), & A_2 (f) &= \frac{d}{df} \left( \frac{\dot{p}}{fp} \right), & A_3 (f) &= \frac{d}{df} \left( \frac{1}{f^2} \right), \\
B_1 (g) &= \frac{d}{dg} \left( \frac{gr}{r^2} \right), & B_2 (g) &= \frac{d}{dg} \left( \frac{g}{r} \right), & B_3 (g) &= \frac{d}{dg} \left( \frac{1}{r} \right).
\end{align*}
\]

If $B_2 = 0$, i.e., $r = c_1 g$, $c_1 \in \mathbb{R}$, $c_1 \neq 0$, then (5.5) yields the following polynomial equation $g$:

\[ \left( \frac{f\dot{p}}{p} - 1 \right) g^2 + \frac{2\lambda}{c_1 f^2} \left( \frac{f\dot{p}}{p} - 1 \right) + \frac{1}{p^2} = 0. \]

The fact that the coefficient of the term $g^2$ in (5.8) must vanish leads to the contradiction $\frac{1}{p^2} = 0$ and so we deduce $B_2 \neq 0$. Nevertheless, due to $A_3 \neq 0$, (5.6) can be divided by the product $A_3 B_2$ as follows:

\[ \left( \frac{A_1 (f)}{A_3 (f)} \right) \left( \frac{B_1 (g)}{B_2 (g)} \right) + 2\lambda \left( \frac{A_2 (f)}{A_3 (f)} - \frac{B_3 (g)}{B_2 (g)} \right) = 0, \]

where the terms $A_4, A_5, B_4, B_5$ must be constant for every $f$ and $g$. Since $A_4 = c_1$ and $A_5 = c_2$, by (5.7), we derive

\[ \frac{f\dot{p}}{p} = \frac{c_1}{f^2} + c_3 \]

and

\[ \frac{\dot{p}}{fp} = \frac{c_2}{f^2} + c_4, \quad c_1, \ldots, c_4 \in \mathbb{R}. \]

After equalizing (5.10) and (5.11), we find

\[ \frac{\dot{p}}{p} = \frac{c_2}{f}, \quad c_2 = c_3, \]

where $c_2$ must be non-vanishing. Otherwise, considering the situation that $\dot{p} = 0$, $p(f) = c_5 \in \mathbb{R}$, $c_5 \neq 0$, into (5.5) gives

\[ \frac{1}{c^2_5} + g^2 - 2\frac{gr}{r} - \left( \frac{2\lambda}{r} \right) \frac{1}{f^2} = 0. \]

The coefficient of the term $\frac{1}{f^2}$ in (5.13) cannot vanish and this leads to a contradiction. So, by (5.12), we derive $A_1 = 0$ and (5.9) reduces to

\[ c_2 B_2 (g) - B_3 (g) = 0. \]
An integration of (5.14) yields
\begin{equation}
68\end{equation}
c_{2} \frac{g}{r} - \frac{1}{\dot{r}} = c_{6}, \ c_{6} \in \mathbb{R}.
\end{equation}
Substituting (5.12) and (5.15) into (5.5) leads to
\begin{equation}
\frac{1}{p^{2}} + \frac{2\lambda c_{6}}{f^{2}} + g^{2} + (c_{2} - 2) \frac{gr}{r} = 0.
\end{equation}
By revisiting (5.12), we obtain
\begin{equation}
p = c_{7} f c_{2}, \ c_{7} \in \mathbb{R}, \ c_{7} \neq 0\end{equation}
and considering this one into (5.16)
\begin{equation}
\frac{1}{c_{2}^{2} f c_{2}} + \frac{2\lambda c_{6}}{f^{2}} + g^{2} + (c_{2} - 2) \frac{gr}{r} = 0.
\end{equation}
Due to the fact that \( f \) is an independent variable in (5.17), we conclude
\begin{equation}
c_{2} = 1 \text{ and } \frac{1}{c_{2}^{2}} + 2\lambda c_{6} = 0.
\end{equation}
Thereby, (5.17) reduces to
\begin{equation}
g^{2} - \frac{gr}{r} = 0.
\end{equation}
Comparing (5.19) with (5.15) yields \( c_{6} = 0 \) which contradicts with (5.18).

6. Some examples

We illustrate several examples relating to the factorable surfaces of type 2 in \( \mathbb{E}^{3} \) with \( K, H \) constants.

**Example 6.1.** Consider the factorable surfaces of type 2 in \( \mathbb{E}^{3} \) given by
\begin{enumerate}
\item \( \Phi_{3} : x(y, z) = y \tan z, \ (y, z) \in [0, \frac{\pi}{4}], \) (isotropic minimal),
\item \( \Phi_{3} : x(y, z) = -\sqrt{z}, \ (y, z) \in [0, 2\pi], \) (\( H = -1 \)),
\item \( \Phi_{3} : x(y, z) = -\frac{z}{4z}, \ (y, z) \in [1, 1.4] \times [1, 2\pi], \) (isotropic flat),
\item \( \Phi_{3} : x(y, z) = \frac{z}{2}, \ (y, z) \in [1, \pi] \times [1, 2\pi], \) (\( K = -1 \)).
\end{enumerate}
These surfaces can be respectively drawn by as in Figs. 1-4.
Acknowledgement. The author would like to thank the referee for his/her careful reading and useful suggestions. All figures in this paper are plotted by using Wolfram Mathematica 7.0.

References

MUBITTIN EVREN AYDIN
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
FIRAT UNIVERSITY
ELAZIG, 23200, TURKEY

E-mail address: meaydin@firat.edu.tr