Finslerian Hypersurface and Generalized $\beta$—Conformal Change of Finsler Metric

Shiv Kumar Tiwari* and Anamika Rai
Department of Mathematics, K. S. Saket Post Graduate College, Ayodhya, Faizabad-224 123, India
e-mail: sktiwarisaket@yahoo.com and anamikarai2538@gmail.com

Abstract. In the present paper, we have studied the Finslerian hypersurfaces and generalized $\beta$—conformal change of Finsler metric. The relations between the Finslerian hypersurface and the other which is Finslerian hypersurface given by generalized $\beta$—conformal change have been obtained. We have also proved that generalized $\beta$—conformal change makes three types of hypersurfaces invariant under certain conditions.

1. Introduction

Let $(M^n, L)$ be an $n$—dimensional Finsler space on a differentiable manifold $M^n$ equipped with the fundamental function $L(x, y)$. In 1984, Shibata [12] introduced the transformation of Finsler metric:

$$L(x, y) = f(L, \beta),$$

where $\beta = b_i(x) y^i$, $b_i(x)$ are components of a covariant vector in $(M^n, L)$ and $f$ is positively homogeneous function of degree one in $L$ and $\beta$. This change of metric is called a $\beta$—change. In 2013, Prasad, B. N. and Kumari, Bindu [10] have considered the $\beta$—change of Finsler metric. In the year 2014 [13], we studied generalized $\beta$—change defining as

$$L(x, y) \rightarrow \mathcal{L}(x, y) = f(L, \beta^1, \beta^2, \ldots, \beta^m),$$

where $f$ is any positively homogeneous function of degree one in $L, \beta^1, \beta^2, \ldots, \beta^m$, where $\beta^1, \beta^2, \ldots, \beta^m$ are linearly independent one-form.

* Corresponding Author.
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The conformal theory of Finsler spaces has been initiated by M. S. Knebelman [7] in 1929 and has been investigated in detail by many authors [1, 2, 3, 6] etc. The conformal change is defined as

\[ L(x, y) \rightarrow e^{\sigma(x)} L(x, y), \]

where \( \sigma(x) \) is a function of position only and known as conformal factor.

We also studied the generalized \( \beta \)-conformal change of Finsler metric by taking

\[ L = f(e^{\sigma(x)} L(x, y), \beta^1, \beta^2, \ldots, \beta^m), \]

where \( f \) is any positively homogeneous function of degree one in \( e^{\sigma L}, \beta^1, \beta^2, \ldots, \beta^m \).

On the other hand, in 1985, M. Matsumoto investigated the theory of Finslerian hypersurface [8]. He has defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds.

In the year 2009, B. N. Prasad and Gauri Shanker [11] studied the Finslerian hypersurfaces and \( \beta \)-change of Finsler metric and obtained different results in his paper. In the present paper, using the field of linear frame [5, 4, 9], we shall consider Finslerian hypersurfaces given by a generalized \( \beta \)-conformal change of a Finsler metric. Our purpose is to give some relations between the original Finslerian hypersurface and the other which is Finslerian hypersurface given by generalized \( \beta \)-conformal change. We have also obtained that a generalized \( \beta \)-conformal change makes three types of hypersurfaces invariant under certain conditions.

### 2. Finslerian Hypersurfaces

Let \( M^n \) be an \( n \)-dimensional manifold and \( F^n = (M^n, L) \) be an \( n \)-dimensional Finsler space equipped with the fundamental function \( L(x, y) \) on \( M^n \). The metric tensor \( g_{ij}(x, y) \) and Cartan’s \( C \)-tensor \( C_{ijk}(x, y) \) are given by

\[ g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}, \quad C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}, \]

respectively and we introduce the Cartan’s connection \( \Gamma = (F^i_{jk}, N^i_j, C^i_{jk}) \) in \( F^n \).

A hypersurface \( M^{n-1} \) of the underlying smooth manifold \( M^n \) may be parametrically represented by the equation \( x^i = x^i(u^\alpha) \), where \( u^\alpha \) are Gaussian coordinates on \( M^{n-1} \) and Greek indices vary from 1 to \( n-1 \). Here, we shall assume that the matrix consisting of the projection factors \( B^i_{\alpha\beta} = \frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial u^\beta} \) is of rank \( n-1 \). The following notations are also employed:

\[ B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \quad B^i_{\alpha\beta} = v^\alpha B^i_{\alpha\beta}. \]

If the supporting element \( y^i \) at a point \( (u^\alpha) \) of \( M^{n-1} \) is assumed to be tangential to \( M^{n-1} \), we may then write \( y^i = B^i_{\alpha}(u)v^\alpha \), i.e. \( v^\alpha \) is thought of as the supporting
element of $M_{n-1}$ at the point $(u^a)$. Since the function \( L(u,v) = L(x(u), y(u,v)) \) gives rise to a Finsler metric of $M_{n-1}$, we get a \((n-1)\)-dimensional Finsler space $F^{n-1} = \{ M_{n-1}, L(u,v) \}$.

At each point $(u^a)$ of $F^{n-1}$, the unit normal vector $N^i(u, v)$ is defined by
\[
(2.1) \quad g_{ij} B^i_\alpha N^j = 0, \quad g_{ij} N^i N^j = 1.
\]

If $B^i_\alpha, N_i$ is the inverse matrix of $(B^i_\alpha, N^i)$, we have
\[
B^i_\alpha B^j_\beta = \delta^i_\beta, \quad B^i_\alpha N_i = 0, \quad N_i N_i = 1 \quad \text{and} \quad B^i_\alpha B^j_\beta + N^i N_j = \delta^i_j.
\]

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get
\[
(2.2) \quad B^i_\alpha = g^{\alpha\beta} g_{ij} B^j_\beta, \quad N_i = g_{ij} N^j.
\]

For the induced Cartan’s connection $ICT = (F^\alpha_{\beta\gamma}, N^\alpha_{\beta}, C^\alpha_{\beta\gamma})$ on $F^{n-1}$, the second fundamental $h-$tensor $H_{\alpha\beta}$ and the normal curvature $H_\alpha$ are respectively given by [9]
\[
(2.3) \quad H_{\alpha\beta} = N_i (B^i_\alpha + F^i_{jk} B^j_\alpha B^k_\beta) + M_\alpha H_\beta,
\quad H_\alpha = N_i (B^i_\alpha + N^i B^j_\beta),
\]

where
\[
M_\alpha = C_{ijk} B^i_\alpha N^j N^k.
\]

Contracting $H_{\alpha\beta}$ by $v^\alpha$, we immediately get $H_{\beta\alpha} = H_{\alpha\beta} v^\alpha = H_\beta$. Furthermore the second fundamental $v-$tensor $M_{\alpha\beta}$ is given by [8]
\[
(2.4) \quad M_{\alpha\beta} = C_{ijk} B^i_\alpha B^j_\beta N^k.
\]

3. Finsler Space with Generalized $\beta-$Conformal Change

Let $(M^n, L)$ be a Finsler space $F^n$, where $M^n$ is an $n-$dimensional differentiable manifold equipped with a fundamental function $L$. A change in fundamental metric $L$, defined by equation (1.4), is called generalized $\beta-$conformal change, where $\sigma(x)$ is conformal factor and function of position only and $\beta^1, \beta^2, \ldots, \beta^m$ all are linearly independent one-form and defined as $\beta^r = b^r_i y^i$.

Homogeneity of $f$ gives
\[
(3.1) \quad e^{\sigma} L f_0 + f_r \beta^r = f,
\]

where the subscripts ‘0’ and ‘r’ denote the partial derivative with respect to $L$ and $\beta^r$ respectively. The letters $r, s, t, r'$ and $s'$ vary from 1 to $m$ throughout the paper. Summation convention is applied for the indices $r, s, t, r'$ and $s'$. If we write $F^n = (M^n, L)$, then the Finsler space $\tilde{F}^n$ is said to be obtained from $F^n$ by
generalized $\beta-$conformal change. The quantities corresponding to $\mathcal{F}^n$ are denoted by putting bar on those quantities.

To find the relation between fundamental quantities of $(M^n, L)$ and $(M^n, \mathcal{L})$, we use the following results:

\begin{align}
\dot{\partial}_i \beta^r &= b^r_i, \quad \dot{\partial}_i L = l_i, \quad \dot{\partial}_i l_i = L^{-1}h_{ij},
\end{align}

where $\dot{\partial}_i$ stands for $\frac{\partial}{\partial y^i}$ and $h_{ij}$ are components of angular metric tensor of $(M^n, L)$ given by

\begin{align}
h_{ij} &= g_{ij} - l_i l_j = L \dot{\partial}_i \dot{\partial}_j L.
\end{align}

Differentiating (3.1) with respect to $L$ and $\beta^s$ respectively, we get

\begin{align}
\dot{e}^\sigma L f_{00} + f_{0r} \beta^r = 0
\end{align}

and

\begin{align}
\dot{e}^\sigma L f_{0s} + f_{rs} \beta^r = 0.
\end{align}

The successive differentiation of (1.4) with respect to $y^i$ and $y^j$ give

\begin{align}
\dot{\partial}_i &= e^\sigma f_{0i} + f_r b^r_i,
\end{align}

\begin{align}
\dot{\partial}_i m^r_j &= e^\sigma f_{0i} h_{ij} + e^{2\sigma} f_{00} l_i l_j + e^\sigma f_{0r} (b^r_i l_j + b^r_j l_i) + f_{fr} b^r_i b^s_j.
\end{align}

Using equations (3.3) and (3.4) in equation (3.6), we have

\begin{align}
\bar{h}_{ij} &= e^\sigma f_{00} h_{ij} + e^{2\sigma} f_{00} l_i l_j + e^\sigma f_{0r} (b^r_i l_j + b^r_j l_i) + f_{fr} b^r_i b^s_j.
\end{align}

If we put $m^r_j = b^r_i - \frac{\beta^r_j}{L} l_i$, equation (3.7) may be written as

\begin{align}
\bar{h}_{ij} &= e^\sigma f_{00} h_{ij} + f_{fr} m^r_i m^s_j.
\end{align}

From equations (3.5) and (3.8), we get the following relation between metric tensors of $(M^n, L)$ and $(M^n, \mathcal{L})$

\begin{align}
\bar{g}_{ij} &= e^\sigma f_{00} g_{ij} + e^\sigma \left( e^\sigma f_{00} - \frac{f_{00}}{L} \right) l_i l_j + f_{fr} m^r_i m^s_j

+ e^\sigma f_{0r} (b^r_i l_j + b^r_j l_i) + f_{fr} b^r_i b^s_j.
\end{align}

Now,

\begin{align}
(a) \quad \dot{\partial}_i m^r_j &= -\frac{1}{L} \left( m^r_i l_j + \frac{\beta^r}{L} h_{ij} \right),
\end{align}

\begin{align}
(b) \quad \dot{\partial}_i f &= e^\sigma f_{0i} + f_r b^r_i,
\end{align}

\begin{align}
(c) \quad \dot{\partial}_i f_{rs} &= e^\sigma f_{rs0} l_i + f_{rst} b^s_i.
\end{align}
Differentiating equation (3.8) with respect to $y^k$ and using equations (3.2), (3.3), (3.4), (3.5), (3.9) and (3.10), we get

\begin{equation}
\overline{C}_{ijk} = p_0 C_{ijk} + p_1 (h_{ij} m^r_k + h_{jk} m^r_i + h_{ki} m^r_j) + p_2 m^r_i m^s_j m^t_k,
\end{equation}

where

\begin{equation}
\begin{aligned}
p_0 &= e^\sigma \frac{f f_0}{L} C_{ijk}, \\
p_1 &= e^\sigma \frac{f_0 f_r + f f_0}{2 L}, \\
p_2 &= \frac{1}{2} (f rs f_t + f st f_r + f tr f_s + f f rs t).
\end{aligned}
\end{equation}

4. Hypersurfaces Given by a Generalized $\beta$–Conformal Change

Consider a Finslerian hypersurface $F^{n-1} = \{M^{n-1}, \bar{L}(u, v)\}$ of the $F^n$ and another Finslerian hypersurface $F^{n-1} = \{M^{n-1}, L(u, v)\}$ of the $F^n$ given by generalized $\beta$–conformal change. Let $N^i$ be the unit vector at each point of $F^{n-1}$ and $(B^a_i, N_i)$ be the inverse matrix of $(B^a_i, N_i)$. The function $B^a_i$ may be considered as components of $(n-1)$ linearly independent tangent vectors of $F^{n-1}$ and they are invariant under generalized $\beta$–conformal change. Thus, we shall show that a unit normal vector $\overline{N}(u, v)$ of $\overline{F}^{n-1}$ is uniquely determined by

\begin{equation}
\overline{g}_{ij} B^a_i N^j = 0, \quad \overline{g}_{ij} \overline{N}^i \overline{N}^j = 1.
\end{equation}

Contracting (3.9) by $N^i N^j$ and paying attention to (2.1) and the fact that $l_i N^i = 0$, we have

\begin{equation}
\overline{g}_{ij} N^i N^j = p_0 + p (b^r_i b^s_j N^i N^j),
\end{equation}

where $p = f f_{rs} + f_r f_s$. Therefore, we obtain

\begin{equation}
\overline{g}_{ij} \left( \pm \frac{N^i}{\sqrt{p_0 + p (b^r_i b^s_j N^i N^j)}} \right) \left( \pm \frac{N^j}{\sqrt{p_0 + p (b^r_i b^s_j N^i N^j)}} \right) = 1.
\end{equation}

Hence, we can put

\begin{equation}
\overline{N}^i = \frac{N^i}{\sqrt{p_0 + p (b^r_i b^s_j N^i N^j)}},
\end{equation}

where we have chosen the positive sign in order to fix an orientation.

Using equations (3.9), (4.3) and from first condition of (4.1), we have

\begin{equation}
B^a_i (2 p_1 L l_i + p b^r_i) \sqrt{\frac{b^r_j N^j}{p_0 + p (b^r_i b^s_j N^i N^j)}} = 0.
\end{equation}
If \( B_i^i(2p_iL_i + pb_i^n) = 0 \), then contracting it by \( v^\alpha \) and using \( y^i = B_i^i v^\alpha \), we get \( L = 0 \) or \( \beta^\alpha = 0 \) which is a contradiction with the assumption that \( L > 0 \). Hence \( b_j \), \( N_j = 0 \). Therefore equation (4.3) is written as

\[
(4.5) \quad N^i = \frac{N^i}{\sqrt{p_0}}.
\]

Summarizing the above, we obtain

**Proposition 4.1.** For a field of linear frame \( (B_1^i, B_2^i, \ldots, B_{n-1}^i; N^i) \) of \( F^n \), there exists a linear frame \( (B_1^i, B_2^i, \ldots, B_{n-1}^i, N^i = \frac{N^i}{\sqrt{p_0}}) \) of \( \overline{F^n} \) such that (4.1) is satisfied along \( F^{n-1} \) and then \( b_i \) is tangential to both of the hypersurfaces \( F^{n-1} \) and \( \overline{F^{n-1}} \).

The quantities \( B_\alpha^i \) are uniquely defined along \( F^{n-1} \) by

\[
B_\alpha^i = \overline{\gamma}^{\alpha\beta} \overline{g}_{ij} B_\beta^j
\]

where \( \overline{\gamma}^{\alpha\beta} \) is the inverse matrix of \( \overline{g}_{\alpha\beta} \). Let \( (\overline{B}_i^i, \overline{N}^i) \) be the inverse matrix of \( (B_\alpha^i, N^i) \), then we have

\[
B_\alpha^i \overline{B}_j^i = \delta_\alpha^j, \quad B_\alpha^i \overline{N}_i = 0, \quad \overline{N}^i \overline{N}_i = 1.
\]

Furthermore \( B_\alpha^i \overline{B}_j^i + \overline{N}^i \overline{N}_j = \delta_i^j \). We also get \( \overline{N}_i = \overline{g}_{ij} \overline{N}_j \) which in view of (3.5), (3.9) and (4.5) gives

\[
(4.6) \quad \overline{N}_i = \sqrt{p_0} N_i.
\]

We denote the Cartan’s connection of \( F^n \) and \( \overline{F^n} \) by \( (\overline{F}^i_{jk}, N^i_j, C^i_{jk}) \) and \( (\overline{F}^i_{jk}, N^i_{\overline{j}}, \overline{C}^i_{\overline{j}k}) \) respectively and put \( D^i_{jk} = \overline{F}^i_{jk} - F^i_{jk} \) which will be called difference tensor. We choose the vector field \( b^i \) in \( F^n \) such that

\[
(4.7) \quad D^i_{jk} = A_{jk} b^i + B_{jk} l^i + \delta^i_j D_k + \delta^i_k D_j,
\]

where \( A_{jk} \) and \( B_{jk} \) are components of a symmetric covariant tensor of second order and \( D_i \) are components of a covariant vector. Since \( N_i b_j = 0 \), \( N_i l^i = 0 \) and \( \delta^i_j N_i B_\alpha^i = 0 \), from (4.7), we get

\[
(4.8) \quad N_i D^i_{jk} B_\beta^k + B_\beta^k B^i_{jk} = 0 \quad \text{and} \quad N_i D^i_{\overline{k}k} B_\overline{\beta}^k = 0.
\]

Therefore, from (2.3) and (4.6), we get

\[
(4.9) \quad \overline{H}_\alpha = \sqrt{p_0} H_\alpha.
\]

If each path of a hypersurface \( F^{n-1} \) with respect to the induced connection also a path of the enveloping space \( F^n \), then \( F^{n-1} \) is called a hyperplane of the first
A hyperplane of the first kind is characterized by $H_\alpha = 0$ [8]. Hence from (4.9), we have

**Theorem 4.1.** If $b^r_i(x)$ be a vector field in $F^n$ satisfying (4.7), then a hypersurface $F^{n-1}$ is a hyperplane of the first kind if and only if the hypersurface $\mathcal{F}^{n-1}$ is a hyperplane of the first kind.

Next contracting (3.11) by $B^i_\alpha N^j N^k$ and paying attention to (4.5), $m^r_i N^i = 0$, $h_{jk} N^j N^k = 1$ and $h_{ij} B^i_\alpha N^j = 0$, we get

$$\overline{M}_\alpha = M_\alpha + \frac{p_1}{p_0} m^r_i B^i_\alpha.$$  

From (2.3), (4.6), (4.8), we have

(4.10)  

$$\overline{\Pi}_{\alpha\beta} = \sqrt{p_0} H_{\alpha\beta}.$$  

If each $h-$path of a hypersurface $F^{n-1}$ with respect to the induced connection is also $h-$path of the enveloping space $F^n$, then $F^{n-1}$ is called a hyperplane of the second kind. A hyperplane of the second kind is characterized by $H_{\alpha\beta} = 0$ [8]. Since $H_{\alpha\beta} = 0$ implies that $H_\alpha = 0$ from (4.9) and (4.10), we have the following:

**Theorem 4.2.** If $b^r_i(x)$ be a vector field in $F^n$ satisfying (4.7), then a hypersurface $F^{n-1}$ is a hyperplane of the second kind if and only if the hypersurface $\mathcal{F}^{n-1}$ is a hyperplane of the second kind.

Finally contracting (3.11) by $B^i_\alpha B^j_\beta N^k$ and paying attention to (4.5), we have

(4.11)  

$$\overline{M}_{\alpha\beta} = \sqrt{p_0} M_{\alpha\beta}.$$  

If the unit normal vector of $F^{n-1}$ is parallel along each curve of $F^{n-1}$, then $F^{n-1}$ is called a hyperplane of third kind. A hyperplane of the third kind is characterized by $H_{\alpha\beta} = 0, M_{\alpha\beta} = 0$ [8]. From (4.10) and (4.11), we have:

**Theorem 4.3.** If $b^r_i(x)$ be a vector field in $F^n$ satisfying (4.7), then a hypersurface $F^{n-1}$ is a hyperplane of the third kind if and only if the hypersurface $\mathcal{F}^{n-1}$ is a hyperplane of the third kind.

References


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