CYCLIC CODES OVER THE RING
\( F_p[u, v, w]/\langle u^2, v^2, w^2, uv - vu, vw - wv, uw - wu \rangle \)

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Abstract. Let \( R_{u^2, v^2, w^2, p} \) be a finite non chain ring \( F_p[u, v, w]/\langle u^2, v^2, w^2, uv - vu, vw - wv, uw - wu \rangle \), where \( p \) is a prime number. This ring is a part of family of Frobenius rings. In this paper, we explore the structures of cyclic codes over the ring \( R_{u^2, v^2, w^2, p} \) of arbitrary length. We obtain a unique set of generators for these codes and also characterize free cyclic codes. We show that Gray images of cyclic codes are 8-quasicyclic binary linear codes of length \( 8n \) over \( F_p \). We also determine the rank and the Hamming distance for these codes. At last, we have given some examples.

1. Introduction

Cyclic codes are a key family of linear codes because of their lavish algebraic structures and practical accomplishment. A considerable attention has been paid to cyclic codes over rings in the early 1990’s because of their affluent applications to design error-correcting coding schemes for wireless communication system. The classification of cyclic codes of length \( n \) over chain rings as well as over some non chain rings, when \( \gcd(n, p) = 1 \), where \( p \) is the characteristic of the ring, has been completely discussed in the literature. On the other hand, the classification of codes when \( p \) divides \( n \) is still not complete. Cyclic codes have been extensively studied over various finite chain rings in [1–5,10]. More latterly, cyclic codes over finite non-chain rings have also been contemplated. However, the analysis on non-chain rings seems to be challenging as the algebraic structure does not allow to give nice and compact presentation of linear codes over these rings. Yildiz and Karadeniz in [12] characterized cyclic codes of odd length over the non-chain ring \( F_2[u, v]/\langle u^2, v^2, uv - vu \rangle \) and obtained some good binary codes under two Gray maps. Sobhani and Molakarimi in [11] characterized the cyclic codes over the ring \( F_{2^m}[u, v]/\langle u^2, v^2, uv - vu \rangle \), they also determined \( F_{2^m}\)-basis as well as the mass the formula for the number of these codes. The authors of [8] extended these studies in more general way for cyclic codes.
codes over the ring $\mathbb{F}_p[u,v]/(u^2, v^2, uv - vu)$ and presented some ternary optimal codes as gray images. In [6], S. T. Dougherty et al. considered a family of rings $F_2[u_1,u_2,\ldots,u_k]/(u_1^2, u_2^2,\ldots,u_k^2, u_iu_j - u_ju_i)$ and studied one-generator cyclic codes.

The purpose of this paper is to obtain the structure theorems for cyclic codes over the non-chain $R_{u^2,v^2,w^2,p}$ ring in more general setting. The paper is organized as follows: In Section 2, we find a generating set of polynomials along with the conditions on these generators for the cyclic codes over the ring $R_{u^2,v^2,w^2,p}$ of length $n$. We also discuss here generating polynomials for cases of free cyclic codes and $n$ relatively prime to $p$. In Section 3, we prove some lemmas, in which we express certain types of polynomials over $R_{u^2,v^2,w^2,p}$ as linear combinations of generators of cyclic codes over $R_{u^2,v^2,w^2,p}$. By help of these lemmas, we derive a minimal spanning set, we also calculate ranks of these codes. In Section 4, we obtain the minimum distance of corresponding codes. In Section 5.1, we will show that gray images of cyclic codes are 8-quasicyclic binary linear code of length $8n$ over $\mathbb{F}_p$. In Section 6, we discuss some examples of cyclic codes over the ring $R_{u^2,v^2,w^2,p}$ of length 3, 4, 5 over $F_3, F_2, F_5$ respectively.

2. The structures of cyclic codes over the ring $R_{u^2,v^2,w^2,p}$

Let $R_{u^2,v^2,w^2,p} = \mathbb{F}_p[u,v]/(u^2, v^2, w^2, vu - uv, wv - vw, uv - vu)$, where $p$ is a prime number and $n$ is a positive integer. We can write $R_{u^2,v^2,w^2,p}$ as $R_{u^2,v^2,w^2,p} = R_{u^2,v^2,w^2,p} + wR_{u^2,v^2,w^2,p}, w^2 = 0$, where $R_{u^2,v^2,w^2,p} = \mathbb{F}_p + uv\mathbb{F}_p + vw\mathbb{F}_p + uw\mathbb{F}_p$ and $w^2 = 0, v^2 = 0$. Let $R_{u^2,v^2,w^2,p,n} = R_{u^2,v^2,w^2,p}[x]/(x^n - 1)$. Let $C$ be a cyclic code of length $n$ over the ring $R_{u^2,v^2,w^2,p}$. We can also consider $C$ as an ideal in the ring $R_{u^2,v^2,w^2,p,n}$. We define the map $\psi : R_{u^2,v^2,w^2,p} \to R_{u^2,v^2,w^2,p}$ by $\psi(\alpha + w\beta) = \alpha$, where $\alpha, \beta \in R_{u^2,v^2,w^2,p}$. Clearly the map $\psi$ is a surjective ring homomorphism. Let $R_{u^2,v^2,w^2,p,n} = R_{u^2,v^2,w^2,p}[x]/(x^n - 1)$. We extend this homomorphism to a homomorphism $\phi$ from $C$ to the ring $R_{u^2,v^2,w^2,p,n}$ defined by

$$\phi(c_0 + c_1x + \cdots + c_{n-1}x^{n-1}) = \psi(c_0) + \psi(c_1)x + \cdots + \psi(c_{n-1})x^{n-1},$$

where $c_i \in R_{u^2,v^2,w^2,p}$. Let $J = \{r(x) \in R_{u^2,v^2,w^2,p,n}[x] : wr(x) \in \ker\phi\}$. We see that $J$ is an ideal of $R_{u^2,v^2,w^2,p,n}$. Hence, we can consider $J$ as a cyclic code over $R_{u^2,v^2,p}$. We know from Theorem 3.1 of [8] that any ideal of $R_{u^2,v^2,w^2,p,n}$ is of the form $(g(x) + up_1(x) + vq_1(x) + uvr_1(x), uaw_1(x) + uvq_2(x) + uvw_2(x) + uvw_3(x), uvv_3(x))$. Now we assume that $B_1 = g(x) + up_1(x) + vq_1(x) + uvr_1(x), B_2 = uvq_2(x) + uvw_2(x) + uvw_3(x), B_3 = uvw_3(x)$. So $J = \langle B_1, B_2, B_3 \rangle$. Therefore, we can write $\ker\phi = \langle wB_1, wB_2, wB_3, wB_4 \rangle$. Since $\phi$ is a surjective homomorphism, the image $\Im\phi$ is an ideal of $R_{u^2,v^2,w^2,p,n}$. Hence, $\Im\phi$ is a cyclic code over $R_{u^2,v^2,w^2,p}$. Again we can write $\Im\phi$ as above. That is, $\Im\phi = \langle B_1', B_2', B_3', B_4' \rangle$. Therefore, the code $C$ over the ring $R_{u^2,v^2,w^2,p}$ can be written as $C = \langle A_1, A_2, \ldots, A_8 \rangle$, where, $A_1$’s are defined as follows:

$$A_1 = f_1(x) + uf_1,2(x) + vf_1,3(x) + wvf_1,4(x) + wvf_1,5(x) + wuvf_1,6(x)$$
+ uvwf(x) + vwf(x) + uvf(x) + vf(x) + uvf(x) + uvf(x) + uvf(x) + uvf(x).

Throughout this paper we use ideals $A_1, A_2, \ldots, A_8$ for above polynomials.

For an ideal $C$ of the ring $R_{a^2, v^2, w^2} = R_{a^2, v^2, w^2, p, n} / \langle x^n - 1 \rangle$, we define the residue and the torsion of an ideal $C$ as

$$\text{Res}(C) = \{ a \in R_{a^2, v^2, w^2, p, n} | \exists b \in R_{a^2, v^2, w^2, p, n} : a + wb \in C \}$$

and

$$\text{Tor}(C) = \{ a \in R_{a^2, v^2, w^2, p, n} | wa \in C \}.$$  

It can be easily shown that when $C$ is an ideal of $R_{a^2, v^2, w^2, p, n}$, $\text{Res}(C)$ and $\text{Tor}(C)$ both are ideals of $R_{a^2, v^2, w^2, p, n}$. And it is easy to show that $\text{Res}(C) = \text{Im} \phi$ and $\text{Tor}(C) = J$. Now we define eight ideals associated to $C$.

\begin{align*}
(2.2) \quad & C_1 = \text{Res}(\text{Res}(\text{Res}(C))) \\
& = C \mod \langle u, v, w \rangle = \langle f_1(x) \rangle, \\
(2.3) \quad & C_2 = \text{Tor}(\text{Res}(\text{Res}(C))) \\
& = \{ f(x) \in \mathbb{F}_p[x] | uf(x) \in C \mod \langle v, w \rangle \} = \langle f_2(x) \rangle, \\
(2.4) \quad & C_3 = \text{Res}(\text{Tor}(\text{Res}(C))) \\
& = \{ f(x) \in \mathbb{F}_p[x] | vf(x) \in C \mod \langle v, w \rangle \} = \langle f_3(x) \rangle, \\
(2.5) \quad & C_4 = \text{Tor}(\text{Tor}(\text{Res}(C))) \\
& = \{ f(x) \in \mathbb{F}_p[x] | uvf(x) \in C \mod \langle v \rangle \} = \langle f_4(x) \rangle, \\
(2.6) \quad & C_5 = \text{Res}(\text{Res}(\text{Tor}(C))) \\
& = \{ f(x) \in \mathbb{F}_p[x] | w(x) \in C \mod \langle v, w \rangle \} = \langle f_5(x) \rangle, \\
(2.7) \quad & C_6 = \text{Tor}(\text{Res}(\text{Tor}(C))) \\
& = \{ f(x) \in \mathbb{F}_p[x] | uvf(x) \in C \mod \langle v \rangle \} = \langle f_6(x) \rangle, \\
(2.8) \quad & C_7 = \text{Res}(\text{Tor}(\text{Tor}(C))) \\
& = \{ f(x) \in \mathbb{F}_p[x] | w(x) \in C \mod \langle v, w \rangle \} = \langle f_7(x) \rangle, \\
(2.9) \quad & C_8 = \text{Tor}(\text{Tor}(\text{Tor}(C))) \\
& = \{ f(x) \in \mathbb{F}_p[x] | uvf(x) \in C \} = \langle f_8(x) \rangle.
\end{align*}
These are ideals of $\mathbb{F}_p[x]/(x^n - 1)$, hence principal ideals. Throughout this paper we use $C_1, C_2, \ldots, C_8$ for above ideals.

**Theorem 1.** Any ideal $C$ of the ring $R_{u^2, u^2, u^2, p, n}$ is uniquely generated by the polynomials $A_1, A_2, \ldots, A_8$ with $f_{i,j}(x) = 0$ or $\deg(f_{i,j}(x)) < \deg(f_j(x))$, where $A_i, f_i$ and $f_{i,j}$ are defined as above.

**Proof.** Proof is similar to the proof of Theorem 1 [11]. \qed

**Theorem 2.** Let $C = (A_1, A_2, \ldots, A_8)$ be an ideal of the ring $R_{u^2, u^2, u^2, p, n}$. Then we must have

1. $f_i(x)|f_i(x)$ for $1 \leq i \leq 7$; $f_j(x)|f_1(x)|(x^n - 1)$ for $2 \leq j \leq 7$;
2. $f_i(x)|f_2(x); f_4(x)|f_3(x); f_6(x)|f_5(x); f_8(x)|f_7(x); f_7(x)|f_5(x); f_2(x)|f_3(x);$
3. $f_{i+1}(x)|f_{i+1}(x)\left(\frac{x^n-1}{f_i(x)}\right)$ for $1 \leq i \leq 7$;
4. For a fixed $j$, $1 \leq j \leq 7, f_{i+j}(x)|\frac{x^n-1}{f_i(x)} f_{i+j}(x)\left(\frac{x^n-1}{f_{i+j}(x)}\right)$ for $1 \leq i \leq 8 - j$;
5. $f_i(x)|f_{i-1}(x)\left(\frac{x^n-1}{f_i(x)} f_{i-1}(x)\right)$ for $3 \leq i \leq 8$;
6. $f_i(x)|f_{i-1}(x)\left(\frac{x^n-1}{f_i(x)} f_{i-1}(x)\right)$ for $4 \leq i \leq 8$;
7. $f_i(x)|f_{i-1}(x)\left(\frac{x^n-1}{f_i(x)} f_{i-1}(x)\right)$ for $i \in \{5, 6, 7, 8\}$, where
   \[A = \left(\frac{f_{i-1}(x) - f_{i-1}(x)}{f_{i-1}(x)}\right)\]
   \[B = \left(\frac{f_{i-1}(x) - f_{i-1}(x)}{f_{i-1}(x)}\right)\]
8. $f_i(x)|f_{i-1}(x)\left(\frac{x^n-1}{f_i(x)} f_{i-1}(x)\right)$ for $i \in \{6, 7, 8\}$, where
   \[A = \left(\frac{f_{i-1}(x) - f_{i-1}(x)}{f_{i-1}(x)}\right)\]
   \[B = \left(\frac{f_{i-1}(x) - f_{i-1}(x)}{f_{i-1}(x)}\right)\]
9. $f_i(x)|f_{i-1}(x)\left(\frac{x^n-1}{f_i(x)} f_{i-1}(x)\right)$ for $i \in \{7, 8\}$, where
   \[A = \left(\frac{f_{i-1}(x) - f_{i-1}(x)}{f_{i-1}(x)}\right)\]
   \[B = \left(\frac{f_{i-1}(x) - f_{i-1}(x)}{f_{i-1}(x)}\right)\]
Proof. (1) We have uwA2 ∈ C. Therefore, uwf2(x) ∈ C. This gives f2(x) ∈ Cw = ⟨f8(x)⟩. Thus, f8(x)|f2(x). Similarly, if we take uwA3, wA4, uwA5, vA6 and uA7, we get f8(x)|f1(x) for 3 ≤ i ≤ 7.

(2) We have vA2 ∈ C. Therefore, uwf2(x) ∈ C mod w. This gives f2(x) ∈ C4 = ⟨f4(x)⟩. Thus, f4(x)|f2(x). Similarly, if we take uA3, uA5, wA2, vA5, wA3 and take mod by w, uw, vuv, uwv, uwv respectively, we get the other conditions of (2).

\[
D = \left( \frac{f_{i-5,1-3}(x) - A_{f_{i-5,1-3}(x)} - Bf_{i-4,1-3}(x)}{f_{i-3}(x)} \right),
\]
\[
E = \left( \frac{f_{i-5,1-2}(x) - A_{f_{i-5,1-2}(x)} - Bf_{i-4,1-2}(x) - Df_{i-3,1-2}(x)}{f_{i-2}(x)} \right)\quad \text{and}
\]
\[
F = \left( \frac{f_{i-5,1-1}(x) - A_{f_{i-5,1-1}(x)} - Bf_{i-4,1-1}(x) - Df_{i-3,1-1}(x) - Ef_{i-2,1-1}(x)}{f_{i-1}(x)} \right).
\]

\[(10) \quad f_8(x) | \frac{f_{n-i}^{i-1}}{f_{j-1}(x)} f_{1,8}(x) - Af_{2,8}(x) - Bf_{3,8}(x) - Df_{4,8}(x) - Ef_{5,8}(x)
\]
\[\quad - Ff_{6,8}(x) - Gf_{7,8}(x), \quad \text{where}
\]
\[A = \left( \frac{f_{i,2}(x)}{f_{j,2}(x)} \right),
\]
\[B = \left( \frac{f_{i,1}(x)}{f_{j,1}(x)} - A_{f_{i,1}(x)} \right),
\]
\[D = \left( \frac{f_{i,4}(x) - A_{f_{i,4}(x)} - Bf_{i,4}(x)}{f_{j,4}(x)} \right),
\]
\[E = \left( \frac{f_{i,5}(x) - A_{f_{i,5}(x)} - Bf_{i,5}(x) - Df_{i,5}(x)}{f_{j,5}(x)} \right),
\]
\[F = \left( \frac{f_{i,6}(x) - A_{f_{i,6}(x)} - Bf_{i,6}(x) - Df_{i,6}(x) - Ef_{i,6}(x)}{f_{j,6}(x)} \right)\quad \text{and}
\]
\[G = \left( \frac{f_{i,7}(x) - A_{f_{i,7}(x)} - Bf_{i,7}(x) - Df_{i,7}(x) - Ef_{i,7}(x) - Ff_{i,7}(x)}{f_{j,7}(x)} \right).
\]

(11) \( f_i(x) | f_{i-2,1-1}(x) \) for \( i \in \{4, 6, 8\} \);

(12) \( f_i(x) | \frac{f_{i,2}(x) - f_{i-1,1}(x) f_{i-1,1}(x)}{f_{j,2}(x)} \) for \( i \in \{4, 6, 8\} \);

(13) \( f_i(x) | f_{i-1,1-4}(x) - f_{i-2,1}(x) f_{i-1,1}(x) \) for \( i \in \{7, 8\} \);

(14) \( f_i(x) | f_{i-1,1-4}(x) - f_{i-2,1}(x) f_{i-1,1}(x) + f_{i-3,1-5}(x) - f_{i-2,1}(x) f_{i-1,1}(x) \) for \( i \in \{7, 8\} \);

(15) \( f_8(x) | f_{4,5}(x) \) and \( f_7(x) | f_{3,5}(x) \);

(16) \( f_6(x) | f_{2,5}(x) \);

(17) \( f_9(x) | f_{5,6}(x) - f_{j,6}(x) f_{7,8}(x) \);

(18) \( f_9(x) | f_{4,6}(x) - f_{j,6}(x) f_{7,8}(x) \);

(19) \( f_8(x) | f_{5,6}(x) = f_{j,6}(x) f_{7,8}(x) \);

(20) \( f_9(x) | f_{4,6}(x) - f_{j,6}(x) f_{7,8}(x) - A_{f_{6,8}(x)} - Bf_{7,8}(x) \), where

\[A = \left( \frac{f_{i,3}(x) - f_{i,2}(x) f_{5,8}(x)}{f_{j,2}(x)} \right) \quad \text{and} \quad B = \left( \frac{f_{i,3}(x) - f_{i,1}(x) f_{5,8}(x) - (A)f_{6,8}(x)}{f_{j,2}(x)} \right).\]
(3) For $1 \leq i \leq 7$, we have $\frac{z_{i-1}}{f(x)} A_i \in C$. Therefore, $\frac{z_{i-1}}{f(x)} f_{i+1}(x) \in C_{i+1} = (f_{i+1}(x))$. Hence, $f_{i+1}(x) | \frac{z_{i-1}}{f(x)} f_{i+1}(x)$.

(4) For $j = 1$, Condition 4 is reduced to Condition 3. For $j = 2$ and for $1 \leq i \leq 6$, we have $\frac{z_{i-1}}{f(x)} f_{i+1}(x) \in C$. This with Condition 3 gives $\frac{z_{i-1}}{f(x)} f_{i+1}(x) f_{i+2} \in C_{i+2} = (f_{i+2}(x))$. Hence, $f_{i+2}(x) \frac{z_{i-1}}{f(x)} f_{i+1}(x) f_{i+2}$. This proves the condition for $j = 2$. Similarly for other value of $j$ we can prove Condition 4.

(5) For $i = 3$, we have

$$\left( \frac{z_{i-1}}{f(x)} A_1 - \frac{z_{i-1}}{f(x)} f_{j-2}(x) A_2 \right)$$

$$= v \frac{z_{i-1}}{f(x)} (f_1(x) - f_{x+2}(x)) + w \frac{z_{i-1}}{f(x)} (f_{x+4}(x) - \frac{f_2(x)}{f_2(x)} f_{x+4}(x))$$

$$+ w \frac{z_{i-1}}{f(x)} (f_1(x) - f_{x+2}(x)) + w \frac{z_{i-1}}{f(x)} (f_{x+4}(x) - \frac{f_2(x)}{f_2(x)} f_{x+4}(x))$$

$$+ u w \frac{z_{i-1}}{f(x)} (f_1(x) - f_{x+2}(x)) + u w \frac{z_{i-1}}{f(x)} (f_{x+4}(x) - \frac{f_2(x)}{f_2(x)} f_{x+4}(x)) \in C.$$
Since therefore, where, (9) For (8) For Similarly, we get the results for rest of the values of i.

\[ x^n - 1 \left( f_{1.5}(x) - f_{1.2}(x) \frac{f_2(x)}{f_{5}(x)} + \left( f_{1.3}(x) - \frac{f_{2.3}(x)}{f_{3}(x)} f_{2.3}(x) \right) \right) \]

\[ - \frac{x^n}{f_1(x)} \left( f_{1.4}(x) - f_{1.2}(x) \frac{f_2(x)}{f_{5}(x)} + \left( f_{1.3}(x) - \frac{f_{2.3}(x)}{f_{3}(x)} f_{2.3}(x) \right) \right) \frac{f_{2.5}(x)}{f_{4}(x)} \]
\[ \in C_5. \]

Similarly, we get the results for rest of the values of i.

(8) For i = 6, we have \( \left( x^n - 1 \right) A_1 - AA_2 - BA_3 + DA_4 + EA_5 \in C. \)

Since \( uv \left( x^n - 1 \right) f_{1.6}(x) - Af_{2.6}(x) - Bf_{3.6}(x) + Df_{4.6}(x) + Ef_{5.6}(x) \in C \) mod \( vw \), therefore,

\[ x^n - 1 \left( f_{1.6}(x) - \frac{f_{2.6}(x)}{f_{3}(x)} f_{2.6}(x) - Af_{3.6}(x) - Bf_{4.6}(x) - Df_{5.6}(x) \right) \]
\[ \Rightarrow f_6(x) \left( x^n - 1 \right) f_{1.6}(x) - \frac{f_{2.6}(x)}{f_{3}(x)} f_{2.6}(x) - Af_{3.6}(x) - Bf_{4.6}(x) - Df_{5.6}(x) \]

where,

\[ A = \frac{x^n - 1}{f_1(x)} \left( f_{1.2}(x) \frac{f_{1.2}(x)}{f_{2}(x)} f_{2.3}(x) \right), \]

\[ B = \frac{x^n - 1}{f_1(x)} \left( f_{1.3}(x) \frac{f_{1.2}(x)}{f_{2}(x)} f_{2.3}(x) \right), \]

\[ C = \frac{x^n - 1}{f_1(x)} \left( f_{1.4}(x) \frac{f_{1.2}(x)}{f_{2}(x)} f_{2.3}(x) \right), \]

\[ D = \frac{x^n - 1}{f_1(x)} \left( f_{1.5}(x) \frac{f_{1.2}(x)}{f_{2}(x)} f_{2.3}(x) + Ef_{5.6}(x) \right) \]

Similarly, we get the results for rest of the values of i.

(9) For i = 7, we have \( x^n - 1 \left( A_1 - AA_2 - BA_3 - DA_4 - EA_5 - FA_6 \right) \in C. \)

Since \( uvw \left( x^n - 1 \right) f_{1.7}(x) - Af_{2.7}(x) - Bf_{3.7}(x) - Df_{4.7}(x) - Ef_{5.7}(x) - Ff_{6.7}(x) \)
\[ \in C \) mod \( uvw \), therefore,

\[ x^n - 1 \left( f_{1.7}(x) - Af_{2.7}(x) - Bf_{3.7}(x) - Df_{4.7}(x) - Ef_{5.7}(x) - Ff_{6.7}(x) \right) \]
\[ \Rightarrow f_7(x) \left( x^n - 1 \right) f_{1.7}(x) - Af_{2.7}(x) - Bf_{3.7}(x) - Df_{4.7}(x) - Ef_{5.7}(x) - Ff_{6.7}(x) \]

where,

\[ A = \left( f_{1.2}(x) \frac{f_{1.2}(x)}{f_{2}(x)} \right), \]

\[ B = \left( f_{1.3}(x) - Af_{2.3}(x) \right), \]

\[ D = \left( f_{1.4}(x) - Af_{2.4}(x) - Bf_{3.4}(x) \right). \]
\[ E = \left( \frac{f_{1,5}(x) - Af_{2,5}(x) - Bf_{3,5}(x) - Df_{4,5}(x)}{f_{5}(x)} \right) \] and
\[ F = \left( \frac{f_{1,6}(x) - Af_{2,6}(x) - Bf_{3,6}(x) - Df_{4,6}(x) - Ef_{5,6}(x)}{f_{6}(x)} \right). \]

Similarly, we get the results for rest of the values of \( i \).

(10) We have
\[ \frac{z^{n-1}}{f_{1}(x)} \left( A_1 - AA_2 - BA_3 - DA_4 - EA_5 - FA_6 - GA_7 \right) \]
\[ = uvw \frac{z^{n-1}}{f_{1}(x)} * \left( f_{1,8}(x) - Af_{2,8}(x) - Bf_{3,8}(x) - Df_{4,8}(x) - Ef_{5,8}(x) \right. \]
\[ \left. - Ff_{6,8}(x) - Gf_{7,8}(x) \right) \in C. \]

Therefore,
\[ \frac{z^{n-1}}{f_{1}(x)} \left( f_{1,8}(x) - Af_{2,8}(x) - Bf_{3,8}(x) - Df_{4,8}(x) - Ef_{5,8}(x) \right. \]
\[ \left. - Ff_{6,8}(x) - Gf_{7,8}(x) \right) \in C. \]

\[ \Rightarrow f_{8}(x) \frac{z^{n-1}}{f_{1}(x)} * \left( f_{1,8}(x) - Af_{2,8}(x) - Bf_{3,8}(x) - Df_{4,8}(x) - Ef_{5,8}(x) \right. \]
\[ \left. - Ff_{6,8}(x) - Gf_{7,8}(x) \right), \]

where
\[ A = \left( \frac{f_{1,2}(x)}{f_{2}(x)} \right), \]
\[ B = \left( \frac{f_{1,3}(x) - Af_{2,3}(x)}{f_{3}(x)} \right), \]
\[ D = \left( \frac{f_{1,4}(x) - Af_{2,4}(x) - Bf_{3,4}(x)}{f_{4}(x)} \right), \]
\[ E = \left( \frac{f_{1,5}(x) - Af_{2,5}(x) - Bf_{3,5}(x) - Df_{4,5}(x)}{f_{5}(x)} \right), \]
\[ F = \left( \frac{f_{1,6}(x) - Af_{2,6}(x) - Bf_{3,6}(x) - Df_{4,6}(x) - Ef_{5,6}(x)}{f_{6}(x)} \right) \text{ and} \]
\[ G = \left( \frac{f_{1,7}(x) - Af_{2,7}(x) - Bf_{3,7}(x) - Df_{4,7}(x) - Ef_{5,7}(x) - Ff_{6,7}(x)}{f_{7}(x)} \right). \]

(11) For \( i = 4 \), we have \( uA_3 = uvf_{2,3}(x) + uvf_{2,5}(x) + uvf_{2,7}(x) \in C \). Therefore, \( uvf_{2,3}(x) \in C \mod w \Rightarrow f_{2,3}(x) \in C_4 \Rightarrow f_{4}(x)|f_{2,3}(x) \). Similarly we get the results for rest of the values of \( i \).

(12) For \( i = 4 \), we have \( uA_3 = \frac{f_{1}(x)}{f_{5}(x)} A_3 \in C_4 \mod w \). Since \( uvf_{1,2}(x) = f_{2,3}(x)f_{3,4}(x) \)
\[ \in C \mod w, \text{ therefore,} \]
\[ \left( f_{1,2}(x) - \frac{f_{1}(x)}{f_{5}(x)} f_{3,4}(x) \right) \in C_4 \Rightarrow f_{4}(x) \left| \left( f_{1,2}(x) - \frac{f_{1}(x)}{f_{5}(x)} f_{3,4}(x) \right) \right. \]
\[ \left. \right|. \text{ Similarly, we get the results for rest of the values of } i. \]
(13) For i = 7, we have \( wA_2 - \frac{f_2(x)}{f_6(x)} A_6 \in C \). Since \( vw(f_{2,3}(x) - \frac{f_2(x)}{f_6(x)} f_{6,7}(x)) \in C \) mod \( w \), therefore, \( f_7(x) | (f_{2,3}(x) - \frac{f_2(x)}{f_6(x)} f_{6,7}(x)) \). Similarly we get the result for \( i = 8 \).

(14) For \( i = 7 \), we have \( wA_1 - \frac{f_4(x)}{f_6(x)} A_5 - \frac{f_{1,2}(x)}{f_6(x)} f_{5,6}(x) A_6 \in C \). Since
\[
vw(f_{1,3}(x) - \frac{f_4(x)}{f_6(x)} f_{6,7}(x) - \frac{f_{1,2}(x)}{f_6(x)} f_{5,6}(x) f_{6,7}(x)) \in C \text{ mod } uwv,
\]
therefore,
\[
\left( f_{1,3}(x) - \frac{f_4(x)}{f_6(x)} f_{5,7}(x) - \frac{f_{1,2}(x)}{f_6(x)} f_{5,6}(x) f_{6,7}(x) \right) \in C_7
\]
\[
\Rightarrow f_7(x) \left( f_{1,3}(x) - \frac{f_4(x)}{f_6(x)} f_{5,7}(x) - \frac{f_{1,2}(x)}{f_6(x)} f_{5,6}(x) f_{6,7}(x) \right).
\]
Similarly, we get the results for rest of the values of \( i \).

(15) We have \( vA_1 \in C \). Since \( vw f_{4,5}(x) \in C \) mod \( uwv \), therefore, \( f_{4,5}(x) \in C_7 \Rightarrow f_7(x) | f_{4,5}(x) \). Similarly by taking \( vA_3 \) we can show \( f_7(x) | f_{3,5}(x) \).

(16) We have \( uwA_2 = uwv f_{2,5}(x) \in C \). Therefore,
\[
f_{2,5}(x) \in C_8 \Rightarrow f_8(x) | f_{2,5}(x).
\]

(17) We have \( vA_3 - \frac{f_{4,5}(x)}{f_7(x)} A_7 \equiv wuv \left( f_{3,6}(x) - \frac{f_{4,5}(x)}{f_7(x)} f_{7,8}(x) \right) \in C_8 \). Therefore,
\[
\left( f_{3,6}(x) - \frac{f_{4,5}(x)}{f_7(x)} f_{7,8}(x) \right) \in C_8 \Rightarrow f_8(x) | \left( f_{3,6}(x) - \frac{f_{4,5}(x)}{f_7(x)} f_{7,8}(x) \right).
\]

(18) We have \( vA_4 - \frac{f_{4,5}(x)}{f_7(x)} A_7 = wuv \left( f_{4,6}(x) - \frac{f_{4,5}(x)}{f_7(x)} f_{7,8}(x) \right) \in C \). Therefore,
\[
\left( f_{4,6}(x) - \frac{f_{4,5}(x)}{f_7(x)} f_{7,8}(x) \right) \in C_8 \Rightarrow f_8(x) | \left( f_{4,6}(x) - \frac{f_{4,5}(x)}{f_7(x)} f_{7,8}(x) \right).
\]

(19) We have \( vA_5 - \frac{f_5(x)}{f_7(x)} A_7 = wuv \left( f_{5,6}(x) - \frac{f_5(x)}{f_7(x)} f_{7,8}(x) \right) \in C \). Therefore,
\[
\left( f_{5,6}(x) - \frac{f_5(x)}{f_7(x)} f_{7,8}(x) \right) \in C_8 \Rightarrow f_8(x) | \left( f_{5,6}(x) - \frac{f_5(x)}{f_7(x)} f_{7,8}(x) \right).
\]

(20) We have \( wA_1 - \frac{f_1(x)}{f_7(x)} A_5 - AA_6 - BA_7 = wuv \left( f_{1,4}(x) - \frac{f_1(x)}{f_7(x)} f_{5,8}(x) - A f_{6,8}(x) - B f_{7,8}(x) \right) \in C \). Therefore,
\[
\left( f_{1,4}(x) - \frac{f_1(x)}{f_7(x)} f_{5,8}(x) - A f_{6,8}(x) - B f_{7,8}(x) \right) \in C_8
\]
\[
\Rightarrow f_8(x) \left( f_{1,4}(x) - \frac{f_1(x)}{f_7(x)} f_{5,8}(x) - A f_{6,8}(x) - B f_{7,8}(x) \right).
\]
where \( A = \left( \frac{f_{1,2}(x) - f_{1,4}(x) f_{5,8}(x)}{f_6(x)} \right) \) and \( B = \left( \frac{f_{3,3}(x) - f_{3,5}(x) f_{5,7}(x) - A f_{6,7}(x)}{f_6(x)} \right). \) □
Theorem 3. If \( C = (A_1, A_2, \ldots, A_n) \) is a cyclic code over the ring \( \mathbb{F}_p[x]/(x^n - 1) \), then \( C \) is a free cyclic code if and only if \( f_1(x) = f_8(x) \). In this case, we have \( C = (A_1) \) and \( A_1(x^n - 1) \) in \( \mathbb{F}_p[x]/(x^n - 1) \).

Proof. Let \( f_1(x) = f_8(x) \). Since \( f_8(x) = f_1(x) \), we have \( f_8(x)f_1(x) = f_1(x)f_8(x) \). Therefore, we have \( f_1(x) = f_8(x) \). Hence, \( f_1(x) = f_8(x) \). Let \( B_1 = f_1(x) + uf_1 draws,
$C_4 = \text{Tor}(\text{Tor}(C_{u,v})) = \langle a_3(x) \rangle$ (see Page 165 of [8]). Hence, $a_3(x)\frac{x^n-1}{g(x)}\frac{x^n-1}{a_1(x)}f_1(x)$.

Since $n$ is relatively prime to $p$, $x^n-1$ can be uniquely factored as product of distinct irreducible factors. Therefore, we must have $\gcd\left(a_3(x), \frac{x^n-1}{g(x)}\frac{x^n-1}{a_1(x)}\right) = 1$. This gives $a_3(x)|r_1(x)$. But, from Theorem 3.1 of [8], we have $\deg(r_1(x)) < \deg(a_3(x))$. This gives $r_1(x) = 0$. Thus we have proved the following theorem.

**Theorem 4.** Let $C_{u,v}$ be a cyclic code over the ring $R_{u^2,v^2,p}$ of length $n$. If $n$ is relatively prime to $p$, then we have $C_{u,v} = \langle f_1(x) + uf_2(x), vf_3(x) + uf_4(x) \rangle$ with $f_4(x)|f_2(x)|f_1(x)|(x^n - 1)$ and $f_4(x)|f_3(x)|f_1(x)|(x^n - 1)$. We also have the conditions $f_3(x)|f_1(x), f_6(x)|f_2(x)$ and $f_7(x)|f_5(x)$. Since $n$ is relatively prime to $p$, $x^n - 1$ can be uniquely factored as product of distinct irreducible factors. Therefore, we must have $\gcd\left(f_1(x), \frac{x^n-1}{f_2(x)}\right) = 1$ for $1 \leq k \leq j$. This gives $f_{1+k}(x)|f_1(x)$.

From Theorem 1, we have $\deg(f_{1+k}(x)) < \deg(f_1(x))$ for $4 \leq j \leq 7$. This gives $f_{1,1+j}(x) = 0$ for $4 \leq j \leq 7$. Similarly, from Condition 4 of Theorem 2, for $i = 3$ and $2 \leq j \leq 5$, we can show that $f_{3,3+j}(x) = 0$. Thus we have proved the following theorem.

**Theorem 5.** Let $C = \langle A_1, A_2, \ldots, A_k \rangle$ be a cyclic code over the ring $R_{u^2,v^2,w^2,p}$ of length $n$. If $n$ is relatively prime to $p$, then we have $C = \langle f_1(x) + uf_2(x), vf_3(x) + uf_4(x), w(f_5(x) + uf_6(x)), w(vf_7(x) + uf_8(x)) \rangle$ with the conditions:

$f_4(x)|f_2(x)|f_1(x)|(x^n - 1), f_4(x)|f_3(x)|f_1(x), f_5(x)|f_6(x)|f_5(x)|(x^n - 1), f_6(x)|f_7(x)|f_5(x)|f_1(x), f_6(x)|f_2(x)$ and $f_7(x)|f_5(x)$.

**3. Ranks and minimal spanning sets**

We follow Dougherty and Shiromoto [7, page 401] for the definition of the rank of a code $C$. We first prove the number of lemmas that we use to find the rank and the minimal spanning set of cyclic codes over $R_{u^2,v^2,w^2,p}$. 
Lemma 1. Let \( C \) be a cyclic code over the ring \( R_{s^2,u^2, w^2,p} \). If \( C = \langle A_1, A_2, \ldots, A_8 \rangle \), then polynomials in \( C \) in the following forms can be written as follows:

1. \( w(p_0(x) + up_1(x) + vp_2(x) + uv p_3(x)) = q_5(x)A_5 + q_6(x)A_6 + q_7(x)A_7 + q_8(x)A_8 \),
2. \( w(up_1(x) + vp_2(x) + uv p_3(x)) = q_6(x)A_6 + q_7(x)A_7 + q_8(x)A_8 \),
3. \( w(vp_2(x) + uv p_3(x)) = q_7(x)A_7 + q_8(x)A_8 \),
4. \( w(uvp_3(x)) = q_8(x)A_8 \),
for some \( q_i(x) \in F_p[x], 5 \leq i \leq 8 \).

Proof. (1) Let \( A' = w(p_0(x) + up_1(x) + vp_2(x) + uv p_3(x)) \in C \). Thus, \( p_0(x) \in C_5 = \langle f_5(x) \rangle \). This gives \( p_0(x) = q_5(x) f(x) \) for some \( q_5(x) \in F_p[x] \). Therefore, \( A' - q_5(x)A_5 = w((p_1(x) - q_5(x)f_5,6(x)) + u(p_2(x) - q_5(x)f_5,7(x)) + uv(p_3(x) - q_5(x)f_5,8(x))) \in C \). Thus, \( p_1(x) - q_5(x)f_5,6(x) \in C_6 = \langle f_6(x) \rangle \). Therefore, \( (p_1(x) - q_5(x)f_5,6(x)) = q_6(x)f_6(x) \) for some \( q_6(x) \in F_p[x] \). Again,

\[
A' - q_5(x)A_5 - q_6(x)A_6 = w(0) + uv(p_3(x) - q_5(x)f_5,8(x) - q_1(x)f_6,8(x)) \in C \).
\]
Thus, \( (p_2(x) - q_5(x)f_5,7(x) - q_1(x)f_6,7(x)) \in C_7 = \langle f_7(x) \rangle \). Therefore, \( (p_2(x) - q_5(x)f_5,7(x) - q_1(x)f_6,7(x)) = q_7(x)f_7(x) \) for some \( q_7(x) \in F_p[x] \). Again,

\[
A' - q_5(x)A_5 - q_6(x)A_6 - q_7(x)A_7 = w(0) + uv(p_3(x) - q_5(x)f_5,8(x) - q_6(x)f_6,8(x) - q_7(x)f_7,8(x)) \in C \).
\]
Thus, \( (p_3(x) - q_5(x)f_5,8(x) - q_6(x)f_6,8(x) - q_7(x)f_7,8(x)) \in C_8 = \langle f_8(x) \rangle \). Therefore, \( (p_3(x) - q_5(x)f_5,8(x) - q_6(x)f_6,8(x) - q_7(x)f_7,8(x)) = q_8(x)f_8(x) \) for some \( q_8(x) \in F_p[x] \). That is, \( A' - q_5(x)A_5 - q_6(x)A_6 - q_7(x)A_7 - q_8(x)A_8 = 0 \Rightarrow A' = q_5(x)A_5 + q_6(x)A_6 + q_7(x)A_7 + q_8(x)A_8 \). This proves Statement (1). The proof of other cases are similar to the proof of Statement (1).

\( \square \)

Lemma 2. Let \( C \) be a cyclic code over the ring \( R_{s^2,u^2, w^2,p} \). If \( C = \langle A_1, A_2, \ldots, A_8 \rangle \) and \( \deg(f_i(x)) = t_i, 1 \leq i \leq 8 \), then the following conditions hold:

1. \( x^{t_1-t_6} A_8 = c_i u_i A_1 + q_8(x) A_8 \), \( 1 \leq i \leq 7 \), where \( \deg(q_8(x)) < t_4 - t_8 \),
2. \( u_1 = u, u_2 = v, u_3 = u, u_4 = w, u_5 = w, u_6 = w \),
3. \( x^{t_5-t_7} A_7 = c_5 u_5 A_5 + q_5(x) A_7 - q_6(x) A_8 \), where \( \deg(q_5(x)) < t_5 - t_7 \),
4. \( x^{t_6-t_7} A_7 = c_6 u_6 A_6 + q_6(x) A_6 - q_7(x) A_8 \), where \( \deg(q_6(x)) < t_5 - t_7 \),
5. \( x^{t_7-t_6} A_6 = c_7 u_7 A_7 - q_7(x) A_6 - q_7(x) A_7 - q_8(x) A_8 \), where \( \deg(q_7(x)) < t_5 - t_6 \),
6. \( c_3 u_3 A_3 = c_2 u_2 A_2 - q_5(x) A_6 - q_7(x) A_7 - q_8(x) A_8 \), where \( \deg(q_5(x)) < t_2 - t_6 \),
7. \( x^{t_5-t_6} A_6 = c_1 u_1 A_1 - q_5(x) A_6 - q_7(x) A_7 - q_8(x) A_8 \), where \( \deg(q_5(x)) < t_1 - t_5 \),
8. \( x^{t_1-t_5} A_5 = c_1 u_1 A_1 + q_5(x) A_5 + q_6(x) A_6 + q_7(x) A_7 + q_8(x) A_8 \), where \( \deg(q_5(x)) < t_1 - t_5 \),
\( c_i \in F_p \) and \( q_i(x), q'_i(x) \in F_p[x] \).
Proof. (1) From Condition (1) of Theorem 2, we have $f_i(x) | f_i(x)$, $1 \leq i \leq 7$. Thus, $f_i(x) = s_i(x) f_i(x)$ for some $s_i(x) \in \mathbb{F}_p[x]$. This can be written as $f_i(x) = (s_{i0} + x s_{i1} + \cdots + x^{t_i-1} s_{i(t_i-1)}) f_i(x)$, where $s_{ij} \in \mathbb{F}_p$. Clearly $s_{i(t_i-1)} \neq 0$. Therefore, $u s_{i-1} A_i - s_i(x) A_i = u v w f_i(x) - s_i(x) f_i(x) = 0$. This gives $x^{t_i-1} A_i - s_i(x) A_i - s_{i(t_i-1)}(s_{i0} + x s_{i1} + \cdots + x^{t_i-1} s_{i(t_i-1)-1}) A_i x^{t_i-1} A_i - s_i(x) A_i - s_{i(t_i-1)}(s_{i0} + x s_{i1} + \cdots + x^{t_i-1} s_{i(t_i-1)-1}) A_i$. Hence, $x^{t_i-1} A_i = c_i u s_{i-1} A_i + q_i(x) A_i$, where $\deg(q_i(x)) < t_i - t_8$.

(2) From Condition (2) of Theorem 2, we have $f_7(x) | f_7(x)$. Thus, $f_7(x) = s_7(x) f_7(x)$ for some $s_7(x) \in \mathbb{F}_p[x]$. This can be written as $f_7(x) = (s_{70} + x s_{71} + \cdots + x^{t_7-1} s_{7(t_7-1)}) f_7(x)$, where $s_7(x) \in \mathbb{F}_p$. This together with Condition (3) of Lemma 1, we get $v A_5 - s_5(x) A_7 = w(u v f_5,6(x) - s_5(x) f_7,8(x)) = q_5(x) A_8$. Thus,

$$s_5(x) A_7(x) = v A_5 - q_5(x) A_8,$$

This can be written as $x^{t_5-7} A_7 = s^{-1}_{5(t_5-7)} v A_5 - s^{-1}_{5(t_5-7)}(s_{50} + x s_{51} + \cdots + x^{t_5-1} s_{5(t_5-1)-1}) A_7(x) - s^{-1}_{5(t_5-7)}(s_{50} + x s_{51} + \cdots + x^{t_5-1} s_{5(t_5-1)}) A_8(x)$. Thus,

$$x^{t_5-7} A_7 = c_5 v A_5 - q_5^2(x) A_7 - q_5^2(x) A_8,$$

where $\deg(q_5^2(x)) = t_5 - t_7 - 1 < t_5 - t_7$.

(3) The proof is similar to Condition 2.

(4) The proof is similar to Condition 2.

(5) From Condition (2) of Theorem 2, we have $f_6(x) | f_6(x)$. Thus, $f_6(x) = s_6(x) f_6(x)$ for some $s_6(x) \in \mathbb{F}_p[x]$. This can be written as $f_6(x) = (s_{60} + x s_{61} + \cdots + s_{6(t_6-1)}) f_6(x)$, where $s_6(x) \in \mathbb{F}_p$. This together with Condition (3) of Lemma 1, we get $u A_5 - s_5(x) A_6 = w(u v f_5,6(x) - s_5(x) f_7,8(x)) = q_6(x) A_8$. Thus,

$$s_6(x) A_6(x) = u A_5 - q_7(x) A_7 - q_6(x) A_8,$$

This can be written as $x^{t_5-6} A_6 = s^{-1}_{6(t_5-6)} u A_5 - s^{-1}_{6(t_5-6)}(s_{60} + x s_{61} + \cdots + x^{t_5-1} s_{6(t_5-1)-1}) A_6 - s^{-1}_{6(t_5-6)} q_7(x) A_7 - s^{-1}_{6(t_5-6)} q_6(x) A_8$, i.e.,

$$x^{t_5-6} A_6 = c_6 u A_5 - q_6^2(x) A_6 - q_6^2(x) A_7 - q_6^2(x) A_8,$$

where $\deg(q_6^2(x)) = t_5 - t_6 - 1 < t_5 - t_6$.

(6) The proof is similar to Condition 5.

(7) The proof is similar to Condition 5.

(8) By the division algorithm, we have

$$x^{t_1-5} (f_5(x) + u f_5,6(x) + v f_5,7(x) + u v f_5,8(x)) = c_1 A_1 + (p_0(x) + up_1(x) + vp_2(x) + wp_3(x) + wp_4(x) + uw p_5(x) + vwp_6(x) + u v w p_7(x)),$$

where $\deg(p_0(x)) < \deg(f_1(x)) = t_1$. Multiplying Eq. (3.5) by $w$ and applying Condition 1 of Lemma 1 gives $x^{t_1-5} A_5 - c_1 w A_1 = w(p_0(x) + up_1(x) + vp_2(x) + wp_3(x)) = q_5(x) A_5 + q_6(x) A_6 + q_7(x) A_7 + q_8(x) A_8$. That is,

$$x^{t_1-5} A_5 = c_1 w A_1 + q_5(x) A_5 + q_6(x) A_6 + q_7(x) A_7 + q_8(x) A_8.$$
We have $p_0(x) = q_5(x)f_5(x)$, thus, $\deg(q_5(x)) + \deg(f_5(x)) = \deg(p_0(x)) < t_1$. Hence, $\deg(q_5(x)) < t_1 - t_5$.

**Theorem 6.** Let $C$ be a cyclic code of length $n$ over $\mathbb{F}_2^n$. If $C = \langle A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8 \rangle$ with $t_2 = \deg(f_i(x)), 1 \leq i \leq 8, t'_4 = \min\{t_2, t_3\}, t'_6 = \min\{t_2, t_3\}, t'_7 = \min\{t_4, t_5, t_6\}$ and $t'_8 = \min\{t_4, t_5, t_6\}$, then $C$ has rank $n + 2t_4 + t'_4 + t'_6 + t'_7 + t'_8 - t_2 - t_3 - t_4 - t_5 - t_6 - t_7 - t_8$. The minimal spanning set $B$ of the code $C$ is $B = \{A_1, xA_1, \ldots, x^{n-t_1-1}A_1, A_2, xA_2, \ldots, x^{t_1-t_2-1}A_2, A_3, xA_3, \ldots, x^{t_1-t_2-1}A_3, A_4, xA_4, \ldots, x^{t_1-t_2-1}A_4, A_5, xA_5, \ldots, x^{n-t_2-1}A_5, A_6, xA_6, \ldots, x^{n-t_2-1}A_6, A_7, xA_7, \ldots, x^{t_2-t_3-1}A_7, A_8, xA_8, \ldots, x^{t_2-t_3-1}A_8 \}$.

**Proof.** It is sufficient to show that $B$ spans the set $B' = \{A_1, xA_1, \ldots, x^{n-t_1-1}A_1, A_2, xA_2, \ldots, x^{n-t_2-1}A_2, A_3, xA_3, \ldots, x^{n-t_2-1}A_3, A_4, xA_4, \ldots, x^{n-t_2-1}A_4, A_5, xA_5, \ldots, x^{n-t_2-1}A_5, A_6, xA_6, \ldots, x^{n-t_2-1}A_6, A_7, xA_7, \ldots, x^{n-t_3-1}A_7, A_8, xA_8, \ldots, x^{n-t_3-1}A_8 \}$. To show $B$ spans $B'$, we write the set $B'$ as $B' = B_1 \cup B_2$, where $B_1 = \{A_1, xA_1, \ldots, x^{n-t_1-1}A_1, A_2, xA_2, \ldots, x^{n-t_2-1}A_2, A_3, xA_3, \ldots, x^{n-t_2-1}A_3, A_4, xA_4, \ldots, x^{n-t_2-1}A_4 \}$ and $B_2 = \{A_5, xA_5, \ldots, x^{n-t_2-1}A_5, A_6, xA_6, \ldots, x^{n-t_2-1}A_6, A_7, xA_7, \ldots, x^{n-t_3-1}A_7, A_8, xA_8, \ldots, x^{n-t_3-1}A_8 \}$. We first show that $B$ spans $B_2$ and then we show that $B$ spans $B_1$. To show $B$ spans $B_2$ we divide the proof in twelve cases.

**Case (1).** Let $t'_5 = t_7, t'_4 = t_5$ and $t'_6 = t_5$. We first show that the element $x^{t_2-t_3}A_8 \in B_2 - B$ is linear combinations of some elements of $B$ and then we show that other elements of the set $B_2 - B$ are linear combinations of elements of $B$. From Statement 1 of Lemma 2,

$$x^{t_2-t_3}A_8 = c_7uA_7 + q_5(x)A_8,$$

where $\deg(q_5(x)) < t_7 - t_8$. Therefore, $x^{t_2-t_3}A_8 \in \text{Span}(B)$. Multiplying Equation (3.7) by $x$, $x^2$, $x^3$, ... $x^{t_2-t_3-1}$ and then putting the value of $x^{t_2-t_3}A_8$ in the equation obtained, we can show that $x^{t_2-t_3+1}A_8, x^{t_2-t_3+2}A_8, \ldots, x^{t_2-t_3-1}A_8 \in \text{Span}(B)$. From Statement 1 of Lemma 2, we have

$$x^{t_2-t_3}A_8 = c_5uvA_5 + q_5(x)A_8,$$

where $\deg(q_5(x)) < t_5 - t_8$. Therefore, $x^{t_2-t_3}A_8 \in \text{Span}(B)$. Arguing as above, we can show that the terms $x^{t_2-t_3+1}A_8, x^{t_2-t_3+2}A_8, \ldots, x^{t_2-t_3-1}A_8 \in \text{Span}(B)$. Again, from Statement 1 of Lemma 2, we have

$$x^{t_1-t_3}A_8 = c_5uweA_5 + q_5(x)A_8,$$

where $\deg(q_5(x)) < t_1 - t_8$. As above, we can show that $x^{t_1-t_3}A_8, x^{t_1-t_3+1}A_8, \ldots, x^{t_1-t_3-1}A_8 \in \text{Span}(B)$. Now we show that $x^{t_2-t_3}A_7 \in \text{Span}(B)$. From Statement 2 of Lemma 2, we have

$$x^{t_2-t_3}A_7 = c_7vA_5 - q_5'(x)A_7 - q_5'(x)A_8,$$

where $\deg(q_5'(x)) < (t_5 - t_7)$. In the above discussion, we have shown that $x^iA_8 \in \text{Span}(B)$ for $1 \leq i \leq n - t_8 - 1$. Clearly, $q_5'(x)A_8 \in \text{Span}(B)$. And, also the term $c_7vA_5, q_5'(x)A_7 \in \text{Span}(B)$ (since $\deg(q_5'(x)) < (t_5 - t_7)$). Therefore,
\(x^{t_5-t_7}A_7 \in \text{Span}(B)\). As above, after putting the value of \(x^{t_5-t_7}A_7\) in the equation obtained by multiplying Equation (3.10) by \(x, x^2, \ldots, x^{t_1-t_5-1}\) successively, we can show that \(x^{t_5-t_7+1}A_7, x^{t_5-t_7+2}A_7, \ldots, x^{t_5-t_7-1}A_7 \in \text{Span}(B)\).

From Statement 4 of Lemma 2, we have

\[
(3.11) \quad x^{t_1-t_7}A_7 = c_1 u w A_1 - q'_6(x)A_7 - q'_6(x)A_8,
\]

where \(\deg(q'_6(x)) < t_1 - t_7\). As above, we can show that \(x^{t_1-t_7}A_7, x^{t_1-t_5+1}A_7, \ldots, x^{n-t_5-1}A_7 \in \text{Span}(B)\). Now we show that \(x^{t_5-t_6}A_6 \in \text{Span}(B)\). From Statement 5 of Lemma 2, we have

\[
(3.12) \quad x^{t_5-t_6}A_6 = c_5 w A_5 - q'_6(x)A_6 - q'_6(x)A_7 - q'_6(x)A_8,
\]

where \(\deg(q'_6(x)) < (t_5 - t_6)\). In the above discussion, we have shown that \(x^{t_5-t_6}A_6 \in \text{Span}(B)\) for \(1 \leq i \leq n - t_5 - 1\). Clearly, \(q'_7(x)A_7, q'_6(x)A_8 \in \text{Span}(B)\). Therefore, \(x^{t_5-t_6}A_6 \in \text{Span}(B)\).

In a similar way, as above, after putting the value of \(x^{t_5-t_6}A_6\) in the equation obtained by multiplying Equation (3.12) by \(x, x^2, x^3, \ldots, x^{t_1-t_5-1}\) successively, we can show that \(x^{t_5-t_6+1}A_6, x^{t_5-t_6+2}A_6, \ldots, x^{t_5-t_6-1}A_6 \in \text{Span}(B)\). From Statement 7 of Lemma 2, we have

\[
(3.13) \quad x^{t_1-t_5}A_5 = c_1 u w A_1 - q'_6(x)A_5 - q'_6(x)A_6 - q'_6(x)A_8,
\]

where \(\deg(q'_6(x)) < t_1 - t_5\). Again as above, we can show that \(x^{t_1-t_5+1}A_5, x^{t_1-t_5+2}A_6, \ldots, x^{n-t_5-1}A_6 \in \text{Span}(B)\). Now we show that the next term \(x^{t_1-t_5}A_5 \in \text{Span}(B)\). From Statement 8 of Lemma 2, we have

\[
(3.14) \quad x^{t_1-t_5}A_5 = c_1 u w A_1 + q_5(x)A_5 + q_6(x)A_6 + q_7(x)A_7 + q_8(x)A_8,
\]

where \(\deg(q_5(x)) < t_1 - t_5\). In the above discussion, we have shown that \(x^{t_1-t_5}A_5 \in \text{Span}(B)\) for \(1 \leq i \leq n - t_5 - 1\). Clearly, \(q_5(x)A_6, q_7(x)A_7, q_8(x)A_8 \in \text{Span}(B)\). Therefore, \(x^{t_1-t_5}A_5 \in \text{Span}(B)\) (since \(\deg(q_8(x)) < t_1 - t_5\)).

Multiplying Equation (3.14) by \(x, x^2, x^3, \ldots, x^{n-t_5-1}\) and then putting the value of \(x^{t_1-t_5}A_5\) in the equation obtained, we can show that the terms \(x^{t_1-t_5+1}A_5, x^{t_1-t_5+2}A_5, \ldots, x^{n-t_5-1}A_5 \in \text{Span}(B)\).

**Case (2A).** Let \(t'_5 = t_4, t'_7 = t_3\) and \(t'_6 = t_2\). Let \(t'_4 = t_3\). As in Case 1, by using Statement 1 of Lemma 2 for \(i = 4, 3, 1\) successively, we can show that \(x^{t_4-t_5}A_8, x^{t_4-t_5+1}A_8, \ldots, x^{n-t_5-1}A_8 \in \text{Span}(B)\). Similarly, as in Case 1, by using Statements 3 and 4 of Lemma 2 successively, we can show that \(x^{t_3-t_7}A_7, x^{t_3-t_7+1}A_7, \ldots, x^{n-t_7-1}A_7 \in \text{Span}(B)\). Again, as in Case 1, by using Statement 6 and then Statement 7 of Lemma 2 successively, we can show that \(x^{t_2-t_6}A_6, x^{t_2-t_6+1}A_6, \ldots, x^{n-t_6-1}A_6 \in \text{Span}(B)\). In a similar fashion, as in Case 1, by using Statement 8 of Lemma 2, we can show that \(x^{t_1-t_5}A_5, x^{t_1-t_5+1}A_5, \ldots, x^{n-t_5-1}A_5 \in \text{Span}(B)\).

**Case (2B).** Let \(t'_5 = t_4, t'_7 = t_3, t'_6 = t_2\) and \(t'_4 = t_2\). As in Case 1, by using Statement 1 of Lemma 2 for \(i = 4, 2, 1\) successively, we can show that \(x^{t_4-t_5}A_8, x^{t_4-t_5+1}A_8, \ldots, x^{n-t_5-1}A_8 \in \text{Span}(B)\). Similarly, as in Case 1, by using Statements 3, 4, 6, 7 and then 8 of Lemma 2 successively, we can show that the rest of elements belongs to \(\text{Span}(B)\).
Case (3). Let \( t'_5 = t_4, t'_7 = t_3 \) and \( t'_6 = t_5 \). As in Case 1, by using Statement 1 of Lemma 2 for \( i = 6, 5 \) and 1, successively, we can show that
\[ x^{t'_5-t_4}A_8, x^{t'_5-t_6+1}A_8, \ldots, x^{t'_1-t_4}A_8 \in \text{Span}(B). \]
Similarly, as in Case 1, by using Statements 3 and 4 of Lemma 2, successively, we can show that
\[ x^{t'_7-t_3}A_7, x^{t'_7-t_5+1}A_7, \ldots, x^{t'_1-t_3}A_7 \in \text{Span}(B). \]
Again, as in Case 1, by using Statement 5 and then Statement 7 of Lemma 2, successively, we can show that
\[ x^{t'_6-t_5}A_6, x^{t'_6-t_7+1}A_6, \ldots, x^{t'_1-t_5}A_6 \in \text{Span}(B). \]
In a similar fashion, as in Case 1, by using Statement 8 of Lemma 2, we can show that
\[ x^{t'_1-t_5}A_5, x^{t'_1-t_6+1}A_5, \ldots, x^{t'_1-t_6}A_5 \in \text{Span}(B). \]

The remaining cases are as follows:

Case (4). If \( t'_8 = t_7, t'_6 = t_3 \) and \( t'_5 = t_5 \).

Case (5). If \( t'_8 = t_7, t'_6 = t_3 \) and \( t'_5 = t_2 \).

Case (6). If \( t'_8 = t_7, t'_7 = t_5 \) and \( t'_5 = t_5 \).

Case (7). If \( t'_8 = t_6, t'_7 = t_3 \) and \( t'_5 = t_2 \).

Case (8). If \( t'_8 = t_6, t'_7 = t_5 \) and \( t'_5 = t_2 \).

Case (9). If \( t'_8 = t_6, t'_7 = t_5 \) and \( t'_5 = t_5 \).

Case (10). If \( t'_8 = t_4, t'_7 = t_5 \) and \( t'_5 = t_5 \), \((10A)\): \( t'_4 = t_3 \), \((10B)\): \( t'_4 = t_2 \).

Case (11). If \( t'_8 = t_4, t'_7 = t_3 \) and \( t'_5 = t_5 \), \((11A)\): \( t'_4 = t_3 \), \((11B)\): \( t'_4 = t_2 \).

Case (12). If \( t'_8 = t_4, t'_7 = t_5 \) and \( t'_5 = t_2 \), \((12A)\): \( t'_4 = t_3 \), \((12B)\): \( t'_4 = t_2 \).

In a similar way as above, by using Lemma 2, we can show that \( B \) spans \( B_1 \) in these cases.

Now we show that \( B \) spans \( B_1 \). From Equation (2.1), we have a homomorphism \( \phi : C \to R_{u,v,p,n} \). Therefore, \( C/\ker \phi \simeq \phi(C) \) and \( \phi(C) \) is a cyclic code over the ring \( R_{u,v,p} \). Thus, we have \( C/\ker \phi \) as a cyclic code over \( R_{u,v,p} \). Therefore, from Theorem 4.1 of [8], the minimal spanning set \( B_0 \) of the code \( C/\ker \phi \) is \( \{A_1 + \ker \phi, A_4 + \ker \phi, \ldots, x^{t'_1-t_4-1}A_1 + \ker \phi, A_2 + \ker \phi, A_3 + \ker \phi, x^2 + \ker \phi, \ldots, x^{t'_1-t_2-1}A_2 + \ker \phi, A_3 + \ker \phi, x^3 + \ker \phi, \ldots, x^{t'_1-t_3-1}A_3 + \ker \phi, A_4 + \ker \phi, x^4 + \ker \phi, \ldots, x^{t'_1-t_4-1}A_4 + \ker \phi \} \). To show \( B \) spans \( B_1 \), we only show that \( x^{t'_1-t_2}A_2 \in \text{Span}(B) \). In a similar way, we can show that \( x^{t'_1-t_2+1}A_2, \ldots, x^{n-t_2-1}A_2, x^{t'_1-t_2}A_4, \ldots, x^{n-t_4-1}A_4 \in \text{Span}(B) \). Since \( B_0 \) spans \( C/\ker \phi \), we can write \( x^{t'_1-t_2}A_2 + \ker \phi \) as a \( R_{u,v,p} \) linear combination of the elements of \( B_0 \), i.e.,
\[ x^{t'_1-t_2}A_2 + \ker \phi = \sum_{i=0}^{n-t_4-1} \alpha_i(x^iA_1 + \ker \phi) + \cdots + \sum_{i=0}^{t'_1-t_4-1} \alpha_i(x^iA_4 + \ker \phi), \]
where \( \alpha_i \in R_{u,v,p} \). Thus, \( x^{t'_1-t_2}A_2 = \left( \sum_{i=0}^{n-t_4-1} \alpha_i(x^iA_1) + \cdots + \sum_{i=0}^{t'_1-t_4-1} \alpha_i(x^iA_4) \right) \in \ker \phi \). Since \( \ker \phi = \text{Span}(B_2) \) and \( B \) spans \( B_2 \), we get \( x^{t'_1-t_2}A_2 \in \text{Span}(B) \). Similarly, we can show that \( x^{t'_1-t_2+1}A_2, \ldots, x^{n-t_4-1}A_4 \in \text{Span}(B) \). This shows that \( B \) spans \( B_1 \). It is easy to see that any elements of the spanning set \( B \) can not be written as the linear combination of its preceding elements and other elements in the spanning set \( B \). Here we only show that \( x^{t'_1-t_2}A_2 \) can not be written as linear combinations of others element of spanning set \( B \). The proof is similar for the rest. Suppose, if possible \( x^{t'_1-t_2}A_2 \) can be written as
linear combinations of the other element of the spanning set $B$. Then we have

$$x^{t_1-t_3-1} A_3 = \sum_{i=0}^{n-t_1-1} \alpha_{1i} x^i A_1 + \sum_{i=0}^{t_1-t_2-1} \alpha_{2i} x^i A_2 + \sum_{i=0}^{t_1-t_3-2} \alpha_{3i} x^i A_3$$

$$+ \sum_{i=0}^{t_1-t_4-1} \alpha_{4i} x^i A_4 + \sum_{i=0}^{t_1-t_5-1} \alpha_{5i} x^i A_5 + \sum_{i=0}^{t_1-t_6-1} \alpha_{6i} x^i A_6$$

$$+ \sum_{i=0}^{t_1-t_7-1} \alpha_{7i} x^i A_7 + \sum_{i=0}^{t_1-t_8-1} \alpha_{8i} x^i A_8,$$

where $\alpha_{ji} = \beta_{ji}^{(i)} + uf_{j2}^{(i)} + uv\beta_{j3}^{(i)} + uw\beta_{j4}^{(i)} + v\beta_{j5}^{(i)} + vw\beta_{j6}^{(i)} + uvw\beta_{j7}^{(i)} + uvw\beta_{j8}^{(i)} \in \mathbb{F}_p$.

(Note that $i$ is not a power of $\beta$ it is a notation.) We have

$$x^{t_1-t_3-1} (vf_3(x) + uvf_{3,4}(x) + wf_{3,5}(x) + uvw_{3,6}(x) + vwf_{3,7}(x) + uvw_{3,8}(x))$$

$$= f_1(x) \sum_{i=0}^{n-t_1-1} \beta_{1i}^{(i)} x^i + uf_1(x) \sum_{i=0}^{n-t_2-1} \beta_{12}^{(i)} x^i + uf_1(x) \sum_{i=0}^{n-t_3-1} \beta_{13}^{(i)} x^i$$

$$+ uf_2(x) \sum_{i=0}^{t_1-t_2-2} \beta_{21}^{(i)} x^i + v f_1(x) \sum_{i=0}^{n-t_1-1} \beta_{23}^{(i)} x^i + v f_1(x) \sum_{i=0}^{t_1-t_3-2} \beta_{24}^{(i)} x^i$$

$$+ v f_3(x) \sum_{i=0}^{t_1-t_2-1} \beta_{31}^{(i)} x^i + v f_3(x) \sum_{i=0}^{t_1-t_3-1} \beta_{32}^{(i)} x^i$$

$$+ uwm_2(x) + wm_3(x) + uw(m_4(x) + uvwm_5(x) + uvwm_6(x)),$$

where $m_2(x), \ldots, m_6(x)$ are polynomials in $\mathbb{F}_p[x]$. By comparing both sides, we have $\beta_{11}^{(i)} = 0$, $\beta_{12}^{(i)} = 0$ for $0 \leq i \leq n-t_1-1$, $\beta_{21}^{(i)} = 0$ for $0 \leq i \leq t_1 - t_2 - 1$, and $x^{t_1-t_3-1} f_3(x) = f_1(x) \sum_{i=0}^{n-t_1-1} \beta_{13}^{(i)} x^i + f_3(x) \sum_{i=0}^{n-t_1-1} \beta_{23}^{(i)} x^i$.

Note that $\deg(x^{t_1-t_3-1} f_3(x)) = t_1 - 1$ but $\deg(f_1(x) \sum_{i=0}^{n-t_1-1} \beta_{13}^{(i)} x^i) \geq t_1$ and $\deg(f_3(x) \sum_{i=0}^{n-t_1-1} \beta_{23}^{(i)} x^i) \leq t_1 - 2$. Hence, this gives a contradiction.

**Theorem 7.** Let $n$ be a positive integer relatively prime to $p$ and $C$ be a cyclic code of length $n$ over the ring $R_{p^2, x, x^2}$. If $C = \langle f_1(x) + uf_2(x), v f_1(x) + uvf_2(x), w f_1(x) + uf_2(x), w(v f_2(x) + uvf_2(x)) \rangle$ with $t_i = \deg(f_i(x))$, $1 \leq i \leq 8$, and $t_7 = \min\{t_3, t_5\}$, then $C$ has rank $n + t_1 + t_2 - t_3 - t_5 - t_7$. The minimal spanning set $B$ of the code $C$ is $B = \{f_1(x) + uf_2(x), x f_1(x) + uf_2(x), \ldots, x^{t_1-t_3-1} f_1(x) + uf_2(x), v f_3(x) + uvf_4(x), x(v f_4(x) + uvf_4(x)), \ldots, x^{t_1-t_3-1} v f_3(x) + uvf_4(x), w f_3(x) + uf_5(x), x w f_3(x) + uvf_4(x)), \ldots, x^{t_1-t_3-1} w f_3(x) + uvf_4(x), v f_4(x) + uvf_4(x), w f_4(x) + uvf_4(x), \ldots, x^{t_1-t_3-1} w f_4(x) + uvf_4(x))\}$.

**Proof.** The proof is similar to the above theorem. \qed
4. Minimum distance

Let $n$ be a positive integer not relatively prime to $p$. Let $C$ be a cyclic code of length $n$ over $R_{u^2,v^2,w^2,p}$. From Eq. (2.9), we have \( C_8 = \{ f(x) \in F_p[x] \mid uvwf(x) \in C \} = \{ f(x) \} \). Also, we know that $C_8$ is a cyclic code over $F_p$.

**Theorem 8.** Let $n$ be a positive integer not relatively prime to $p$. If $C = (A_1, A_2, \ldots, A_8)$ is a cyclic code of length $n$ over $R_{u^2,v^2,w^2,p}$, then $w_H(C) = w_H(C_8)$.

**Proof.** Let $M(x) = m_0(x) + \beta m_1(x) + \alpha m_2(x) + \gamma m_3(x) + \delta m_4(x) + \mu m_5(x) + \nu m_6(x) + \upsilon m_7(x) \in C$, where $m_0(x), m_1(x), \ldots, m_7(x) \in F_p[x]$. We have $uvwM(x) = uvwm_0(x), wH(uvwM(x)) \leq wH(M(x))$ and $uvwC$ is subcode of $C$ with $w_H(uvwC) \leq w_H(C)$. Thus, $w_H(uvwC) = w_H(C)$. Therefore, it is sufficient to focus on the subcode $uvwC$ in order to prove the theorem. Since $w_H(C_8) = w_H(uvwC)$, we get $w_H(C) = w_H(C_8)$. \( \square \)

**Definition.** Let $m = b_{l-1}p^{l-1} + b_{l-2}p^{l-2} + \cdots + b_1p + b_0, b_i \in F_p, 0 \leq i \leq l-1$, be the $p$-adic expansion of $m$.

1. If $b_{l-i} \neq 0$ for all $1 \leq i \leq q, q < l$, and $b_{l-i} = 0$ for all $i, q + 1 \leq i \leq l$, then $m$ is said to have a $p$-adic length $q$ zero expansion.
2. If $b_{l-i} \neq 0$ for all $1 \leq i \leq q, q < l$, $b_{l-q-1} = 0$ and $b_{l-i} \neq 0$ for some $i, q + 2 \leq i \leq l$, then $m$ is said to have $p$-adic length $q$ non-zero expansion.
3. If $b_{l-i} \neq 0$ for $1 \leq i \leq l$, then $m$ is said to have a $p$-adic length $l$ expansion or $p$-adic full expansion.

**Lemma 3.** Let $C$ be a cyclic code over $R_{u^2,v^2,w^2,p}$ of length $p^l$, where $l$ is a positive integer. Let $C = (f(x))$ where $f(x) = (x^p)^{l-1} - h(x), 1 \leq b < p$. If $h(x)$ generates a cyclic code of length $p^{l-1}$ and Hamming distance $d$, then the Hamming distance $d(C)$ of $C$ is $(b + 1)d$.

**Proof.** For $c \in C$, we have $c = (x^p)^{l-1}h(x)m(x)$ for some $m(x) \in \frac{R_{u^2,v^2,w^2,p}[x]}{(x^p-1)}$. Since $h(x)$ generates a cyclic code of length $p^{l-1}$, we have

\[
\begin{align*}
w(c) &= w((x^p)^{l-1}h(x)m(x)) \\
&= w(x^{(p-1)(l-1)}h(x)m(x)) + w(b^lC_0x^{p-1}(b-1)h(x)m(x)) + \cdots \\
&\quad + w(b^{l-1}C_{l-1}x^{p-1}h(x)m(x)) + w(h(x)m(x)).
\end{align*}
\]

Thus, $d(C) = (b + 1)d$. \( \square \)

**Theorem 9.** Let $C$ be a cyclic code over $R_{u^2,v^2,w^2,p}$ of length $p^l$, where $l$ is a positive integer. Then, $C = (A_1, A_2, \ldots, A_8)$ where $f_1(x) = (x-1)^{t_1}, f_2(x) = (x-1)^{t_1}, \ldots, f_8(x) = (x-1)^{t_8}$ for some $t_1 > t_2, t_3 > t_4 > t_7 > 0, t_2 > t_6, t_3 > t_7$ and $t_A > t_5 > t_6, t_7 > t_8 > 0$. 


By Lemma 3, the subcode generated by \( f \) has a p-adic length q zero expansion or full expansion (l = q), then \( d(C) = \langle b_{l-1} + 1 \rangle \cdots \langle b_{l-q} + 1 \rangle \).

(b) If \( t_s \) has a p-adic length q non-zero expansion, then \( d(C) = 2(b_{l-1} + 1) \cdots (b_{l-q} + 1) \).

Proof. The first claim easily follows from Theorem 2. From Theorem 8, we see that \( d(C) = d(C_b) = d((x-1)^{t_s}) \). Hence, we only need to determine the minimum weight of \( C_b = \langle (x-1)^{t_s} \rangle \).

(1) If \( t_s \leq p^{l-1} \), then \( (x-1)^{t_s} = (x-1)^{p^{l-1}} - 1 \in C \).

Thus, \( d(C) = 2 \).

(2) Let \( t_s > p^{l-1} \). (a) If \( t_s \) has a p-adic length q zero expansion, we have \( t_s = b_{l-1}p^{l-1} + b_{l-2}p^{l-2} + \cdots + b_1 p + b_0 \), and \( f_s(x) = (x-1)^{t_s} = (x^{p^{l-1}} - 1)^{b_{l-1}}(x^{p^{l-2}} - 1)^{b_{l-2}} \cdots (x^{p^0} - 1)^{b_0} \). Let \( h(x) = (x^{p^{l-1}} - 1)^{b_{l-1}} \). Then \( h(x) \) generates a cyclic code of length \( p^{l-q+1} \) and minimum distance \( b_{l-q+1} \). By Lemma 3, the subcode generated by \( (x^{p^{l-q+1}} - 1)^{b_{l-q+1}} h(x) \) has minimum distance \( b_{l-q+1} + 1 \). By induction on \( q \), we can see that the code generated by \( f_s(x) \) has minimum distance \( b_{l-1} + 1(b_{l-2} + 1) \cdots (b_{l-q} + 1) \).

Thus, \( d(C) = (b_{l-1} + 1)(b_{l-2} + 1) \cdots (b_{l-q} + 1) \).

(b) If \( t_s \) has a p-adic length q non-zero expansion, we have \( t_s = b_{l-1}p^{l-1} + b_{l-2}p^{l-2} + \cdots + b_1 p + b_0, b_{l-q-1} = 0 \).

Let \( r = b_{l-1}p^{l-1} + b_{l-2}p^{l-2} + \cdots + b_1 p + b_0 \) and \( h(x) = (x-1)^r = (x^{p^{l-q-1}} - 1)^{b_{l-1}}(x^{p^{l-q-2}} - 1)^{b_{l-2}} \cdots (x^{p^0} - 1)^{b_0} \). Since \( r < p^{l-q-1} \), we have \( p^{l-q-1} = r + j \) for some non-zero \( j \). Thus, \( (x-1)^{p^{l-q-1}} h(x) = (x^{p^{l-q-1}} - 1) \in C \). Hence, the subcode generated by \( h(x) \) has minimum distance 2. By Lemma 3, the subcode generated by \( (x^{p^{l-q-1}} - 1)^{b_{l-q-1}} h(x) \) has minimum distance \( 2(b_{l-q} + 1) \). By induction on \( q \), we can see that the code generated by \( f_s(x) \) has minimum distance \( 2(b_{l-1} + 1)(b_{l-2} + 1) \cdots (b_{l-q} + 1) \). Thus, \( d(C) = 2(b_{l-1} + 1)(b_{l-2} + 1) \cdots (b_{l-q} + 1) \). \( \square \)

Let \( C \) be a cyclic code of length \( n \) over the ring \( R_{a_2, a_1, a_2, p} \). Let \( d_L \) be the Lee distance of a code \( C \). By Theorem 1 of [9], we have

\[
\left\lfloor \frac{d_L - 1}{8} \right\rfloor \leq n - \log_p |C|.
\]

If \( C \) is a free code, then we get

\[
\left\lfloor \frac{d_L - 1}{8} \right\rfloor \leq n - \text{Rank}(C).
\]
5. The Gray map

Let $w_L$ and $w_H$ denote the Lee weight and the Hamming weight respectively. We define the Lee weight as follows:

$$w_L(\alpha) = w_H(\phi_L(\alpha)) \text{ for all } \alpha \in R_{u^2,v^2,w^2, p},$$

where the Gray map $\phi_L : R_{u^2,v^2,w^2, p} \rightarrow \mathbb{F}_p^8$ is defined as follows:

$$\phi_L(\alpha_1 + \omega \alpha_2 + v \alpha_3 + w \alpha_4 + w \alpha_5 + w \alpha_6 + w^2 \alpha_7 + w^2 \alpha_8) = (\alpha_8, \alpha_6 + \alpha_8, \alpha_7 + \alpha_8, \alpha_4 + \alpha_8, \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_2 + \alpha_4 + \alpha_6 + \alpha_8, \alpha_3 + \alpha_4 + \alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7 + \alpha_8).$$

The Gray map naturally extend to $R_{u^2,v^2,w^2, p}^n$ as distance preserving isometry $\phi_L : (R_{u^2,v^2,w^2, p}^n, \text{ Lee weight}) \rightarrow (\mathbb{F}_p^n, \text{ Hamming weight})$ as follows

$$\phi_L(\alpha_1, \alpha_2, \ldots, \alpha_n) \rightarrow (\phi_L(\alpha_1), \phi_L(\alpha_2), \ldots, \phi_L(\alpha_n)), \forall \alpha_i \in R_{u^2,v^2,w^2, p}.$$

By linearity of the map $\phi_L$ we obtain the following theorem.

**Theorem 10.** If $C$ is a linear code over $R_{u^2,v^2,w^2, p}$ of length $n$, size $p^k$ and minimum lee weight $d$, then $\phi_L(C)$ is a $p$-ary linear code with parameters $[8n, k, d]$.

5.1. Gray images of cyclic codes over $R_{u^2,v^2,w^2, p}$

**Definition.** Let $T$ be the cyclic shift such that $T(c_0, c_1, \ldots, c_{n-1}) = (c_{n-1}, c_0, \ldots, c_{n-2})$. Then a linear code $C$ is an $l$-quasicyclic code of length $n$ if it is invariant under $T^l$.

**Lemma 4.** Let $T$ be the cyclic shift. Then

$$\phi \circ T = T^8 \circ \phi_L.$$

**Proof.** Let $\tilde{T} = (r_0, r_1, \ldots, r_{n-1}) \in R_{u^2,v^2,w^2, p}^n$ and $\phi_L$ be the Gray map defined in 2.2. Then

$$(\phi \circ T)(\alpha) = (\phi_L(r_{n-1}), \phi(r_0), \ldots, \phi_L(r_{n-2})).$$

We also know that

$$\phi_L(r_0, r_1, \ldots, r_{n-1}) = (\phi_L(r_0), \phi_L(r_1), \ldots, \phi_L(r_{n-1})), $$

where each $\phi_L(r_i)$ is of length 8. Therefore, if we apply the cyclic shift eight times, the whole of $\phi_L(r_{n-1})$ will shift from the end to the beginning, which means we will get

$$(T^8 \circ \phi_L)(\alpha) = (\phi_L(r_{n-1}), \phi_L(r_0), \ldots, \phi_L(r_{n-2})).$$

Hence, we get the result. 

**Theorem 11.** Let $C$ be a cyclic code of length $n$ over the ring $R_{u^2,v^2,w^2, p}$. Then $\phi_L(C)$ is a $8$-quasicyclic binary linear code of length $8n$ over $\mathbb{F}_p$. 

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Proof. Let $C$ be a cyclic code over $R_{u^2,v^2,w^2,p}$. Then we know that $T(C) = C$. Now applying $\phi_L$ to both sides we get
$$\phi_L(T(C)) = \phi_L(C).$$
But, by Lemma 4, we know that $\phi \circ T = T^8 \circ \phi_L$. Therefore,
$$\phi_L(C) = \phi_L(T(C)) = (\phi_L \circ T)(C) = T^8(\phi_L(C))$$
which implies that $\phi_L(C)$ is invariant under $T^8$. Hence, $\phi_L(C)$ is a 8-quasicyclic code. \hfill \Box

6. Examples

Example 1. Cyclic codes of length 4 over the ring $R_{u^2,v^2,w^2,2}$. We have
$$x^4 - 1 = (x - 1)^4$$
over $F_2$. Let $g = x - 1$, some of the non zero cyclic codes of length 4 over the ring $R_{u^2,v^2,w^2,2}$ with generator polynomials, rank and minimum distance are given in Table 1.

<table>
<thead>
<tr>
<th>Non-zero generator polynomials</th>
<th>Rank</th>
<th>$d(C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(vuw^2 + (c_0 + c_1x)uvw)$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$(vwg + c_0uvw)$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$(uwg^4 + c_1vuw^4 + c_0uvwg^2)$</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>$(uwg^4 + c_0uvw(c_2 + c_3x), vwg^4 + (c_0 + c_1x)uvw)$</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$(wg^4 + c_2wg^2 + c_1vuw^2 + c_4uvw, uvwg^2)$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$(wg + c_1uw + c_2vw)$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$(uwg^4 + c_1uwg^4 + c_2vuw + c_3uvwg, uvwg^2 + c_0uvwg)$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$(vg^4 + c_1uwg + c_1vuwg, uvwg^2 + c_0wg, wg^4, uw, vw)$</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>$(g^2 + v + u + c_1w, vg + u, wg + c_1w, uv + c_1w, uw, vw, vuv)$</td>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>

Example 2. Cyclic codes of length 3 over the ring $R_{u^2,v^2,w^2,3}$. We have
$$x^3 - 1 = (x - 1)^3$$
over $F_3$. Let $g = x - 1$, some of the non zero cyclic codes of length 3 over the ring $R_{u^2,v^2,w^2,3}$ with generator polynomials, rank and minimum distance are given in Table below:

Example 3. Cyclic codes of length 5 over the ring $R_{u^2,v^2,w^2,5}$. We have
$$x^5 - 1 = (x - 1)^5$$
over $F_5$. Let $g = x - 1$, some of the non zero cyclic codes of length 5 over the ring $R_{u^2,v^2,w^2,5}$ with generator polynomials, rank and minimum distance are given in Table below:
Table 2. Non zero cyclic codes of length 3 over $R_{u^2,v^2,w^2,3}$.

<table>
<thead>
<tr>
<th>Non-zero generator polynomials</th>
<th>Rank</th>
<th>$d(C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(uw^2 + vwg, vwg^3)$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$(uw^2 + c_1uw + c_1vw, uw^2 + c_1vw)$</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>$(uw^2 + c_1uw + c_1vw, uw + c_1uw)$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$(uw^2 + c_1uw + c_1vw, uw + c_0vw)$</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>$(uv^3 + c_1uv, v + c_1vw, uv + c_1vw)$</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>$(uv^3 + c_1uv, v + c_1vw, v + c_0vw, uw + c_0vw)$</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 3. Non zero cyclic codes of length 5 over $R_{u^2,v^2,w^2,5}$.

<table>
<thead>
<tr>
<th>Non-zero generator polynomials</th>
<th>Rank</th>
<th>$d(C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(uw^4 + vwg^3)$</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>$(uw^4 + vwg^4 + uwv, vwg^4 + uwv, vwg^4)$</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$(uw^4 + c_1uw + uvw(u + c_1x), uw^2 + c_1x)uwv)$</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$(uw^4 + uvw(c_0 + c_1x) + uvw(c_0 + c_1x), uvw(c_0 + c_1x))$</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>$(uw^4 + uvw(c_0 + c_1x) + uvw(c_0 + c_1x), uvw(c_0 + c_1x)) + uvw(c_0 + c_1x)$</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>$(uw^4 + uvw(c_0 + c_1x) + uvw(c_0 + c_1x), uvw(c_0 + c_1x))$</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>$(uw^4 + uvw(c_0 + c_1x) + uvw(c_0 + c_1x), uvw(c_0 + c_1x))$</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>$(uw^4 + uvw(u + c_1x), uvw(u + c_1x), uvw(u + c_1x), uvw(u + c_1x))$</td>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>

References


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