UPPERS TO ZERO IN POLYNOMIAL RINGS OVER GRADED DOMAINS AND UM-t-DOMAINS

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Abstract. Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain, $H$ be the set of nonzero homogeneous elements of $R$, and $\star$ be a semistar operation on $R$. The purpose of this paper is to study the properties of quasi-Prüfer and UM-t-domains of graded integral domains. For this reason we study the graded analogue of $\star$-quasi-Prüfer domains called gr-$\star$-quasi-Prüfer domains. We study several ring-theoretic properties of gr-$\star$-quasi-Prüfer domains. As an application we give new characterizations of UM-t-domains. In particular it is shown that $R$ is a gr-$t$-quasi-Prüfer domain if and only if $R$ is a UM-t-domain if and only if $R_P$ is a quasi-Prüfer domain for each homogeneous maximal $t$-ideal $P$ of $R$. We also show that $R$ is a UM-t-domain if and only if $H$ is a $t$-splitting set in $R[X]$ if and only if each prime $t$-ideal $Q$ in $R[X]$ such that $Q \cap H = \emptyset$ is a maximal $t$-ideal.

1. Introduction

Gilmer characterized Prüfer domains as integrally closed domains such that each prime ideal of the polynomial ring contained in an extended prime is extended [20, Theorem 19.15]. The later condition is called a quasi-Prüfer domain, see [6] and [16, Chapter 6]. Thus an integral domain $D$ is a Prüfer domain if and only if $D$ is integrally closed and quasi-Prüfer. As a $t$-operation analogue it is well-known that $D$ is a Prüfer $v$-multiplication domain ($PvMD$) if and only if $D$ is an integrally closed UM-t-domain [23, Proposition 3.2]. Recall that $D$ is called a UM-t-domain [23], if every upper to zero in $D[X]$ is a maximal $t$-ideal and has been studied by several authors (see [8], [10], [12], [14] and [31]). In [9], Chang and Fontana unified quasi-Prüfer and UM-t-domains by introducing the notion of $\star$-quasi-Prüfer domain, where $\star$ is a semistar operation on a domain. In this paper, we study quasi-Prüfer and UM-t-domain properties of graded integral domains. (Relevant definitions are reviewed in the sequel.)

Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded (commutative) integral domain graded by an arbitrary torsionless grading monoid $\Gamma$. In [1], Anderson-Anderson defined the
graded analogue of some classical domains in *Multiplicative Ideal Theory* like a graded-Pr"{u}fer domain, graded GCD-domain and graded GGCD-domain. It is known that $R$ is a graded-Pr"{u}fer domain (resp., graded GCD-domain, graded GGCD-domain) if and only if $R$ is a Pr"{u}fer domain (resp., GCD-domain, GGCD-domain) [1, Theorem 6.4, Corollary 6.7 and Proposition 6.6]. In [5], Anderson and Chang had begun an investigation on graded integral domains including graded integral domains with a unit of nonzero degree. They defined $R$ to be a graded-Pr"{u}fer domain if each nonzero finitely generated homogeneous ideal of $R$ is invertible, and gave an example of a graded-Pr"{u}fer domain which is not Pr"{u}fer [5, Example 3.6]. Then the author in [32] gave some characterizations of graded-Pr"{u}fer domains.

For $a \in R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$, denote by $C(a)$ the ideal of $R$ generated by homogeneous components of $a$. In [5] and [32], the authors used $C(a)$ to investigate properties of graded integral domains. Since there was not the role of an indeterminate, in most of the results, the base ring was required to have a unit of nonzero degree or to satisfy some other related condition (see [5, Section 1]). Because of this consideration, the author in [33], introduced a homogeneous content ideal for polynomial rings over graded domains to make use of the role of an indeterminate. For a polynomial $f = a_0 + a_1X + \cdots + a_nX^n \in R[X]$, define the homogeneous content ideal of $f$ by $A_f := \bigoplus_{\alpha \in C} (a_\alpha)$. Using $A_f$ we no longer need to assume that the base ring has a unit of nonzero degree.

The main purpose of this paper is to study the quasi-Pr"{u}fer and UM-domain properties of graded integral domains. For this reason in Section 2 we introduced the graded analogue of quasi-Pr"{u}fer domains called gr-quasi-Pr"{u}fer domains and make use of the homogeneous content ideal $A_f$. Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain. Then $R$ is called a gr-quasi-Pr"{u}fer domain in case, if $Q$ is a prime ideal in $R[X]$ and $Q \subseteq P[X]$ for some homogeneous quasi-prime ideal $P$ of $R$, then $Q = (Q \cap R)[X]$. When $\ast = d$ the identity operation on $R$, then we call the gr-d-quasi-Pr"{u}fer domain a gr-quasi-Pr"{u}fer domain. It is shown that $R$ is a gr-quasi-Pr"{u}fer domain if and only if each upper to zero in $R[X]$ contains a nonzero polynomial $g \in R[X]$ with $A_g = R^*$, if and only if for each upper to zero $Q$ in $R[X]$, $A_Q^* = R^*$. It is also shown that $R$ is a gr-quasi-Pr"{u}fer domain if and only if $\text{NA}(R, \ast)$ is a quasi-Pr"{u}fer domain if and only if every prime ideal of $\text{NA}(R, \ast)$ is extended from a homogeneous prime ideal of $R$. As an application, in Section 3, we give several new characterizations of UM-domains. In particular, we show that $R$ is a UM-domain if and only if $R_P$ is a quasi-Pr"{u}fer domain for each homogeneous prime $t$-ideal $P$ of $R$ if and only if $R$ is a gr-$t$-quasi-Pr"{u}fer domain (see Theorem 3.2). Also we show that $R$ is a UM-domain if and only if $H$ (the multiplicative set of nonzero homogeneous elements of $R$) is a $t$-splitting set in $R[X]$ if and only if each prime $t$-ideal $Q$ in $R[X]$ such that $Q \cap H = \emptyset$ is a maximal $t$-ideal (see Theorem 3.6). We also connect gr-quasi-Pr"{u}fer domains to UM-domains. More precisely, if $\ast$ is a (semi)star operation on $R$, it is shown that $R$ is a gr-$\ast_f$-quasi-Pr"{u}fer domain if and only if $R$ is a UM-domain and
\( \tilde{\tau} \) and \( w \) coincide on nonzero homogeneous ideals of \( R \) (see Theorem 3.9). In particular \( R \) is a gr-quasi-Pr"ufer domain if and only if \( R \) is a UMT-domain and \( d \) and \( w \) coincide on nonzero homogeneous ideals of \( R \). Hence if \( R \) is a one dimensional graded domain, then \( R \) is a gr-quasi-Pr"ufer domain if and only if \( R \) is a quasi-Pr"ufer domain. Finally, we give an example of a gr-quasi-Pr"ufer domain that is not a quasi-Pr"ufer domain (see Example 3.14).

To facilitate the reading of the paper, we review some basic facts on semistar operations on (graded) integral domains. Let \( \Gamma \) be a nonzero torsionless grading monoid, that is, \( \Gamma \) is a commutative cancellative monoid (written additively), and \( (\Gamma) = \{a - b \mid a, b \in \Gamma\} \) be the quotient group of \( \Gamma \); so \( (\Gamma) \) is a torsionfree abelian group. It is known that a cancellative monoid is torsionless if and only if it can be given a total order compatible with the monoid operation [28, page 123]. Let \( R = \bigoplus_{a \in \Gamma} R_a \) be a \( \Gamma \)-graded integral domain. That is, \( \deg(x) = \alpha \) for each \( 0 \neq x \in R_\alpha \) and \( \deg(0) = 0 \), and thus each nonzero \( f \in R \) can be written uniquely as \( f = x_{\alpha_1} + \cdots + x_{\alpha_n} \) with \( \deg(x_{\alpha_i}) = \alpha_i \) and \( \alpha_1 < \cdots < \alpha_n \). A nonzero \( x \in R_\alpha \) for all \( \alpha \in \Gamma \) is said to be homogeneous, and so if \( H = \bigcup_{\alpha \in \Gamma}(R_\alpha \setminus \{0\}) \), then \( H \) is the saturated multiplicative set of nonzero homogeneous elements of \( R \). Then \( R_H = \bigoplus_{a \in (\Gamma)}(R_{H\alpha})_a \), called the homogeneous quotient field of \( R \), is a \((\Gamma)\)-graded integral domain whose nonzero homogeneous elements are units. An integral ideal \( I \) of \( R \) is said to be homogeneous if \( I = \bigoplus_{a \in \Gamma}(I \cap R_\alpha) \). A fractional ideal \( I \) of \( R \) is homogeneous if \( sI \) is an integral homogeneous ideal of \( R \) for some \( s \in H \) (thus \( I \subseteq R_H \)). An overring \( T \) of \( R \), with \( R \subseteq T \subseteq R_H \) will be called a homogeneous overring if \( T = \bigoplus_{a \in (\Gamma)}(T \cap (R_{H\alpha}))_a \). Thus \( T \) is a \((\Gamma)\)-graded integral domain with \( T_\alpha = T \cap (R_{H\alpha})_a \) for all \( \alpha \in (\Gamma) \). For more on graded integral domains and their divisibility properties, see [2], [28].

Let \( D \) be an integral domain with quotient field \( K \). Let \( \mathcal{F}(D) \) denote the set of all nonzero \( D \)-submodules of \( K \), \( \mathcal{F}(\mathcal{D}) \) be the set of all nonzero fractional ideals of \( D \), and \( f(\mathcal{D}) \) be the set of all nonzero finitely generated fractional ideals of \( D \). Obviously, \( f(\mathcal{D}) \subseteq \mathcal{F}(\mathcal{D}) \subseteq \mathcal{F}(\mathcal{D}) \). As in [29], a semistar operation on \( D \) is a map \( \star : \mathcal{F}(\mathcal{D}) \rightarrow \mathcal{F}(\mathcal{D}), E \mapsto ^{\star}E^* \), such that, for all \( 0 \neq x \in K \), and for all \( E, F \in \mathcal{F}(\mathcal{D}) \), the following properties hold: \((\star 1) \) \((xE)^* = x^{\star}E^*\); \((\star 2) \) \( E \subseteq F \) implies that \( ^{\star}E^* \subseteq ^{\star}F^*\); \((\star 3) \) \( E \subseteq ^{\star}E^*\); and \((\star 4) \) \( ^{\star}E^{\star} := (E^{\star})^* = E^* \).

A semistar operation \( \star \) is called a (semi)star operation on \( D \), if \( D^{\star} = D \). Let \( \star \) be a semistar operation on \( D \). For every \( E \in \mathcal{F}(\mathcal{D}) \), put \( ^{\star}E = \bigcup E^* \), where the union is taken over all \( F \in f(D) \) with \( F \subseteq E \). It is easy to see that \( \star \) is a semistar operation on \( D \). If \( \star = \star_f \), then \( \star \) is said to be a semistar operation of finite type. We say that a nonzero ideal \( I \) of \( D \) is a quasi-\( \star \)-ideal of \( D \), if \( I^{\star} \cap D = \{1\} \); a quasi-\( \star \)-prime (ideal of \( D \)), if \( I \) is a prime quasi-\( \star \)-ideal of \( D \); and a quasi-\( \star \)-maximal (ideal of \( D \)), if \( I \) is maximal in the set of all proper quasi-\( \star \)-ideals of \( D \). Each quasi-\( \star \)-maximal ideal is a prime ideal. It was shown in [15, Lemma 4.20] that if \( D^{\star} \neq K \), then each proper quasi-\( \star_f \)-ideal of \( D \) is contained in a quasi-\( \star_f \)-maximal ideal of \( D \). We denote by \( \text{QMax}^\star(D) \) (resp.,
ideals to homogeneous ones. It is known that by \( h \) and that if \( \tilde{\star} \)

serving \([2, \text{Proposition 2.5}]\), and \( \tilde{\star} \) serving \([29, \text{page 6}]\).

Given a semistar operation \( \star \) on \( D \), it is possible to construct a semistar operation \( \star \), which is defined as follows, for each \( E \in \mathcal{F}(D) \), \( E^\star := \bigcap_{F \in \mathcal{QMax}^\star(D)} E D_F \).

The most widely studied (semi)star operations on \( D \) have been the identity \( d \), \( \tilde{d} \), and \( \tilde{\star} \) operations, where \( A^{-1} := (A^{-1})^{-1} \), with \( A^{-1} := (D : A) := \{ x \in K \mid x A \subseteq D \} \). We usually use these operations without subscripts. If \( \star \) is a (semi)star operation on \( D \), then \( d \leq \star \leq v \).

Let \( \star \) be a semistar operation on \( D \). Recall from \([17]\) that, \( D \) is called a \( \text{Pr"ufer} \ \star \text{-multiplication domain} \) (for short, a \( \text{P} \\star \text{MD} \)) if each nonzero finitely generated ideal of \( D \) is \( \star \text{-invertible} \); i.e., if \( (I^{-1})^\gamma = D^\star \) for all \( I \in \mathcal{I}(D) \).

When \( \star = v \), we recover the classical notion of \( \text{P} \text{rMD} \); when \( \star = d_D \), the identity (semi)star operation, we recover the notion of \( \text{Pr"ufer} \) domain.

Let \( \star \) be a semistar operation on a graded integral domain \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \). We say that \( \star \) is homogeneous preserving if \( \star \) sends homogeneous fractional ideals to homogeneous ones. It is known that \( d \), \( t \), and \( v \) are homogeneous preserving \([2, \text{Proposition 2.5}]\), \( \tilde{\varnothing} \) is homogeneous preserving \([32, \text{Proposition 2.3}]\), and that if \( \star \) is homogeneous preserving, then so is \( \star f \) \([32, \text{Lemma 2.4}]\).

Denote by \( h\text{-QSpec}^\star(R) \) the homogeneous elements of \( \text{QSpec}^\star(R) \) and let \( h\text{-QMax}^\star(R) \) denote the set of ideals of \( R \) which are maximal in the set of all proper homogeneous quasi-\( \star \)-ideals of \( R \) (if \( \star \) is a (semi)star operation we denote these sets by \( h\text{-Spec}^\star(R) \) and \( h\text{-Max}^\star(R) \) respectively). It is shown that if \( R^\star \subseteq R_H \) and \( \star = \star f \) homogeneous preserving, then \( h\text{-QMax}^\star(R)(\subseteq h\text{-QSpec}^\star(R)) \) is nonempty, each proper homogeneous quasi-\( \star f \)-ideal is contained in a homogeneous maximal quasi-\( \star f \)-ideal \([32, \text{Lemma 2.1}]\), and \( h\text{-QMax}^{\star f}(R) = h\text{-QMax}^\star(R) \) \([32, \text{Proposition 2.5}]\).

2. Graded \( \star \text{-quasi-Pr"ufer domains} \)

Let \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \) be a graded integral domain with quotient field \( K \), \( H \) be the set of nonzero homogeneous elements of \( R \), \( X \) be an indeterminate over \( K \), and \( \star \) be a semistar operation on \( R \) such that \( R^\star \subseteq R_H \). The following is the key definition in this paper.

**Definition 2.1.** The graded integral domain \( R \) is called a gr-\( \star \)-quasi-Pr"ufer domain in case, if \( Q \) is a prime ideal in \( R[X] \) and \( Q \subseteq P[X] \) for some \( P \in h\text{-QSpec}^\star(R) \), then \( Q = (Q \cap R)[X] \). When \( \star = d \) the identity operation on \( R \), then we call the gr-\( d \)-quasi-Pr"ufer domain a gr-\( d \)-quasi-Pr"ufer domain.
It can be seen that if \( R \) has trivial grading \( \Gamma = \{0\} \), then a gr\(\ast\)-quasi-Prüfer domain is the same as a \(\ast\)-quasi-Prüfer domain [9]

It is clear from the definition that if \( R \) is a \(\ast\)-quasi-Prüfer domain, then it is a gr\(\ast\)-quasi-Prüfer domain. Assume that \( \ast_1 \leq \ast_2 \) are two semistar operations on \( R \). It is easy to see that if \( R \) is a gr\(\ast\)-\(\ast_1\)-quasi-Prüfer domain, then \( R \) is a gr\(\ast\)-\(\ast_2\)-quasi-Prüfer domain, since \( h\text{-}Q\text{Spec}^{\ast_2}(R) \subseteq h\text{-}Q\text{Spec}^{\ast_1}(R) \)

Assume that \( L \) is a fractional ideal of \( R[X] \) such that \( L \subseteq R_H[X] \), and set \( A_L := \sum_{f \in L} A_f \). It is easy to see that \( L \subseteq A_L[X] \). By an upper to zero in \( R[X] \), we mean a nonzero prime ideal \( Q \) of \( R[X] \) such that \( Q \cap R = 0 \).

**Proposition 2.2.** Let \( \ast \) be a semistar operation on a graded integral domain \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \). Then the following statements are equivalent:

1. \( R \) is a gr\(\ast\)-quasi-Prüfer domain.
2. Let \( Q \) be an upper to zero in \( R[X] \), then \( A_Q \not\subseteq P \) for each \( P \in h\text{-}Q\text{Spec}^{\ast}(R) \).
3. Let \( Q \) be an upper to zero in \( R[X] \), then \( Q \not\subseteq P[X] \) for each \( P \in h\text{-}Q\text{Spec}^{\ast}(R) \).
4. \( R_P \) is a quasi-Prüfer domain for each \( P \in h\text{-}Q\text{Spec}^{\ast}(R) \).
5. \( R_{H \setminus P} \) is a gr\(\ast\)-quasi-Prüfer domain for each \( P \in h\text{-}Q\text{Spec}^{\ast}(R) \).

**Proof.** (1) \( \Rightarrow \) (3). Follows from the definition.

(3) \( \Rightarrow \) (2). If \( Q \) is an upper to zero in \( R[X] \), then by assumption \( Q \not\subseteq P[X] \) for all \( P \in h\text{-}Q\text{Spec}^{\ast}(R) \). Hence \( A_Q \not\subseteq P \) for each \( P \in h\text{-}Q\text{Spec}^{\ast}(R) \), since \( Q \subseteq A_Q[X] \).

(2) \( \Rightarrow \) (1). Assume that \( Q \) is a prime ideal in \( R[X] \) such that \( (Q \cap R)[X] \subseteq Q \subseteq P[X] \) for some \( P \in h\text{-}Q\text{Spec}^{\ast}(R) \). Then we can find an upper to zero \( Q_1 \) in \( R[X] \) such that \( Q_1 \subseteq Q \) by [11, Theorem A]. Thus \( A_{Q_1} \subseteq A_Q \subseteq P \) for some \( P \in h\text{-}Q\text{Spec}^{\ast}(R) \), and this contradicts the hypothesis.

(1) \( \Rightarrow \) (4). Let \( P \in h\text{-}Q\text{Spec}^{\ast}(R) \). If \( Q \) is a prime ideal of \( R_P[X] \) with \( c_{R_P}(Q) \subseteq R_P \), then \( c_{R_P}(Q) \subseteq PR_P \), and hence \( Q \subseteq PR_P[X] \) (where \( c_D(f) \) is the fractional ideal of an integral domain \( D \) generated by the coefficients of \( f \in D[X] \) and \( c_D(Q) = \sum_{f \in Q} c_D(f) \) for \( Q \) an ideal of \( D[X] \)). So \( Q \cap R[X] \subseteq P[X] \), and by (1) we have \( Q \cap R[X] = (Q \cap R)[X] \). Hence \( Q = (Q \cap R_P)[X] \).

Thus \( R_P \) is a quasi-Prüfer domain by [9, Theorem 1.1].

(4) \( \Rightarrow \) (1) is the same as the proof of part (iv)\(\Rightarrow\)(i) of [9, Lemma 2.1].

(1) \( \Rightarrow \) (5). Let \( P \in h\text{-}Q\text{Spec}^{\ast}(R) \). Assume that \( Q \) is a prime ideal of \( R_{H \setminus P}[X] \) and \( Q \subseteq qR_{H \setminus P}[X] \) for some \( qR_{H \setminus P} \in h\text{-}\text{Spec}(R_{H \setminus P}) \). Hence \( Q \cap R[X] \subseteq qR_{H \setminus P}[X] \cap R[X] \subseteq P[X] \). Thus \( Q \cap R[X] = (Q \cap R)[X] \) by (1). Therefore \( Q = (Q \cap R_{H \setminus P})[X] \) and \( R_{H \setminus P} \) is a gr\(\ast\)-quasi-Prüfer domain.

(5) \( \Rightarrow \) (1). Assume that \( Q \) is a prime ideal in \( R[X] \) and \( Q \subseteq P[X] \) for some \( P \in h\text{-}Q\text{Spec}^{\ast}(R) \). Thus \( Q_{H \setminus P} \subseteq PR_{H \setminus P}[X] \). Hence by (5) one has \( Q_{H \setminus P} = (Q_{H \setminus P} \cap R_{H \setminus P})[X] \). Consequently \( Q = (Q \cap R)[X] \) and \( R \) is a gr\(\ast\)-quasi-Prüfer domain.

\(\square\)
Recall from [32], that $R$ is called a graded Prüfer $\ast$-multiplication domain (GP$_\ast$MD) if every nonzero finitely generated homogeneous ideal of $R$ is a $\ast$-invertible. When $\ast = v$ we have the notion of a graded-PcMD(=PcMD) [1]. Also when $\ast = d$, a GPdMD is a graded-Prüfer domain [5].

**Corollary 2.3.** Every GP$_\ast$MD is a gr-$\ast$-quasi-Prüfer domain.

*Proof.* Assume that $R$ is a GP$_\ast$MD. Then for $P \in h$-QSpec$(\tilde{R})$, we have $R_P$ is a valuation domain by [32, Theorem 4.4], and hence $R_P$ is a quasi-Prüfer domain. So that by Proposition 2.2, $R$ is a gr-$\ast$-quasi-Prüfer domain. Since $\tilde{\ast} \leq \ast$ we have $R$ is a gr-$\ast$-quasi-Prüfer domain. □

**Corollary 2.4.** Let $\ast$ be a homogeneous preserving semistar operation on a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ such that $R^\ast \subseteq R_H$. Then $R$ is a gr-$\ast$-$f$-quasi-Prüfer domain if and only if $R$ is a gr-$\ast$-$\tilde{\ast}$-quasi-Prüfer domain.

*Proof.* Use Proposition 2.2, together with the equality $h$-QMax$_{\ast}^{\ast} (R) = h$-QMax$_{\ast}^{\ast} (R)$ of [32, Proposition 2.5]. □

Note that the $t$-operation is a homogeneous preserving star operation and that $w = \tilde{t}$. Thus in particular $R$ is a gr-$t$-quasi-Prüfer domain if and only if $R$ is a gr-$w$-quasi-Prüfer domain. It is shown [9, Corollary 2.4] that $D$ is a $t$-quasi-Prüfer domain if and only if $D$ is a UMt-domain. In Theorem 3.2 we will show that $R$ is a gr-$t$-quasi-Prüfer domain if and only if $R$ is a UMt-domain. Recently Chang defined another notion of graded UMt-domains for graded integral domains $R$ such that $R_H$ is UFD [7]. A graded integral domain $R$ with $R_H$ a UFD is called a graded UMt-domain if every upper to zero in $R$ is a maximal $t$-ideal, in the sense that a prime ideal $U$ in $R$ is called an upper to zero in $R$, if there exists a prime element $f \in R_H$ such that $U = fR_H \cap R$. It is shown in [7, Theorem 3.5] that, if in addition $R$ has a unit of nonzero degree, then $R$ is a UMt-domain if and only if $R$ is a graded UMt-domain.

Assume that $\ast$ is a homogeneous preserving semistar operation on $R$ such that $R^\ast \subseteq R_H$. Then using [32, Lemma 2.1], one has $h$-QMax$_{\ast}^{\ast} (R) \neq \emptyset$ and if $I$ is a homogeneous ideal of $R$, then $I^\ast \ast = R^\ast$ if and only if $I \nsubseteq P$ for all $P \in h$-QMax$_{\ast}^{\ast} (R)$.

**Lemma 2.5.** Let $\ast$ be a homogeneous preserving semistar operation on a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ such that $R^\ast \subseteq R_H$. Then the following statements are equivalent:

1. $R$ is a gr-$\ast$-$f$-quasi-Prüfer domain.
2. Each upper to zero in $R[X]$ contains a nonzero polynomial $g \in R[X]$ with $A_g^\ast = R^\ast$.
3. If $Q$ is an upper to zero in $R[X]$, then $A_Q^\ast = R^\ast$.

*Proof.* (1) $\Leftrightarrow$ (3). Follows from Proposition 2.2, because the property $A_Q \nsubseteq P$ for all $P \in h$-QMax$_{\ast}^{\ast} (R)$ is equivalent to $A_Q^\ast = R^\ast$.

(3) $\Rightarrow$ (2) is obvious.
(2) ⇒ (1) is the same as the proof of part (2⋆) ⇒ (1⋆) of [9, Lemma 2.3]. □

We say that $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ is a graded valuation domain (gr-valuation domain) if either $u \in R$ or $u^{-1} \in R$ for every nonzero homogeneous $u \in R_H$. It is known that a gr-valuation domain $R$ has a unique homogeneous maximal ideal $M$, and in this case, $R_M$ is a valuation domain [32, Lemma 4.3]. It is clear that $R$ is a gr-valuation domain if and only if $R$ is a graded-Prüfer domain with a unique homogeneous maximal ideal. In particular a gr-valuation domain is integrally closed.

The following proposition is the graded version of the celebrated result of Krull [20, Theorem 19.8]. The integral closure of $R$ is denoted by $\bar{R}$.

**Proposition 2.6** (cf. [25, Theorem 2.10]). Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain. Then the integral closure of $R$ in $K$ is the intersection of the family $\{V_\lambda\}_{\lambda \in \Lambda}$ of gr-valuation overrings of $R$. In particular, $\bar{R}$ is a homogeneous overring of $R$.

Let $R$ be a graded integral domain and $\star$ a semistar operation on $R$. By Proposition 2.6, $\bar{R}$ is a homogeneous overring of $R$. Note that $\bar{R}$ may not be a fractional ideal of $R$. However the same proof of [32, Proposition 2.3] shows that $\star$ sends nonzero homogeneous $R$-submodules of $R_H$ to homogeneous ones. Therefore $\bar{R} := (\bar{R})^\star$ is a homogeneous overring of $R$.

For a fractional ideal $I$ of $R$ let $I_h$ denote the fractional ideal of $R$ generated by the set of homogeneous elements of $R$ in $I$.

**Lemma 2.7.** Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain and $T$ a homogeneous overring of $R$. If $J$ is an ideal of $T$, then $(J \cap R)_h = J_h \cap R$.

Proof. The inclusion $(J \cap R)_h \subseteq J_h \cap R$ is clear. Let $x = \sum x_i \in J_h \cap R$ where $x_i$ are homogeneous components of $x$. Then $x_i \in J_h \cap R \subseteq J \cap R$. Therefore $x = \sum x_i \in (J \cap R)_h$. □

Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain, and $T$ be a homogeneous overring of $R$. Let $\star$ and $\star'$ be semistar operations on $R$ and $T$, respectively. Recall from [32] that $T$ is called a homogeneously $(\star, \star')$-linked overring of $R$ if

$$F^* = R^* \Rightarrow (FT)^{\star'} = T^{\star'}$$

for each nonzero homogeneous finitely generated ideal $F$ of $R$.

Let $N(\star) := \{ f \in R[X] \mid f \neq 0 \text{ and } A^*_f = R^* \}$ and set $\text{NA}(R, \star) := R[X]_{N(\star)}$ and $\text{NA}(R) := \text{NA}(R, d)$. Then it is shown in [33], that $\text{NA}(R, \star)$ is compatible with the graded structure of the base ring $R$ and that if $R$ has the trivial grading, then $\text{NA}(R, \star) = \text{Na}(R, \star)$ the usual $\star$-Nagata ring [18]. It is known that $N(\star) = N(\star_f) = N(\tilde{\star}) = R[X] \setminus \bigcup \{ P[X] \mid P \in h\text{-QMax}^\tilde{\star}(R) \}$ and $\text{Max}(\text{NA}(R, \star)) = \{ P \text{NA}(R, \star) \mid P \in h\text{-QMax}^\tilde{\star}(R) \}$ [33, Proposition 2.3].
Proposition 2.8 ([33, Theorem 3.6]). Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and $\star$ be a semistar operation on $R$ such that $R^\star \subseteq R_H$. Then, the following statements are equivalent:

1. $R$ is a GP$\star$MD.
2. Every ideal of $NA(R, \star)$ is extended from a homogeneous ideal of $R$.
3. $NA(R, \star)$ is a Prüfer domain.

In particular if $R$ is a GP$\star$MD, then $R^\widehat{\star}$ is integrally closed.

We are now ready to state and prove the main result of this section which gives some characterizations of gr-$\star_f$-quasi-Prüfer domains.

Theorem 2.9. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and $\star$ be a homogeneous preserving semistar operation on $R$ such that $R^\star \subseteq R_H$. Then, the following statements are equivalent:

1. $R$ is a gr-$\star_f$-quasi-Prüfer domain.
2. Set $\tilde{R} = (\tilde{R})^\widehat{\star}$ and let $\iota : R \hookrightarrow \tilde{R}$ be the canonical embedding, then $\tilde{R}$ is a GP$\tilde{(}\tilde{\star})$ MD.
3. Each homogeneous overring $T$ of $R$ is a gr-$(\star_f)$-quasi-Prüfer domain, where $\iota : R \hookrightarrow T$ is the canonical embedding.
4. Each homogeneously $(\star, \star')$-linked overring $T$ of $R$ is a gr-$\star_f'$-quasi-Prüfer domain.
5. Every prime ideal of $NA(R, \star)$ is extended from a homogeneous prime ideal of $R$.
6. $NA(R, \star_f)$ is a quasi-Prüfer domain.
7. The integral closure of $NA(R, \star_f)$ is a Prüfer domain.
8. $R_P$ is a quasi-Prüfer domain, for each $P \in h$-$QMax^{\star_f}(R)$ (or, for each $P \in h$-$QSpec^{\star_f}(R)$).

Proof. (6) $\Rightarrow$ (8). Let $P \in h$-$QMax^{\star_f}(R)$ (or $P \in h$-$QSpec^{\star_f}(R)$). Then $PNA(R, \star)$ is a maximal ideal of $NA(R, \star)$ [33, Proposition 2.3]. Since $R_P(X) = R_P[X]_{PNA(R, \star)}$ and $NA(R, \star)$ is a quasi-Prüfer domain, then $R_P(X)$ is a quasi-Prüfer domain by [9, Theorem 1.1 (1) $\Leftrightarrow$ (11)]. Then [9, Theorem 1.1 (1) $\Leftrightarrow$ (9)] implies that $R_P$ is a quasi-Prüfer domain.

(8) $\Rightarrow$ (6). Let $Q \in \text{Max}(NA(R, \star))$. Then there exists a $P \in h$-$QMax^{\star_f}(R)$ such that $Q = PNA(R, \star)$ and $NA(R, \star)Q = R_P(X)$ [33, Proposition 2.3]. Thus using [9, Theorem 1.1], one has $NA(R, \star_f)$ is a quasi-Prüfer domain.

(1) $\Leftrightarrow$ (8) is Proposition 2.2.

(6) $\Rightarrow$ (2). Assume that $NA(R, \star_f)(= NA(R, \tilde{\star}))$ is a quasi-Prüfer domain and thus the integral closure $\overline{NA(R, \star)}$ is a Prüfer domain by [9, Theorem 1.1]. Note that $\overline{NA(R, \star)} = \overline{R[X]}_{\overline{N(\tilde{\star})}}$, where $N(\tilde{\star}) = N(\star_f) = \{ y \in R[X] \mid A_y^{\star_f} = R^\star \}$. Set $\star := (\tilde{\star})$. Clearly $\star$ is a (semi)star operation of finite type on $\tilde{R}$. Moreover $NA(R, \star) = \overline{R[X]}_{\overline{N}}$, where $\overline{N} = \{ h \in \tilde{R}[X] \mid (A_h^\star)^{\star_f} = \tilde{R}^\star \}$. Then $\overline{N}$ is a multiplicatively closed subset of $\tilde{R}[X]$ and it is easy to see that $N(\tilde{\star}) \subseteq \overline{N}$ (indeed if $f \in N(\tilde{\star})$, then $A_f^{\star_f} \subseteq R^\star$ and so $(A_f^\star)^{\star_f} = (A_f \overline{R})^\star = \tilde{R}$). Hence
\( \overline{\text{NA}(R, \pi)} \subseteq \text{NA}(\tilde{R}, \ast) \) and so \( \text{NA}(\tilde{R}, \ast) \) is a Prüfer domain by [20, Theorem 26.1]. Therefore \( \tilde{R} \) is a GP*MD by Proposition 2.8.

(2) \( \Rightarrow \) (7). With the notation used in part (6) \( \Rightarrow \) (2), since \( \tilde{R} \) is a GP*MD, we have \( \text{NA}(\tilde{R}, \ast) \) is a Prüfer domain by Proposition 2.8. The conclusion will trivially follows if we show that \( \overline{\text{NA}(R, \pi)} = \text{NA}(\tilde{R}, \ast) \), i.e., \( \tilde{R}[X]_{N(\pi)} = \tilde{R}[X]_{\tilde{N}} \).

Note that \( N(\tilde{R}) = R[X] \setminus \{ P[X] \mid P \in h\text{-QMax}^{\tilde{2}}(R) \} \), \( \tilde{N} = \tilde{R}[X] \setminus \{ Q[X] \mid Q \in h\text{-Max}^{\tilde{2}}(\tilde{R}) \} \) and \( \tilde{R}[X]_{N(\pi)} \subseteq \tilde{R}[X]_{\tilde{N}} \). By [9, Lemma 2.15(b)], the natural embedding \( \iota : R \hookrightarrow \tilde{R} \) verifies \( \tilde{\pi}\text{-INC} \) and \( \tilde{\pi}\text{-GU} \).

Let \( Q \) be a prime ideal of \( \tilde{R} \). We show that \( Q \in h\text{-Max}^{\tilde{2}}(\tilde{R}) \) if and only if \( Q \cap R \in h\text{-QMax}^{\tilde{2}}(R) \). First we show that if \( P = Q \cap R \) is a quasi-\( \tilde{\pi} \)-prime ideal of \( R \), then \( Q \) is a \( \ast \)-prime ideal of \( \tilde{R} \). Since \( Q \in \mathcal{F}(\tilde{R}) \subseteq \mathcal{F}(R) \), we have \( Q^{\ast} = Q^{(2)} \ast = \bigcap_{P \in \text{QSpec}^{\ast}(R)} QRP \). Assume that \( P = Q \cap R \) is a quasi-\( \tilde{\pi} \)-prime ideal of \( R \) and \( x \in Q^{\ast} \). Then \( x \in QRP \). So there exist \( a \in Q \) and \( b \in R \setminus P \) such that \( x = a/b \). Therefore \( xb = a \in Q \) implies that \( x \in Q \), since \( Q \) is a prime ideal of \( \tilde{R} \) and \( b \in \tilde{R} \setminus P \) and \( x \in Q^{\ast} \subseteq (\tilde{R})^{\ast} = \tilde{R} \). Therefore \( Q^{\ast} \subseteq Q \) and so \( Q \) is a \( \ast \)-prime ideal of \( \tilde{R} \). Now assume that \( P := Q \cap R \in h\text{-QMax}^{\tilde{2}}(R) \). Note that \( P = \tilde{P}_h = (Q \cap R)_h = Q_h \cap R \) by Lemma 2.7, and \( Q_h \) is a homogeneous \( \ast \)-prime ideal by [32, Page 186]. If \( Q_h \subseteq Q \), then by \( \tilde{\pi}\text{-INC} \) we have \( Q_h \cap R \subseteq Q \cap R \), that is \( P \subseteq P \) which is a contradiction. Therefore \( Q \) is a homogeneous \( \ast \)-prime ideal of \( \tilde{R} \). Let \( M \in h\text{-Max}^{\tilde{2}}(\tilde{R}) \) such that \( Q \subseteq M \). By \( \tilde{\pi}\text{-INC} \) we have \( P = Q \cap R \subseteq M \cap R \). Therefore \( M \cap R \subseteq (M \cap R)^{\ast} \cap R = (M^{\ast} \cap R^{2}) \cap R \subseteq (M^{\ast} \cap \tilde{R}) \cap R = (M^{\ast} \cap \tilde{R}) \cap R = M \cap R \), which is a contradiction since \( P \in h\text{-QMax}^{\tilde{2}}(R) \).

Conversely, assume that \( Q \in h\text{-Max}^{\tilde{2}}(\tilde{R}) \) and that \( P := Q \cap R \subseteq P' \) for some \( P' \in h\text{-QMax}^{\tilde{2}}(R) \). By \( \tilde{\pi}\text{-GU} \), there exists a \( \ast \)-prime ideal \( Q' \) of \( \tilde{R} \) such that \( Q' \cap R = P' \) and \( Q \subseteq Q' \). Note that using Lemma 2.7 and \( \tilde{\pi}\text{-INC} \), we can assume that \( Q' \) is a homogeneous prime ideal, and this is a contradiction.

From the fact that \( Q \in h\text{-Max}^{\tilde{2}}(\tilde{R}) \) if and only if \( Q \cap D \in h\text{-QMax}^{\tilde{2}}(R) \), it can be seen that the ideals of \( \tilde{R}[X] \) that are maximal with respect to the property of being disjoint from \( N(\tilde{R}) \) are the ideals \( \{ (Q \cap \tilde{R})[X] \mid Q \in h\text{-Max}^{\tilde{2}}(\tilde{R}) \} \). From this, [20, Proposition 4.8 and Theorem 4.10] and [30, Proposition 1.5], it follows easily that \( \tilde{R}[X]_{N(\tilde{R})} = \tilde{R}[X]_{\tilde{N}} \).

(7) \( \Rightarrow \) (6) is true by [9, Theorem 1.1].

(1) \( \Rightarrow \) (4). Assume that \( T \) is a homogeneously \( (\ast, \ast^{\prime}) \)-linked overring of \( R \). Thus \( \text{NA}(T, \ast^{\prime}) \) is an overring of \( \text{NA}(R, \ast) \) by [33, Lemma 2.8]. Since we already proved that (1) is equivalent to (7), we have \( \text{NA}(R, \ast) \) has Prüfer integral closure. Hence \( \text{NA}(T, \ast^{\prime}) \) also has Prüfer integral closure. Therefore \( T \) is a gr-\( \ast^{\prime} \)-quasi-Prüfer domain.

(4) \( \Rightarrow \) (3). Assume that \( T \) is a homogeneous overring of \( R \). Then it can be seen that \( T \) is homogeneously \( (\ast_{f}, (\ast_{f})_{\ast}) \)-linked overring of \( R \). Hence \( T \) is a gr-(\( \ast_{f} \))-quasi-Prüfer domain.
(3) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (5). Let $\Omega = Q \text{NA}(R, *) = QR[X]_{N(*)}$ be a prime ideal of $\text{NA}(R, *)$ for some prime ideal $Q$ of $R[X]$ such that $Q \cap N(*) = \emptyset$. In part (2) $\Rightarrow$ (7), we showed that $\text{NA}(R, *) = \text{NA}(\tilde{R}, *)$. So there exists a prime ideal $L$ of $\text{NA}(\tilde{R}, *)$ such that $L \cap \text{NA}(R, *) = QR[X]_{N(*)}$. Note that we proved (1) $\iff$ (2), hence $\tilde{R}$ is a GP*MD. Thus by Proposition 2.8, there exists a homogeneous prime ideal $L$ of $\tilde{R}$ such that $L = L \text{NA}(\tilde{R}, *)$. Whence $L \text{NA}(\tilde{R}, *) \cap \text{NA}(R, *) = QR[X]_{N(*)}$ and intersecting with $R[X]$ one obtains that $Q = (L \cap R)[X]$. Note that $L \cap R$ is a homogeneous prime ideal of $R$ such that $\Omega = (L \cap R) \text{NA}(R, *)$.

(5) $\Rightarrow$ (1). Suppose that $R$ is not a gr-$*f$-quasi-Prüfer domain. Then by Lemma 2.5, there is an upper to zero $Q$ in $R[X]$ such that $Q \cap N(*) = \emptyset$. Hence $Q \text{NA}(R, *) = QR[X]_{N(*)}$ is a proper prime ideal of $\text{NA}(R, *)$. Note that $Q \text{NA}(R, *) \neq P \text{NA}(R, *)$ for all nonzero homogeneous prime ideals $P$ of $R$, since $Q$ is an upper to zero in $R[X]$. This fact contradicts the assumption (5).

The following corollary is immediate from Theorem 2.9, Proposition 2.2 and Lemma 2.5.

**Corollary 2.10.** Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent:

1. $R$ is a gr-quasi-Prüfer domain.
2. $R$ is a graded-Prüfer domain.
3. Each homogeneous overring $T$ of $R$ is a gr-quasi-Prüfer domain.
4. Every prime ideal of $\text{NA}(R)$ is extended from a homogeneous prime ideal of $R$.
5. $\text{NA}(R)$ is a quasi-Prüfer domain.
6. The integral closure of $\text{NA}(R)$ is a Prüfer domain.
7. $R_P$ is a quasi-Prüfer domain, for each $P \in h\text{-Max}(R)$ (or, for each $P \in h\text{-Spec}(R)$).
8. $R_{P \setminus P}$ is a gr-quasi-Prüfer domain, for each $P \in h\text{-Max}(R)$ (or, for each $P \in h\text{-Spec}(R)$).
9. Each upper to zero in $R[X]$ contains a nonzero polynomial $g \in R[X]$ with $A_g = R$.
10. If $Q$ is an upper to zero in $R[X]$, then $A_Q = R$.

**Remark 2.11.** Let $*$ be a semistar operation on a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ such that $R^* \subsetneq R_H$. Note that by [32, Proposition 2.3], $R^\circ$ is a homogeneous overring of $R$ and let $\iota : R \hookrightarrow R^\circ$. Then exactly by the same way as the proof of [18, Corollary 3.5], one can show that $h\text{-QMax}(R^\circ) = \{QR_Q \cap R^\circ | Q \in h\text{-QMax}^\circ(R)\}$, and hence $\text{NA}(R, *) = \text{NA}(R^\circ, \iota_*)$.

**Proposition 2.12.** Let $*$ be a semistar operation on a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ such that $R^* \subsetneq R_H$. Then the following statements are equivalent:
(1) $R$ is a GP⋆MD.
(2) $R$ is a gr-⋆f-quasi-Prüfer domain and $R_Q$ is integrally closed for all $Q \in h\text{-}Q\text{Max}^+(R)$.
(3) $R$ is a gr-⋆f-quasi-Prüfer domain and $R^*$ is integrally closed.

Proof. (1) $\Rightarrow$ (3) holds by Corollary 2.3 and Proposition 2.8.
(3) $\Rightarrow$ (1). Note that $\text{NA}(R,⋆) = \text{NA}(R^*,(⋆))$ by Remark 2.11. On the other hand $\text{NA}(R^*,(⋆))$ is integrally closed since $R^*$ is integrally closed and $\text{NA}(R,⋆)$ is a quasi-Prüfer domain by Theorem 2.9. Thus $\text{NA}(R,⋆)$ is a Prüfer domain and hence $R$ is a GP⋆MD by Proposition 2.8.
(1) $\iff$ (2) holds by [13, Proposition 3.8] and Corollary 2.3. □

3. Graded integral UMt-domains

Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain and $H$ be the set of nonzero homogeneous elements of $R$. In this section we give several new characterizations of UMt-domains. In particular we show that $R$ is a UMt-domain if and only if $R$ is a gr-t-quasi-Prüfer domain. We also connect the gr-⋆f-quasi-Prüfer domains to UMt-domains for (semi)star operation $⋆$ on $R$.

Let $D$ be an integral domain. Then $D$ is a trivially graded domain with $\Gamma = \{0\}$, and each nonzero element of $D$ is homogeneous, i.e., $H = D \setminus \{0\}$. Hence in this case, a prime ideal $Q$ of $D[X]$ is an upper to zero if and only if $Q \cap H = \emptyset$. Also note that each upper to zero in $D[X]$ is a prime t-ideal.

The following proposition is a useful graded version of the well-known result of Houston and Zafrullah [23, Theorem 1.4] (see also [19, Theorem 3.3]). Recall from [22, Proposition 4.3] that $(I[X])^t = I'[X]$ for each fractional ideal $I$ of $R$.

**Proposition 3.1.** Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain and $Q$ be a prime t-ideal in $R[X]$ such that $Q \cap H = \emptyset$. Consider the following statements.

1. $(A_Q)^t \neq R$.
2. $Q$ is a maximal t-ideal.
3. $Q$ is t-invertible.

Then (1) $\iff$ (2) $\iff$ (3) and, if $Q$ is an upper to zero, then (2) $\Rightarrow$ (3).

**Proof.** (1) $\Rightarrow$ (2). Suppose that $Q$ is not a maximal t-ideal, and let $M$ be a maximal t-ideal of $R[X]$ which contains $Q$. Since the containment is proper, we have that $M \cap R \neq 0$. Then by [23, Proposition 1.1], $M = (M \cap R)[X]$ and $M \cap R$ is a t-ideal of $R$. Since $Q \subseteq M$, $A_Q$ is contained in the t-ideal $M \cap R$, so that $(A_Q)^t \neq R$.

(2) $\Rightarrow$ (1). Since $Q \cap H = \emptyset$ and $A_Q$ is homogeneous, one has $Q \subseteq A_Q[X]$. Then $(A_Q)^t = R$ using [22, Proposition 4.3].

(3) $\Rightarrow$ (2) is true by [23, Proposition 1.3].

Now assume that $Q$ is an upper to zero in $R[X]$. Then (2) $\Rightarrow$ (3) is true by [23, Theorem 1.4]. □
In the following result which is the first main result of this section, we show that $R$ is a gr-$t$-quasi-Prüfer domain if and only if $R$ is a UMT-domain if and only if $R_P$ is a quasi-Prüfer domain, for each homogeneous prime (or maximal) $t$-ideal $P$ of $R,$ which are new characterizations of UMT-domains.

**Theorem 3.2.** Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain. Then the following statements are equivalent:

1. $R$ is a gr-$t$-quasi-Prüfer domain.
2. Let $Q$ be an upper to zero in $R[X],$ then $A_Q \nsubseteq P$ for each $P \in h$-$\text{Spec}^t(R)$.
3. Let $Q$ be an upper to zero in $R[X],$ then $Q \nsubseteq P[X]$ for each $P \in h$-$\text{Spec}^t(R)$.
4. $R_P$ is a quasi-Prüfer domain for each $P \in h$-$\text{Spec}^t(R)$.
5. $R_{H \setminus P}$ is a gr-quasi-Prüfer domain for each $P \in h$-$\text{Spec}^t(R)$.
6. Each upper to zero in $R[X]$ contains a nonzero polynomial $g \in R[X]$ with $(A_g)^t = R$.
7. If $Q$ is an upper to zero in $R[X],$ then $(A_Q)^t = R$.
8. Each upper to zero in $R[X]$ is a $t$-invertible.
9. Each upper to zero in $R[X]$ is a maximal $t$-ideal.
10. $R$ is a UMT-domain.

**Proof.** (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follows from Proposition 2.2.

(1) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) follows from Lemma 2.5.

(7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) follows from Proposition 3.1.

(9) $\Leftrightarrow$ (10) is the definition of UMT-domains [23].

Note that a consequence of Proposition 2.12 is that $R$ is a graded-PvMD if and only if $R$ is an integrally closed gr-$t$-quasi-Prüfer domain. Thus Theorem 3.2 implies the following corollary.

**Corollary 3.3 ([1, Theorem 6.4]).** Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain. Then $R$ is a graded-PvMD if and only if $R$ is a PvMD.

**Proposition 3.4.** Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain. Then the following statements are equivalent:

1. $R$ is a UMT-domain.
2. Every prime ideal of $NA(R,v)$ is extended from a homogeneous prime ideal of $R.$
3. $NA(R,v)$ is a quasi-Prüfer domain.
4. Each homogeneously $(t_R,d_T)$-linked overring $T$ of $R$ is a gr-quasi-Prüfer domain.

**Proof.** The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are immediate from Theorems 2.9 and 3.2, and (1) $\Leftrightarrow$ (4) follows from Theorem 2.9.

Let $D$ be an integral domain. A multiplicative subset $S$ of $D$ is called a $t$-splitting set if each $0 \neq d \in D$ can be written as $dD = (AB)^t,$ where $A$ and
Proof. The following statements are equivalent of Theorem 3.2, one has $R$ invertible for all $0 \neq d \in D$. It is known that $D$ is a UM-domain if and only if $D \setminus \{0\}$ is a $t$-splitting set of $D[X]$ [8, Corollary 2.9]. In the following theorem we generalized this result to the graded case among other things, which is the second main result in this section. Before that we need a lemma and for an ideal $I$ of $R$ set $C(I) := \sum_{a \in I} C(a)$.

**Lemma 3.5.** Let $R = \bigoplus_{\alpha \in F} R_{\alpha}$ be a graded integral domain and $I$ be an ideal of $R$. Then $\mathcal{A}_{I[X]} = C(I)$.

**Proof.** Assume that $a \in I$. Then $aX \in I[X]$ and so $C(a) = A_{aX} \subseteq A_{I[X]}$. Hence $C(I) \subseteq \mathcal{A}_{I[X]}$. Conversely let $f = \sum_{i=0}^{n} a_iX^i \in I[X]$. Then $\mathcal{A}_{f} = \sum_{i=0}^{n} C(a_i) \subseteq C(I)$. Hence $\mathcal{A}_{I[X]} \subseteq C(I)$.

**Theorem 3.6.** Let $R = \bigoplus_{\alpha \in F} R_{\alpha}$ be a graded integral domain. Then the following statements are equivalent:

1. $R$ is a UM-domain.
2. If $Q$ is a prime $t$-ideal in $R[X]$ such that $Q \cap H = \emptyset$, then $(A_Q)^t = R$.
3. Each prime $t$-ideal $Q$ in $R[X]$ such that $Q \cap H = \emptyset$, is a maximal $t$-ideal.
4. $H$ is a $t$-splitting set in $R[X]$.

**Proof.** (1) $\Rightarrow$ (2). Assume that $Q$ is a prime $t$-ideal of $R[X]$ such that $Q \cap H = \emptyset$. If $Q$ is an upper to zero, then $(A_Q)^t = R$ by Theorem 3.2. Otherwise $P := Q \cap R \neq 0$ and $P \cap H = \emptyset$. If $P[X] \subseteq Q$, pick $q \in Q \setminus P[X]$, and let $Q_1$ be an upper to zero in $R[X]$ such that $q \in Q_1 \subseteq Q$ by [11, Theorem A]. Thus using Theorem 3.2, one has $R = (A_{Q_1})^t \subseteq (A_Q)^t \subseteq R$, and then $(A_Q)^t = R$. Now assume that $Q = P[X]$ and that $(A_Q)^t = (C(P))^t \subseteq R$ (Lemma 3.5). Then there exists a homogeneous maximal $t$-ideal $M$ of $R$ such that $(C(P))^t \subseteq M$. So that $Q = P[X] \subseteq M[X]$ and hence $Q \cap N(v) = \emptyset$. Thus $Q \cap N(v) = \emptyset$. Thus $Q \cap N(v) = \emptyset$. Thus $P_{0}$ of $R$ such that $Q \cap N(v) = P_{0}$ of $R$ by Proposition 3.4. By intersecting this last equality with $R$, we have $P = P_{0}$, the desired contradiction, since $P \cap H = \emptyset$.

(2) $\Leftrightarrow$ (3) holds by Proposition 3.1.

(3) $\Rightarrow$ (1). Assume that $Q$ is an upper to zero in $R[X]$. Then $Q \cap H = \emptyset$ and $Q$ is a prime $t$-ideal. Hence $Q$ is a maximal $t$-ideal by (3), which implies that $R$ is a UM-domain by Theorem 3.2.

(4) $\Rightarrow$ (2). Suppose that $H$ is a $t$-splitting set in $R[X]$, and let $Q$ be a prime $t$-ideal of $R[X]$ with $Q \cap H = \emptyset$. For any $0 \neq f \in Q$, let $f[X] = (AB)^t$, where $A$ and $B$ are integral ideals of $R[X]$ such that $A^t \cap sR[X] = sA^t$ for all $s \in H$ and $B^t \cap H \neq \emptyset$. Since $Q \cap H = \emptyset$, $B \not\subseteq Q$, so $A \subseteq Q$. Thus if $s$ is a nonzero homogeneous element in $A_{Q}$, then $(A, s)^t \subseteq (A_{Q}[X])^t = (A_Q)^t[X]$ [22, Proposition 4.3]. Therefore $R = A_{R[X]} = A_{(A,s)^t} \subseteq A_{(A_Q)^t[X]} = (A_Q)^t \subseteq R$ by Lemma 3.5, and then $(A_Q)^t = R$. 

$B$ are integral ideals of $D$ such that $A^t \cap sD = sA^t$ (equivalently, $(A, s)^t = D$) for all $s \in S$ and $B^t \cap S \neq \emptyset$. The notion of $t$-splitting sets was introduced in [3], where it is shown that $S$ is a $t$-splitting set of $D$ if and only if $dD_S \cap D$ is $t$-invertible for all $0 \neq d \in D$. It is known that $D$ is a UM-domain if and only if $D \setminus \{0\}$ is a $t$-splitting set of $D[X]$ [8, Corollary 2.9]. In the following theorem we generalized this result to the graded case among other things, which is the second main result in this section. Before that we need a lemma and for an ideal $I$ of $R$ set $C(I) := \sum_{a \in I} C(a)$.
(2) ⇒ (4). Let $0 ≠ g ∈ R[X]$, and let $J = gR[X]_H ∩ R[X] = gR_H[X] ∩ R[X]$. By [3, Corollary 2.3], to show that $H$ is a $t$-splitting set, it suffices to show that $J$ is $t$-invertible. We first show that $(A_J)^t = R$. Assume $(A_J)^t ⊆ R$, and let $P$ be a maximal $t$-ideal of $R$ containing $(A_J)^t$. Note that we have $J ⊆ A_J[X] ⊆ (A_J)^t[X] ⊆ P[X]$. Let $Q$ be a prime ideal of $R[X]$ minimal over $J$ such that $Q ⊆ P[X]$. Then $Q$ is a $t$-ideal (since $J$ is a $t$-ideal of $R[X]$ by [26, Lemma 3.17]), and since $P$ is homogeneous by [4, Lemma 1.2], we have $(A_Q)^t ⊆ (A_{P[X]})^t = C(P)^t = P$ using Lemma 3.5. Assume that $Q ∩ H = ∅$. Then by the hypothesis $R = (A_Q)^t ⊆ P$ which is a contradiction. Hence we have $Q ∩ H ≠ ∅$. Let $0 ≠ x ∈ Q ∩ H$. Then there are a $y ∉ Q$ and a nonnegative integer $n$ such that $yx^n ∈ J$ [24, Theorem 2.1], whence $y ∈ gR[X]_H ∩ R[X] = J ⊆ Q$. This contradiction shows that $(A_J)^t = R$.

Let $f_1, \ldots, f_n ∈ J$ such that $(A_{f_1} + \cdots + A_{f_n})^t = R$, and let $I = (g, f_1, \ldots, f_n)^t$ (so $IR_H[X] = gR_H[X] = JR_H[X]$ using [22, Proposition 4.3]). Let $M$ be a maximal $t$-ideal of $R[X]$. If $M ∩ H = ∅$, then $M_H$ is a prime ideal of $R_H[X]$, and thus $IR_H[X]_M = (IR_H[X])_{M_H} = (gR_H[X])_{M_H} = (JR_H[X])_{M_H} = JR[X]_M$. If $M ∩ H ≠ ∅$, then $P := M ∩ R ≠ ∅$, and $M = (M ∩ R)[X] = P[X]$ by [23, Proposition 1.1]. Note that $P$ is a homogeneous maximal $t$-ideal of $R$ ([23, Proposition 1.1] and [4, Lemma 1.2]). If $I ⊆ M$, then $R = (A_I)^t ⊆ (A_M)^t ⊆ R$. But by Lemma 3.5, we have $(A_M)^t = (A_{P[X]})^t = (C(P))^t = P$ which is a contradiction. Therefore we have $I ∉ M$. By the same reasoning $J ∉ M$. Hence $IR[X]_M = R[X]_M = JR[X]_M$. Thus $J = I$ by [26, Proposition 2.8], and since $I$ is $t$-locally principal, $I = J$ is $t$-invertible by [26, Corollary 2.7].

**Corollary 3.7** ([8, Corollary 2.9]). Let $D$ be an integral domain. Then $D$ is a UM-domain if and only if $D \setminus \{0\}$ is a $t$-splitting set of $D[X]$.

A saturated multiplicative subset $S$ of $D$ is called a splitting set if for each $0 ≠ d ∈ D$, $d = s′a$ for some $s ∈ S$ and $a ∈ D$ with $aD ∩ s′D = as′D$ for all $s′ ∈ S$. The concept of splitting sets was introduced by Gilmer and Parker [21], where they proved that if $S$ is a splitting set generated by prime elements, then $D$ is a UFD if $D_S$ is a UFD. Note that a $t$-splitting set of a GCD-domain is a splitting set. The following corollary gives a new characterization of (graded) GCD-domains. Recall from [1] that a graded integral domain $R = \bigoplus_{0 ∈ \Gamma} R_α$ is a graded GCD-domain if each pair of nonzero homogeneous elements of $R$ has a GCD.

**Corollary 3.8.** Let $R = \bigoplus_{0 ∈ \Gamma} R_α$ be a graded integral domain. Then the following statements are equivalent:

1. $H$ is a splitting set in $R[X]$.
2. $R$ is a graded GCD-domain.
3. $R$ is a GCD-domain.

**Proof.** (1) ⇒ (2). Assume that $H$ is a splitting set in $R[X]$. Then for each $0 ≠ f ∈ R[X]$, $f = ag$ where $a ∈ H$ and $g ∈ R[X]$ with $(g, s)_v = R[X]$ for all
The following theorem connects the gr-$\star_f$-quasi-Prüfer domains to UM$\mathcal{T}$-domains for (semi)star operation $\star$ on $R$.

**Theorem 3.9.** Assume that $\star$ is a (semi)star operation on $R$. Then the following statements are equivalent:

1. $R$ is a gr-$\star_f$-quasi-Prüfer domain.
2. Each homogeneously $(\star_f,t)$-linked overring of $R$ is a UM$\mathcal{T}$-domain and each element of $h$-$\text{Max}^{*}_{t}(R)$ is a $t_{R}$-ideal.
3. $R$ is a UM$\mathcal{T}$-domain and each element of $h$-$\text{Max}^{\vee}(R)$ is a $t_{R}$-ideal.
4. $R$ is a UM$\mathcal{T}$-domain and, $\tilde{\star}$ and $\tilde{w}_{R}$ coincide on nonzero homogeneous ideals.

**Proof.** (1) $\Rightarrow$ (3). Since $\star_f \leq t_{R}$ and $R$ is a gr-$\star_f$-quasi-Prüfer domain, then $R$ is a gr-$t_{R}$-quasi-Prüfer domain and thus is a UM$\mathcal{T}$-domain by Theorem 3.2. Let $P$ be an element of $h$-$\text{Max}^{\vee}(R)$. By Theorem 2.9, $R_{P}$ is a quasi-Prüfer domain and by [9, Corollary 1.3], $PR_{P}$ is a $t$-ideal of $R_{P}$. Thus $P = PR_{P} \cap R$ is a $t_{R}$-ideal by [26, Lemma 3.17].

(3) $\Rightarrow$ (4). The second part of (3) implies that $h$-$\text{Max}^{\vee}(R) = h$-$\text{Max}^{t_{R}}(R)$. Thus $\tilde{\star}$ and $\tilde{w}_{R}$ coincide on homogeneous ideals using [32, Proposition 2.6].

(4) $\Rightarrow$ (3). If $\tilde{\star}$ and $\tilde{w}_{R}$ coincide on homogeneous ideals, then $h$-$\text{Max}^{\vee}(R) = h$-$\text{Max}^{t_{R}}(R) = h$-$\text{Max}^{\vee}(R)$ by [32, Proposition 2.5]. So that each element of $h$-$\text{Max}^{\vee}(R)$ is a $t_{R}$-ideal.

(3) $\Rightarrow$ (1). Since each element of $h$-$\text{Max}^{\vee}(R)$ is a $t_{R}$-ideal, one has $h$-$\text{Max}^{\vee}(R) = h$-$\text{Max}^{t_{R}}(R)$. Thus $N(\star) = N(t_{R})$ and hence $\text{NA}(R,\star) = \text{NA}(R, t_{R})$. Now Theorem 2.9, completes the proof.

(1) $\Rightarrow$ (2). Assume that $T$ is a homogeneously $(\star_f, t_{T})$-linked overring of $R$. Thus $\text{NA}(T, t_{T})$ is an overring of $\text{NA}(R, \star_f)$ by [33, Lemma 2.8]. Using Theorem 2.9, we have $\text{NA}(R, \star_f)$ has Prüfer integral closure. Hence $\text{NA}(T, t_{T})$ has Prüfer integral closure. Therefore $T$ is a UM$\mathcal{T}$-domain by Proposition 3.4. Moreover each element of $h$-$\text{Max}^{\vee}(R)$ is a $t_{R}$-ideal by (1) $\Rightarrow$ (3).

(2) $\Rightarrow$ (3) is trivial. □

A homogeneous overring $T$ of $R$ is called a **homogeneously $t$-linked overring** of $R$ if, it is homogeneously $(t_{R}, t_{T})$-linked overring of $R$.

**Corollary 3.10.** A graded integral domain $R$ is a UM$\mathcal{T}$-domain if and only if each homogeneously $t$-linked overring of $R$ is a UM$\mathcal{T}$-domain.
The following corollary shows that a graded integral domain $R$ is a gr-quasi-
Prüfer domain if and only if it is a UM$T$-domain and $d_R$ and $w_R$ coincide on
homogeneous ideals.

**Corollary 3.11.** Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain. Then the
following statements are equivalent:

1. $R$ is a gr-quasi-Prüfer domain.
2. Each homogeneous overring $T$ of $R$ is a UM$T$-domain and each element
   of $h$-Max($R$) is a $t$-ideal.
3. $R$ is a UM$T$-domain and each element of $h$-Max($R$) is a $t$-ideal.
4. $R$ is a UM$T$-domain and, $d_R$ and $w_R$ coincide on nonzero homogeneous
   ideals.

From Corollary 3.11, and the fact that height one primes are $t$-ideals we
can show that if $R$ is a one dimensional graded integral domain, then $R$ is a
gr-quasi-Prüfer domain if and only if $R$ is a quasi-Prüfer domain. But it is not
the case in general, see Example 3.14(2).

**Lemma 3.12.** Let $\ast$ be a (semi)star operation on a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$. Then the
following statements are equivalent:

1. $R$ is a GP$\ast$MD.
2. $R$ is a P$v$MD and $\tilde{\ast}$ and $t$ coincide on homogeneous ideals.
3. $R$ is a P$v$MD and $\ast f$ and $t$ coincide on homogeneous ideals.

**Proof.** (1) $\Rightarrow$ (2). Since $R$ is a GP$\ast$MD, and $\ast \leq v$, one has $R$ is a GP$v$MD by [32], and hence $R$ is a P$v$MD by [1, Theorem 6.4] (or Corollary 3.3). Also $h$-Spec$^f(R) \subseteq h$-Spec$^v(R)$. On the other hand if $P \in h$-Spec$^v(R)$, then $R_P$

**Corollary 3.13.** Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$. Set $\tilde{R} = (\tilde{R})^{w_R}$ and let $\tilde{\iota} : R \hookrightarrow \tilde{R}$ be the canonical embedding. Then the following statements are equivalent:

1. $R$ is a gr-$t$-quasi-Prüfer (or a UM$T$-)domain.
2. $\tilde{R}$ is a GP$(w_R)^{t}$MD.
3. $\tilde{R}$ is a P$(w_R)^{t}$MD.
4. $\tilde{R}$ is a P$v_{\tilde{R}}$MD and $(w_R)^{t}$ and $w_{\tilde{R}}(= t_{\tilde{R}})$ coincide on homogeneous ideals.
5. $\tilde{R}$ is a P$v_{\tilde{R}}$MD and $(w_R)^t = w_{\tilde{R}} = t_{\tilde{R}}$.

**Proof.** (1) $\Leftrightarrow$ (2) is true by Theorem 2.9, (2) $\Leftrightarrow$ (4) holds by Lemma 3.12, and (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) holds by [9, Corollary 2.18].
In the following we give an example of a gr-quasi-Pr"ufer domain which is not a quasi-Pr"ufer domain.

Example 3.14. (1) Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$. If $d_R$ and $w_R$ coincide on homogeneous ideals, then we do not have necessarily $d_R = w_R$. Let $D$ be an integral domain, $X$ be an indeterminate over $D$, and $R := D[X, X^{-1}]$. It is shown in [5, Example 3.6], that $R$ is a graded-Pr"ufer domain if and only if $D$ is a Pr"ufer domain and $R$ is a Pr"ufer domain if and only if $D$ is a field. Assume further that $D$ is a non-field Pr"ufer domain. Then $R$ is a graded-Pr"ufer domain. Thus by Lemma 3.12, $R$ is a P$\nu$MD and $d_R$ and $w_R$ coincide on homogeneous ideals. If $d_R = w_R$, then $R$ must be a Pr"ufer domain, and so $D$ is a field, a contradiction.

(2) Assume that $D$ is a non-Pr"ufer quasi-Pr"ufer domain (e.g. $D = K[Y^2, Y^3]$ for a field $K$ and $Y$ an indeterminate over $K$) and set $R := D[X, X^{-1}]$. Then $\overline{R} := \overline{D}[X, X^{-1}]$ is a graded-Pr"ufer domain and so $R$ is a gr-quasi-Pr"ufer domain by Corollary 2.10. Now if $R$ is a quasi-Pr"ufer domain, we have $\overline{D}$ is a field which implies that $D$ is a field which is a contradiction.

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