DENSITY OF THE HOMOTOPY MINIMAL PERIODS OF MAPS ON INFRA-SOLVMANIFOLDS OF TYPE (R)

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ABSTRACT. We study the homotopical minimal periods for maps on infra-solvmanifolds of type (R) using the density of the homotopical minimal period set in the natural numbers. This extends the result of [10] from flat manifolds to infra-solvmanifolds of type (R). We give some examples of maps on infra-solvmanifolds of dimension three for which the corresponding density is positive.

1. Introduction

Let $f : X \to X$ be a self-map on a topological space $X$. We define the following: The set of periodic points of $f$ with minimal period $n$

$$P_n(f) = \text{Fix}(f^n) - \bigcup_{k<n} \text{Fix}(f^k)$$

and the set of homotopy minimal periods of $f$

$$\text{HPer}(f) = \bigcap_{g \simeq f} \{n \in \mathbb{N} \mid P_n(g) \neq \emptyset\}.$$ 

The famous Šarkovskiǐ theorem characterizes the dynamics (minimal periods) of a map of interval [32]. The set of minimal periods of maps on the circle has been completely described in [2]. This led to a problem of study the set of homotopy minimal periods of self-maps. Such an invariant gives an information about rigid dynamics of self-maps. A fundamental question is to determine if the set $\text{HPer}(f)$ of homotopy minimal periods is empty, finite or infinite. This problem was successfully studied in [18] when the space is a torus of any dimension, and this was extended in [14] (see also [15, 27]) to any nilmanifold,
and in [12, 28] and [19] to the special solvmanifolds modeled on Sol³ and Sol⁴ respectively. When \( X \) is the Klein bottle, the same problem was studied in [13, 21, 23, 30], and when \( X \) is an infra-nilmanifold and \( f \) is an expanding map, it was shown in [7, 24, 26, 33] that \( \text{HPer}(f) \) is co-finite.

It is now natural to seek for more information when \( \text{HPer}(f) \) becomes infinite. When \( X \) is a flat manifold, some sufficient conditions on \( X \) and \( f \) for \( \text{HPer}(f) \) to be infinite were found in [10, 29]. For this purpose, the following invariant was considered: The lower density of the homotopy minimal periods of \( f \) is defined to be ([10, Definition 1.1] and [16, Remark 3.1.60])

\[
\text{DH}(f) = \liminf_{n \to \infty} \frac{\#(\text{HPer}(f) \cap [0,n])}{n}.
\]

From the definition, \( \text{DH}(f) \in [0, 1] \). If \( \text{HPer}(f) \) is either empty or finite, then \( \text{DH}(f) = 0 \). So, we are interested in the case when \( \text{HPer}(f) \) is infinite. If one picks randomly a natural number, \( \text{DH}(f) \) is a lower bound for the probability of choosing number in \( \text{HPer}(f) \). Thus, the real number \( \text{DH}(f) \) will bring to us more information about the periods of given map \( f \) when \( \text{HPer}(f) \) is infinite.

The purpose of this paper is to study the lower density of homotopy minimal periods of maps infra-solvmanifolds of type (R). This extends the results in [10] from flat manifolds to infra-solvmanifolds of type (R). We give some examples of maps on infra-solvmanifolds of dimension three for which the corresponding density is positive.

2. Infra-solvmanifolds

Let \( S \) be a connected and simply connected solvable Lie group. A discrete subgroup \( \Gamma \) of \( S \) is a lattice of \( S \) if \( \Gamma \backslash S \) is compact, and in this case, we say that the quotient space \( \Gamma \backslash S \) is a special solvmanifold. Let \( \Pi \subset \text{Aff}(S) \) be a torsion-free finite extension of the lattice \( \Gamma = \Pi \cap S \). That is, \( \Pi \) fits the short exact sequence:

\[
1 \longrightarrow S \longrightarrow \text{Aff}(S) \longrightarrow \text{Aut}(S) \longrightarrow 1
\]

Then \( \Pi \) acts freely on \( S \) and the manifold \( \Pi \backslash S \) is called an infra-solvmanifold. The finite group \( \Phi = \Pi / \Gamma \) is the holonomy group of \( \Pi \) or \( \Pi \backslash S \). It sits naturally in \( \text{Aut}(S) \). Thus every infra-solvmanifold \( \Pi \backslash S \) is finitely covered by the special solvmanifold \( \Gamma \backslash S \). An infra-solvmanifold \( \Pi \backslash S \) is of type (R) if \( S \) is of type (R) or completely solvable, i.e., if \( \text{ad} X : \mathfrak{S} \to \mathfrak{S} \) has only real eigenvalues for all \( X \) in the Lie algebra \( \mathfrak{S} \) of \( S \). It is known that if \( S \) is of type (R), then \( \exp : \mathfrak{S} \to S \) is surjective. The abelian groups \( \mathbb{R}^n \) and the connected, simply connected nilpotent Lie groups are of type (R). Hence the flat manifolds and the infra-nilmanifolds are examples of infra-solvmanifolds of type (R).

We first recall the following:
Lemma 2.1 ([25, Lemma 2.1]). Let $S$ and $S'$ be simply connected solvable Lie groups, and let $\Pi \subset \text{Aff}(S)$ and $\Pi' \subset \text{Aff}(S')$ be finite extensions of lattices $\Gamma = \Pi \cap S$ of $S$ and $\Gamma' = \Pi' \cap S'$ of $S'$, respectively. Then there exist fully invariant subgroups $\Lambda \subset \Gamma$ and $\Lambda' \subset \Gamma'$ of $\Pi$ and $\Pi'$ respectively, which are of finite index, so that any homomorphism $\theta : \Pi \to \Pi'$ restricts to a homomorphism $\Lambda \to \Lambda'$.

When the infra-solvmanifolds are of type (R), we have the following second Bieberbach type result.

Theorem 2.2 ([20, Theorem 2.3]).

1. Any continuous map $f : \Pi \setminus S \to \Pi' \setminus S'$ between infra-solvmanifolds of type (R) has an affine map $(d, D) : S \to S'$ as a homotopy lift.

2. Any continuous map $f : \Gamma \setminus S \to \Gamma' \setminus S'$ between special solvmanifolds of type (R) has a Lie group homomorphism $D : S \to S'$ as a homotopy lift.

When $f$ is a homeomorphism, $D$ can be chosen to be invertible.

Let $f : \Pi \setminus S \to \Pi \setminus S$ be a self-map on the infra-solvmanifold $\Pi \setminus S$ of type (R) with affine homotopy lift $(d, D) : S \to S$. Since $\text{HPer}(f)$ is a homotopy invariant, we may assume that $f$ is induced by the affine map $(d, D)$. The map $f$ induces a homomorphism $\varphi : \Pi \to \Pi$ on the group $\Pi$ of covering transformations of the covering projection $S \to \Pi \setminus S$, which is given by

$$(*) \quad \varphi(\alpha)(d, D) = (d, D)\alpha, \quad \forall \alpha \in \Pi.$$

For any $(a, A) \in \Phi$, let $\varphi(a, A) = (a', A')$; then $A'D = DA$. Thus the homomorphism $\varphi$ induces a function $\bar{\varphi} : \Phi \to \Phi$ given by $\bar{\varphi}(A) = A'$ and this function satisfies $\bar{\varphi}(A)D = DA$ for all $A \in \Phi$. However, in general, $\bar{\varphi}$ is not necessarily a homomorphism.

Recall further that:

Theorem 2.3 ([11, Theorem 6.1]). Let $f : M \to M$ be a self-map on a compact PL-manifold of dimension $\geq 3$. Then $f$ is homotopic to a map $g$ with $P_n(g) = \emptyset$ if and only if $N\text{Per}(f) = 0$.

The infra-solvmanifolds of dimension 1 or 2 are the circle, the torus and the Klein bottle. Theorem 2.3 for dimensions 1 and 2 is verified respectively in [1], [2] and [13, 21, 30]. Immediately we have for any self-map $f$ on an infra-solvmanifold of any dimension,

$$\text{HPer}(f) = \{ k \mid N\text{Per}(f) \neq 0 \}.$$

Recalling from [17] that

$$N\text{Per}(f) = \text{(number of irreducible essential orbits of Reidemeister classes of } f^n) \times n,$$

we have

$$\text{HPer}(f) = \{ k \mid \exists \text{ an irreducible essential fixed point class of } f^k \}.$$
Recall from [6, Propositions 9.1 and 9.3] the following: Let \( f \) be a map on an infra-solvmanifold \( \Pi \setminus S \) of type (R) induced by an affine map \( (d, D) : S \to S \).

For any \( \alpha \in \Pi \), \( \text{Fix}(\alpha(d, D)) \) is an empty set or path connected. Hence every nonempty fixed point class of \( f \) is path connected, and every isolated fixed point class forms an essential fixed point class with index \( \pm \det(I - A_D^x) \) where \( \alpha = (a, A) \). When the infra-solvmanifold \( \Pi \setminus S \) is of type (R), the converse also holds. Namely, every essential fixed point class of \( f \) consists of a single element. Remark that \( (d, D)^k \) induces the map \( f^k \). Any fixed point class of \( f^k \) is of the form \( p(\text{Fix}(\alpha(d, D)^k)) \) for some \( \alpha = (a, A) \in \Pi \). It is essential if and only if it consists of a single element with index \( \pm \det(I - A_D^x) \).

Note further that it is reducible to \( \ell \) if and only if \( \ell \mid k \) and there exists \( \beta \in \Pi \) such that \( p(\text{Fix}(\beta(d, D)^k)) \subset p(\text{Fix}(\alpha(d, D)^k)) \), or equivalently, the Reidemeister class \([\beta]\) of \( f^k \) is boosted up to the Reidemeister class \([\alpha]\) of \( f^{k/\ell} \). This means that \( [\alpha] = [\beta] \varphi^k(\beta) \varphi^{2k}(\beta) \cdots \varphi^{k-\ell}(\beta) \) as the Reidemeister class of \( f^k \). For some \( \gamma \in \Pi \), we thus have \( \alpha = \gamma(\beta) \varphi^k(\gamma) \varphi^{2k}(\gamma) \cdots \varphi^{k-\ell}(\gamma) \varphi^k(\gamma)^{-1} \).

Hence

\[
\alpha = (\gamma/\beta) \varphi^k(\gamma)^{-1}(\varphi^k(\gamma) \varphi^{2k}(\gamma)^{-1}) \cdots (\varphi^{k-\ell}(\gamma) \varphi^k(\gamma)^{-1})
\]

with \( \beta' = \gamma/\beta \). Consequently, the fixed point class \( p(\text{Fix}(\alpha(d, D)^k)) \) is irreducible if and only if for any \( \beta \in \Pi \) and for any \( \ell < k \) with \( \ell \mid k \),

\[
\alpha(d, D)^k \neq (\beta(d, D)^{k/\ell})^{k/\ell}
\]

or

\[
\alpha \neq \beta \varphi^k(\beta) \varphi^{2k}(\beta) \cdots \varphi^{k-\ell}(\beta).
\]

For any endomorphism \( D \) on \( S \), we denote the differential of \( D : S \to S \) by \( D_* : \mathfrak{g} \to \mathfrak{g} \). Now, in conclusion, we can summarize the above observation as follows:

**Theorem 2.4.** Let \( f : \Pi \setminus S \to \Pi \setminus S \) be a self-map on the infra-solvmanifold \( \Pi \setminus S \) of type (R) with an affine homotopy lift \( (d, D) : S \to S \). Let \( \varphi : \Pi \to \Pi \) be the homomorphism induced by \( (d, D) \), i.e., \( \varphi(\alpha)(d, D) = (d, D)\alpha \forall \alpha \in \Pi \). Then

\[
\text{HPer}(f) = \left\{ \begin{array}{ll}
\exists \alpha = (a, A) \in \Pi, \text{ such that } \det(I - A_D^x) \neq 0 \text{ and } \\
\forall \ell < k \text{ with } \ell \mid k, \forall \beta \in \Pi, \\
\alpha(d, D)^k \neq (\beta(d, D)^{k/\ell})^{k/\ell}
\end{array} \right\}
\]

In order to generalize the results of [10] from flat manifolds to infra-solvmanifolds of type (R), we need the following observation which is crucial in our discussion.
Lemma 2.5. Let \( \Lambda \) be a lattice of a connected, simply connected solvable Lie group \( S \) of type \((R)\), and let \( K : S \to S \) be a Lie group homomorphism such that \( K(\Lambda) \subset \Lambda \). For some choice of a linear basis in the Lie algebra \( \mathfrak{S} \) of \( S \), \( K_* \) is an upper block triangular matrix with diagonal blocks integer matrices; in particular \( \det K_* \) is an integer.

Proof. First we assume that \( S \) is nilpotent and thus \( \Lambda \) is a finitely generated torsion-free nilpotent group. The lower central series of \( \Lambda \) is defined inductively via \( \gamma_1(\Lambda) = \Lambda \) and \( \gamma_i+1(\Lambda) = [\Lambda, \gamma_i(\Lambda)] \). The isolator of a subgroup \( H \) of \( \Lambda \) is defined by

\[
\Lambda^\sqrt{H} = \{x \in \Lambda \mid x^k \in H \text{ for some } k \geq 1\}.
\]

It is known ([31, p. 473], [4, Chap. 1] or [22]) that the sequence

\[
\Lambda = \Lambda_1 \supset \Lambda_2 = \sqrt{\gamma_2(\Lambda)} \supset \cdots \supset \Lambda_c = \sqrt{\gamma_c(\Lambda)} \supset \Lambda_{c+1} = 1
\]

forms a central series with \( \Lambda_i/\Lambda_{i+1} \cong \mathbb{Z}^{k_i} \).

Now we can choose a generating set \( a = \{a_1, \ldots, a_c\} \) in such a way that \( \Lambda_i \) is the group generated by \( \Lambda_{i+1} \) and \( a_i = \{a_{i1}, \ldots, a_{in_i}\} \). We refer to \( a = \{a_1, \ldots, a_c\} \) as a preferred basis of \( \Lambda \). Under the diffeomorphism \( \log : S \to S \), the image \( \log a \) of \( a \) is a basis of the vector space \( \mathfrak{S} \). Note also that \( \Lambda_i = \Lambda \cap \gamma_i(S) \) is a lattice of \( \gamma_i(S) \) and a fully invariant subgroup of \( \Lambda \). Since \( K(\Lambda) \subset \Lambda \), it follows that \( K(\Lambda_i) \subset \Lambda_i \) and the differential of \( K \) is expressed as a rational matrix with respect to the basis \( \log a \) of the form

\[
\begin{bmatrix}
K_{c*} & * & \cdots & * \\
0 & K_{c-1*} & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & K_{1*}
\end{bmatrix},
\]

where each square matrix \( K_{i*} \) is an integer matrix, see also [24, Lemma 4.2].

Now we go back to the cases where \( S \) is solvable of type \((R)\). According to [34, Remark 8.2], \( \Lambda \) is a positive polycyclic group and \( S \) is its supersolvable completion. Let \( N \) be the maximal connected nilpotent normal subgroup of \( S \). Then \( \Lambda \cap N \) is the nilradical \( \text{nil}(\Lambda) \) of \( \Lambda \), which is a lattice of \( N \), see [34, Proposition 5.1]. Hence we have the following diagram:

\[
\begin{array}{ccccccc}
1 & \longrightarrow & N & \longrightarrow & S & \longrightarrow & S/N \cong \mathbb{R}^s & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
1 & \longrightarrow & \text{nil}(\Lambda) & \longrightarrow & \Lambda & \longrightarrow & \Lambda/\text{nil}(\Lambda) \cong \mathbb{Z}^s & \longrightarrow & 1.
\end{array}
\]

By the assumption on \( K \), \( K \) restricts to a homomorphism \( \kappa : \Lambda \to \Lambda \). Thus \( \kappa \) and hence \( K \) in turn restricts to \( \kappa' : \text{nil}(\Lambda) \to \text{nil}(\Lambda) \) and then induces a homomorphism \( \overline{\kappa} : \mathbb{Z}^s \to \mathbb{Z}^s \). We choose a preferred basis of \( \text{nil}(\Lambda) \) under which \( K' : N \to N \) yields a rational matrix \( K'_* \) with diagonal blocks integer matrices as above. Now we can complete the set of generators of \( \text{nil}(\Lambda) \) to
a set of generators $a = \{a_0, a_1, \ldots, a_c\}$, called a \textit{preferred basis}, of $\Lambda$ so that $\kappa$ induces an integer matrix $K_\ast$ and so $\kappa$ induces an upper block triangular matrix

$$K_\ast = \begin{bmatrix} K' \ast & * \\ 0 & K_\ast \end{bmatrix}$$

so that all diagonal blocks are integer matrices and hence $\det K_\ast$ is an integer.

\[\Box\]

\textbf{Remark 2.6.} Let $\Lambda$ be a lattice of a connected, simply connected solvable Lie group $S$ of type (R). In the proof of the above lemma, we can choose a preferred basis (generator) $a$ of $\Lambda$ so that $\log a$ is a (linear) basis of the Lie algebra $S$ of $S$ and if $K$ is a homomorphism on $S$ such that $K(\Lambda) \subset \Lambda$, then $K_\ast$ is an upper block triangular matrix with diagonal blocks integer matrices with respect to the ordered basis $\log a$. We also refer to $\log a$ as a \textit{preferred basis} of $\Lambda$ or $S$.

\section{Density of homotopy minimal periods}

In this section, we will generalize the main result of \cite{10} from flat manifolds to infra-solvmanifolds of type (R).

Let $f$ be a self-map on an infra-solvmanifold $\Pi \setminus S$ of type (R) with holonomy group $\Phi$. Let $f$ have an affine homotopy lift $(d, D)$. Recall that $f$ induces a homomorphism $\varphi : \Pi \to \Pi$ satisfying the identity (\star): $\varphi(\alpha)(d, D) = (d, D)\alpha$, $\forall \alpha \in \Pi \subset S \rtimes \text{Aut}(S)$. Let $\Gamma = \Pi \cap S$. It is not necessarily true that $\varphi(\Gamma) \subset \Gamma$. Using Lemma 2.1, we can choose a lattice $\Lambda \subset \Gamma$ of $S$ so that $\varphi(\Lambda) \subset \Lambda$. Thus for any $\lambda = (\lambda, I) \in \Lambda$, we have $\varphi(\lambda) = (\varphi(\lambda), I)$ and so

$$(\varphi(\lambda), I)(d, D) = (d, D)(\lambda, I).$$

Evaluating at the identity $1$ of $S$, we obtain that $\varphi(\lambda) \cdot d = d \cdot D(\lambda)$. Consequently, we have that

$$\varphi|_{\Lambda} = \mu(d)D.$$

Furthermore, for any $(a, A) \in \Pi$, since $\Gamma$ is a normal subgroup of $\Pi$, we have $(a, A)(\gamma, I)(a, A)^{-1} \in \Gamma$: this implies $(\mu(a)A)(\Gamma) \subset \Gamma$ and $(\mu(a)A)(\Lambda) \subset \Lambda$. Consequently, we have homomorphisms $\mu(d)D, \mu(a)A : S \to S$ such that $(\mu(d)D)(\Lambda) \subset \Lambda$ and $(\mu(a)A)(\Lambda) \subset \Lambda$. We have to notice here that it is not necessary to have that $D(\Lambda), A(\Lambda) \subset \Lambda$. By Remark 2.6, we can choose a preferred basis $a$ of $\Lambda$ so that $(\mu(d)D)_* = \text{Ad}(d)D_*$ and $(\mu(a)A)_* = \text{Ad}(a)A_*$ are upper block triangular rational matrices with diagonal blocks integer matrices with respect to the basis $\log a$ of $\mathcal{S}$.

In what follows, we shall denote $\mu(d)D$ and $\mu(a)A$ by $D$ and $A$, respectively. By Lemma 2.5, the differentials of $D$ and $A$ induce rational matrices with integer blocks on the diagonal. By considering only integer blocks on the diagonal, we
obtain integer matrices, denoted by $D_*$ and $A_*$. Hence,

$$D_* = \begin{bmatrix}
D_{c_*} & 0 & \cdots & 0 \\
0 & D_{c-1_*} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & D_{0_*}
\end{bmatrix}.$$ 

This does not change the determinant and the eigenvalue of the differentials of $D$ and $A$. We can call $D_*$ and $A_*$ linearizations of $D$, $(d, D)$ or $f$, and $\alpha = (a, A) \in \Pi$, respectively. We denote the free abelian group of all integer linear combinations of the basis vectors in $\log a = \{\log a_1, \log a_{c-1}, \ldots, \log a_0\}$ by simply $Z = Z_0 \oplus Z_{c-1} \oplus \cdots \oplus Z_0$. Then we have $D_*(Z) \subset Z$, $D_*(Z) \subset Z$ and $A_*(Z) \subset Z$.

In the following, we provide three lemmas that generalize [10, Lemmas 4.3 and 4.5, Proposition 4.6] from flat manifolds to infra-solvmanifolds of type (R). These are essential in proving our main results.

**Lemma 3.1.** Let $M = \Pi \setminus S$ be an infra-solvmanifold of type (R). Let $f$ be a self-map on $M$ with an affine homotopy lift $(d, D)$. Assume that

1. any eigenvalue $\lambda$ of $D_*$ of modulus $1$ is a root of unity, but not $1$;
2. $\det D_* \neq 0, \pm 1$.

Then there exists a positive integer $N_0$ such that

$$|\det \left( \frac{I - D_*^{k\ell}}{I - D_*^\ell} \right)| > 1$$

for all positive integers $k$ and $\ell$, provided their prime divisors are all greater than $N_0$.

**Proof.** Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of $D_*$ counted with multiplicities. We first show that all $1 - \lambda_i^\ell$ are nonzero. In fact, if $1 - \lambda_i^\ell = 0$ for some $i$, then $\lambda_i = 1$ and so $\lambda_i$ is a primitive $\ell_0$-th root of unity for some $\ell_0$ where $1 \leq \ell_0 | \ell$. Since $\lambda_i \neq 1$, $\ell_0 > 1$. If $p$ is a prime divisor of $\ell_0$, then it is a prime divisor of $\ell$ and so $p > N_0$. It follows that $[Q(\lambda_i) : Q] > p - 1 > m$. This contradicts the fact that $[Q(\lambda_i) : Q]$ is smaller than the size $m$ of $D_*$. Thus $I - D_*^\ell$ is invertible and $(I - D_*^{k\ell})/(I - D_*^\ell) = I + D_*^\ell + \cdots + D_*^{(k-1)\ell}$ and

$$\det \left( \frac{I - D_*^{k\ell}}{I - D_*^\ell} \right) = \prod_{i=1}^{m} \frac{1 - \lambda_i^{k\ell}}{1 - \lambda_i^\ell}.$$ 

Let $N_0 > m + 1$ and let $\ell$ be a positive integer all of whose prime divisors are greater than $N_0$. Assume $|\lambda_i| = 1$. By our assumption, $\lambda_i \neq 1$ and is a root of unity. The above argument shows that $1 - \lambda_i^\ell \neq 0$ and by the same reasoning $1 - \lambda_i^{k\ell} \neq 0$; thus $(1 - \lambda_i^{k\ell})/(1 - \lambda_i^\ell)$ is nonzero and finite for each such $k$ and $\ell$. Hence we can choose a constant $\delta > 0$ such that for all such $k$ and $\ell$

$$\left| \frac{1 - \lambda_i^{k\ell}}{1 - \lambda_i^\ell} \right| > \delta.$$
For \( \lambda_i \) with \( |\lambda_i| \neq 1 \), as \( N_0 \to \infty \), we have

\[
\left| \frac{1 - \lambda_i^k}{1 - \lambda_i^\ell} \right| = |1 + \lambda_i^\ell + \lambda_i^{2\ell} + \cdots + \lambda_i^{k-1}| \to \begin{cases} 
1 & \text{when } |\lambda_i| < 1 \\
\infty & \text{when } |\lambda_i| > 1.
\end{cases}
\]

By the assumption that \( |\det D_*| > 1 \), there exists an eigenvalue whose absolute value is bigger than 1. Hence as \( N_0 \to \infty \) we have

\[
\prod_{i=1}^m \left| \frac{1 - \lambda_i^k}{1 - \lambda_i^\ell} \right| \to \infty.
\]

Consequently, for \( N_0 \) large enough, the lemma is proved.

\[\square\]

**Lemma 3.2.** Let \( M = \Pi \setminus S \) be an infra-solvmanifold of type \((R)\). Let \( f \) be a self-map on \( M \) with an affine homotopy lift \((d,D)\). Assume that

1. any eigenvalue \( \lambda \) of \( D_* \) of modulus 1 is a root of unity, but not 1;
2. \( \det D_* \neq 0, \pm 1 \).

Then there exists a positive integer \( N_1 \) such that

\[
|\det(I - D_*^k)| > \sum_{1 < \ell | k} |\det(I - D_*^{k/\ell})|
\]

for all positive integers \( k \), provided all its positive prime divisors are greater than \( N_1 \).

**Proof.** Let \( \lambda_1, \ldots, \lambda_m \) be the eigenvalues of \( D_* \) counted with multiplicities. From our assumptions and hence from the observations in the proof of Lemma 3.1, we have:

- Since \( \det D_* \neq 0 \), all \( \lambda_i \) are nonzero.
- If \( |\lambda_i| = 1 \), then \( \lambda_i \neq 1 \) and \( \lambda_i \) is a root of unity, and \( 1 - \lambda_i^k \neq 0 \) for all \( k \) whose prime divisors are \( > N_0 \) where \( N_0 \) is a positive integer chosen in the previous lemma; hence there are constants \( 0 < \delta_1 < \delta_2 \) such that for all \( \lambda_i \) with \( |\lambda_i| \leq 1 \), we have \( \delta_1 \leq |1 - \lambda_i^k| \leq \delta_2 \) for all \( k \) with this property.

For those eigenvalues with \( |\lambda_i| > 1 \), we claim that there is a sufficiently large \( k \) such that

\[
\sum_{1 < \ell | k} |1 - \lambda_i^{k/\ell}| < |1 - \lambda_i^k|.
\]

Suppose on the contrary that for any \( K > 0 \) there is \( k_0 > K \) such that

\[
|1 - \lambda_i^{k_0}| \leq \sum_{1 < \ell | k_0} |1 - \lambda_i^{k_0/\ell}|.
\]

Then

\[
|1 - \lambda_i^{k_0}| \leq \sum_{1 < \ell | k_0} |1 - \lambda_i^{k_0/2}| < \tau(k_0)|1 - \lambda_i^{k_0/2}|,
\]
where \( \tau(k) \) is the number of all the divisors of \( k \). Since \( \tau(k) \leq 2\sqrt{k} \) (see [16, Exercise 3.2.17]), we have

\[
2\sqrt{k_0} > |1 + \lambda_i^{2n/2}| \geq |\lambda_i|^{k_0/2} - 1,
\]

which contradicts the obvious fact that \( \lim_{k \to 0} \sqrt{k}/(|\lambda_i|^{k/2} - 1) = 0 \).

Therefore we can choose \( N_1 \geq N_0 \) such that if \( k \) is a positive integer whose prime divisors are \( \geq N_1 \), then

\[
\sum_{1 < \ell | k} |\det(I - D_{k/\ell}^\ell)| = \sum_{1 < \ell | k} \left( \prod_{i=1}^m |1 - \lambda_i^{k/\ell}| \right)
\]

\[
\leq \prod_{i=1}^m \left( \sum_{1 < \ell | k} |1 - \lambda_i^{k/\ell}| \right)
\]

\[
< \prod_{i=1}^m |1 - \lambda_i^k| = |\det(I - D_k^k)|.
\]

\[\square\]

**Lemma 3.3.** Let \( M = \Pi \backslash S \) be an infra-solvmanifold of type (R). Let \( f \) be a self-map on \( M \) with an affine homotopy lift \((d,D)\). Assume that

1. any eigenvalue \( \lambda \) of \( D_* \) of modulus 1 is a root of unity, but not 1;
2. \( \det D_* \neq 0, \pm 1 \).

Then there exists a positive integer \( N_2 \) such that the equality

\[
\mathcal{Z} = \bigcup_{1 < \ell | k} (I + D_{\ell}^\ell + \cdots + D_k^{k-\ell})(\mathcal{Z})
\]

is impossible for all positive integers \( k \), provided its positive prime divisors are all greater than \( N_2 \).

**Proof.** Remark that the proof of Lemma 3.1 shows that there exists a positive integer \( N_0 \) such that for all positive integers \( k \) whose prime divisors are greater than \( N_0 \), \( I - D_k^k \) has nonzero determinant if \( \ell | k \). Since \( D_*(\mathcal{Z}) \subset \mathcal{Z} \), we have

\[
(I - D_k^k)(\mathcal{Z}) = (I + D_{\ell}^\ell + \cdots + D_k^{k-\ell})(I - D_k^k)(\mathcal{Z})
\]

\[
\subset (I + D_{\ell}^\ell + \cdots + D_k^{k-\ell})(\mathcal{Z})
\]

for all \( k \) and \( \ell \) with \( \ell | k \). Thus if we had the equality

\[
\mathcal{Z} = \bigcup_{\ell | k, 1 < \ell < k} (I + D_{\ell}^\ell + \cdots + D_k^{k-\ell})(\mathcal{Z})
\]

we would have

\[
\mathcal{Z}/(I - D_k^k)(\mathcal{Z}) = \left( \bigcup_{\ell | k, \ell \neq 1, k} (I + D_{\ell}^\ell + \cdots + D_k^{k-\ell})(\mathcal{Z}) \right)/(I - D_k^k)(\mathcal{Z})
\]

\[
= \bigcup_{\ell | k, \ell \neq 1, k} ((I + D_{\ell}^\ell + \cdots + D_k^{k-\ell})(\mathcal{Z})/(I - D_k^k)(\mathcal{Z})).\]
We remark that \( Z \) is a free abelian group and \( I + D^*_k + D^2 + \cdots + D^{k-\ell} \) defines an injective endomorphism of \( Z \). In particular, we have an isomorphism
\[
Z/(I - D^*_k)(Z)
\]
\[
\cong (I + D^*_k + D^2 + \cdots + D^{k-\ell})Z/(I + D^*_k + D^2 + \cdots + D^{k-\ell})(I - D^*_k)(Z)
\]
\[
= (I + D^*_k + D^2 + \cdots + D^{k-\ell})Z/(I - D^*_k)(Z).
\]
This would imply
\[
Z/(I - D^*_k)(Z) \cong \bigcup_{\ell \in k, \ell \neq 1, k} \big(Z/(I - D^*_k)(Z)\big)
\]
and hence we would have
\[
|\det(I - D^*_k)| \leq \sum_{\ell \in k, \ell \neq 1, k} |\det(I - D^*_k)| \leq \sum_{1 < \ell \in k} |\det(I - D^*_k)|.
\]
This contradicts Lemma 3.2.

Now we are ready to state and prove our main results.

**Theorem 3.4.** Let \( M = \Pi \setminus S \) be an infra-solvable manifold of type \((R)\). Let \( f \) be a self-map on \( M \) with an affine homotopy lift \((d,D)\). Let \( \varphi : \Pi \to \Pi \) be the homomorphism satisfying
\[
\varphi(\alpha)(d,D) = (d,D)\alpha, \quad \forall \alpha \in \Pi.
\]

**Assume that**

1. any eigenvalue \( \lambda \) of \( D_* \) of modulus 1 is a root of unity, but not 1;
2. \( \det D_* \neq 0, \pm 1 \);
3. \( \text{fix}(\varphi : \Phi \to \Phi) = \{ I \} \).

**Then there exists an integer \( N \) with the following property: if \( k \) is a positive integer with prime factorization \( k = p_1^{n_1} \cdots p_s^{n_s} \) such that all \( p_i \)'s are greater than \( N \), then \( k \in \text{HPer}(f) \).**

**Proof.** Choose an integer \( N \) so that \( N \geq \max\{m + 1, N_2, \text{order of } \varphi \} \). Let \( k = p_1^{n_1} \cdots p_s^{n_s} \) be a prime factorization of \( k \) such that all \( p_i \)'s are greater than \( N \). Then we have to show that \( k \in \text{HPer}(f) \). For this purpose, by Theorem 2.4, we need to find \( \alpha = (a, A) \in \Pi \) satisfying:

- \( \det(I - A_1 D^k) \neq 0 \),
- \( \forall \ell < k \) with \( \ell \mid k \), \( \forall \beta \in \Pi \), \( \alpha(d,D)^k \neq (\beta(d,D))^{k/\ell} \).

We will show that we can choose \( \alpha = (a, I) \) in \( \Gamma \subset \Pi \). Recall first that the proof of Lemma 3.1 shows that there exists a positive integer \( N_0 \) such that for all positive integers \( m \) whose prime divisors are larger than \( N_0 \), \( I - D^m \) has nonzero determinant if \( \ell \mid k \). Since \( N \geq N_0 \), \( \det(I - D^k) \neq 0 \) for all \( \ell \mid k \). In particular, \( \det(I - D^k) \neq 0 \). By [25, Lemma 3.3] [9, Theorem 1], we have \( \det(I - D^k) = \det(I - D^k) \neq 0 \).
It remains to prove the second condition. We assume on the contrary that for any $\alpha = (a,I) \in \Gamma$, there exists $\ell < k$ with $\ell \mid k$ and there exist $\beta = (b,B) \in \Pi$ such that $\alpha(d,D)^k = (\beta(d,D)^{k/\ell})^k$, which is equivalent to

\[ \alpha = \beta \varphi^k(\beta) \varphi^k - \ell(\beta). \]

Now we recall that since $D$ is an automorphism, $\varphi$ is the conjugation by $(d,D)$, $\varphi|_I = \mu(d)D$ and $\varphi$ is the conjugation by $D$. The matrix part (the holonomy part) of both sides of (1) yields

\[ I = B\varphi^k(B) \varphi^k(B) \cdots \varphi^k - \ell(B). \]

Taking $\varphi^\ell$, we have

\[ I = \varphi^\ell(B) \varphi^\ell(B) \cdots \varphi^\ell(B) \varphi(B). \]

Hence

\[ \varphi^k(B)^{-1} = B^{-1} = \varphi^\ell(B) \varphi^\ell(B) \cdots \varphi^k - \ell(B). \]

This gives us $\varphi^k(B) = B$. By the choice of $k$, $k$ must be relatively prime to the order $p$ of $\varphi$. Choose $x, y \in \mathbb{Z}$ so that $kx + py = 1$. Since $\varphi = \varphi^{kx + py} = (\varphi^k)^x$, it follows that $\varphi(B) = B$. Since $\text{fix}(\varphi) = \{I\}$ by our assumption, we have $B = I$. Plugging into (1), we have

\[ a = b \varphi^\ell(b) \varphi^\ell(b) \cdots \varphi^k - \ell(b). \]

Since $\varphi|_I = \mu(d)D = D$, we have

\[ a = b \mathbb{D}^\ell(b) \mathbb{D}^\ell(b) \cdots \mathbb{D}^k - \ell(b) \]

for some $\ell < k$ with $\ell \mid k$.

Now we have to show that for any $\ell < k$ with $\ell \mid k$

\[ \{e \mathbb{D}^\ell(e) \mathbb{D}^\ell(e) \cdots \mathbb{D}^k - \ell(e) \mid e \in \Gamma\} \neq \Gamma. \]

Recall in the proof of Lemma 2.5 that $\Gamma$ has a central series

\[ \Gamma = \Gamma_0 \supset \text{nil}(\Gamma) = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_c \supset \Gamma_{c+1} = 1 \]

with $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}^{k_i}$. Since $\mathbb{D}(\Gamma_i) \subset \Gamma_i$, it induces $\mathbb{D}_i : \Gamma_i/\Gamma_{i+1} \to \Gamma_i/\Gamma_{i+1}$. Note also that

\[ \mathbb{D}_i = \begin{bmatrix} \mathbb{D}_0 & 0 & \cdots & 0 \\ 0 & \mathbb{D}_{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbb{D}_0 \end{bmatrix}, \]

where $\mathbb{D}_i$ are integer matrices. Hence some $\mathbb{D}_i$ satisfies the assumptions (1) and (2). By Lemma 3.3, we have

\[ \{\tilde{e} + \mathbb{D}_i(\tilde{e}) + \mathbb{D}_i^2(\tilde{e}) + \cdots + \mathbb{D}_i^{k_\ell}(\tilde{e}) \mid \tilde{e} \in \Gamma_i/\Gamma_{i+1}\} \neq \Gamma_i/\Gamma_{i+1}. \]

This proves our assertion. Hence $a \in \Gamma$ can be chosen so that

\[ a \neq b \varphi^\ell(b) \varphi^\ell(b) \cdots \varphi^k - \ell(b) \]

for any $b \in \Gamma$. This contradiction proves the second condition. \qed
Corollary 3.5. Let $M = \Pi \backslash S$ be an infra-solvmanifold of type $(R)$ and let $f$ be a self-map on $M$ with an affine homotopy lift $(d, D)$. Let $\varphi : \Pi \to \Pi$ be the homomorphism satisfying

$$\varphi(\alpha)(d, D) = (d, D)\alpha, \ \forall \alpha \in \Pi.$$ 

Assume that

1. any eigenvalue $\lambda$ of $D_*$ of modulus 1 is a root of unity, but not 1;
2. $\det D_* \neq 0, \pm 1$;
3. $\text{fix}(\varphi : \Phi \to \Phi) = \{I\}$. 

Then $DH(f)$ is positive.

Proof. By Theorem 3.4, there exists an integer $N$ with the following property: for any positive integer $k = p_1^{n_1} \cdots p_s^{n_s}$ with all $p_i$’s distinct primes and greater than $N$, $k \in \text{HPer}(f)$. Thus

$$\text{HPer}(f) \supset \{k \mid \text{any prime divisor of } k \text{ is } > N\}.$$ 

Let $q_1, \ldots, q_\ell$ be the all prime numbers which are smaller than or equal to $N$. Then the set

$$\{k \mid k \equiv 1 \mod q_1 \cdots q_\ell\}$$

is contained in the set on the right-hand side of the above. For, if $k \equiv 1 \mod q_1 \cdots q_\ell$ and if $p$ is a prime divisor of $k$ with $p \leq N$, then $p = q_j$ for some $j$; thus $q_j \mid k$ and $q_j \mid k - 1$ and hence $q_j = 1$, a contradiction.

Furthermore, we have that $N! \mid k - 1$ implies $q_1 \cdots q_\ell \mid k - 1$. This shows that

$$\text{HPer}(f) \supset \{k \mid k \equiv 1 \mod N!\}$$

and the set on the right-hand side has density $1/N!$. Consequently,

$$DH(f) \geq 1/N! > 0. \quad \square$$

A special solvmanifold is an infra-solvmanifold with the trivial holonomy group. Hence the third condition of Corollary 3.5 on such a manifold is automatically fulfilled. Immediately we have:

Corollary 3.6. Let $f$ be a self-map on a special solvmanifold $M$ with a Lie group homomorphism $D$ as a homotopy lift. Assume that

1. any eigenvalue $\lambda$ of $D_*$ of modulus 1 is a root of unity, but not 1;
2. $\det D_* \neq 0, \pm 1$.

Then $DH(f)$ is positive.

4. Computational results

In this section, we will consider some examples on infra-solvmanifolds up to dimension three. For infra-solvmanifolds up to dimension 3, there are only three possibilities for the solvable Lie group $G$ on which the manifold is modeled. It can be modeled on either the abelian groups $\mathbb{R}^n (n \leq 3)$, the 2-step nilpotent Heisenberg group Nil or the 2-step solvable Lie group Sol.
We can find a complete description of $H\text{Per}(f)$ for maps $f$ on tori in [2] and [18], and on the Klein bottle in [21]. The remaining infra-solvmanifolds of dimension 3 are three-dimensional flat manifolds, infra-nilmanifolds on Nil and infra-solvmanifolds of Sol.

We will give three examples, one from each remaining manifold. For any self-map $f$ on the manifold $\Pi \backslash G$, let $\varphi : \Pi \to \Pi$ be a homomorphism induced by $f$. Consider an affine map $(d, D)$ on $G$ satisfying ($\ast$). To apply Corollary 3.5, we have to consider the case where $D = \mu(d)D$ is invertible. If this is the case, then ($\ast$) says that $\varphi$ is the conjugation by $(d, D)$, that is, $\varphi(\alpha) = (d, D)\alpha(d, D)^{-1}$.

If $\alpha = (a, I) \in \Gamma$, then

$$\varphi(\alpha) = (d, D)(a, I)(d, D)^{-1} = (dD(a)d^{-1}, I) = (\mu(d)D(a), I).$$

Here $\mu(d)$ is the automorphism on $G$ obtained by conjugating by the element $d \in G$. Thus $\varphi(\Gamma) \subseteq \Gamma$ and $\varphi|_{\Gamma} = \mu(d)D = D$, and hence $\bar{\varphi}$ is the conjugation by $D$. In particular $\bar{\varphi} : \Phi \to \Phi$ is an isomorphism.

We start with the following easy observation.

**Lemma 4.1.** Let $\Phi$ be a group with presentation

$$\Phi = \langle x, y \mid x^2 = y^2 = 1, xy = yx \rangle,$$

and let $\psi$ be an isomorphism on $\Phi$. Then $\text{fix}(\psi) = 1$ if and only if $\psi$ satisfies one of the following:

- $\psi(x) = y, \psi(y) = xy$
- $\psi(x) = xy, \psi(y) = x$

### 4.1. A flat manifold of dimension three

We have a complete classification of three-dimensional Bieberbach groups. There are six orientable ones and four nonorientable ones, see the book [35, Theorems 3.5.5 and 3.5.9]. Every group has an explicit representation into $\mathbb{R}^3 \rtimes \text{GL}(4, \mathbb{Z})$ (not into $\mathbb{R}^4 \rtimes O(4)$) in this book. Of course one of them is $\mathfrak{G}_1 = \mathbb{Z}^3$. Let

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and let $t_i = (e_i, I) \in \mathbb{R}^3 \rtimes \text{GL}(3, \mathbb{Z})$. Then $t_1, t_2$ and $t_3$ generate the subgroup $\Gamma$ of $\mathbb{R}^3 \rtimes \text{GL}(3, \mathbb{Z})$, which is isomorphic to the group of all integer vectors of $\mathbb{R}^3$.

Let $\alpha = (a, A), \beta = (b, B)$ and $\gamma = (c, C)$ be elements of $\mathbb{R}^3 \rtimes \text{GL}(3, \mathbb{Z})$,

where

$$a = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad c = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix},$$
Then $A, B, C$ have order 2 and $AB = C = BA$, and
\[
\mathfrak{G}_6 = \left\{ (t_1, t_2, t_3, \alpha, \beta, \gamma) \mid \begin{array}{c}
t_1t_2 = 1, 
\gamma \beta \alpha = t_1t_3, 
\alpha^2 = t_1, \alpha t_2 \alpha^{-1} = t_2^{-1}, \alpha t_3 \alpha^{-1} = t_3^{-1}, 
\beta t_1 \beta^{-1} = t_1^{-1}, \beta t_2 \beta^{-1} = t_2^{-1}, \alpha \beta \gamma = t_3^{-1}, 
\end{array} \right\}.
\]
Thus $\mathfrak{G}_6$ fits the short exact sequence
\[1 \longrightarrow \Gamma \longrightarrow \mathfrak{G}_6 \longrightarrow \Phi \longrightarrow 1,
\]
where $\Phi = (A, B) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Every element of $\mathfrak{G}_6$ can be written uniquely in the form $\alpha^k \beta^m \gamma^n$. We first observe the following: Since $\gamma \beta \alpha = t_1t_3$, we have $\gamma = t_3^{-1}t_1$, and
\[
\beta^m \alpha^k = \begin{cases} 
\alpha^k \beta^m & \text{when } (k, m) = (e, e) \\
\alpha^{-k} \beta^m & \text{when } (k, m) = (e, o) \\
\alpha^k \beta^{-m} & \text{when } (k, m) = (o, e) \\
\alpha^{-k} \beta^{-m} t_3 & \text{when } (k, m) = (o, o)
\end{cases}
\]
and
\[
(\alpha^k \beta^m \gamma^n)^2 = \alpha^k (\beta^m \alpha^k) \beta^m \gamma^n = \begin{cases} 
\alpha^k \beta^{2m} & \text{when } (k, m) = (e, e) \\
\beta^{2m} & \text{when } (k, m) = (e, o) \\
\alpha^k & \text{when } (k, m) = (o, e) \\
t_3^{2n-1} & \text{when } (k, m) = (o, o).
\end{cases}
\]
Let $\varphi : \mathfrak{G}_6 \rightarrow \mathfrak{G}_6$ be any homomorphism that induces an isomorphism $\bar{\varphi}$ on $\Phi$ satisfying $\text{fix}(\bar{\varphi}) = \{1\}$. By Lemma 4.1, we have either $\bar{\varphi}(\alpha) = \beta$, $\bar{\varphi}(\beta) = \alpha \beta$ or $\bar{\varphi}(\alpha) = \alpha \beta$, $\bar{\varphi}(\beta) = \alpha$.

In general, $\varphi$ has the form
\[
\varphi(\alpha) = \alpha^k_1 \beta^m_1 \gamma^n_1, \quad \varphi(\beta) = \alpha^k_2 \beta^m_2 \gamma^n_2, \quad \varphi(\gamma) = \alpha^k_3 \beta^m_3 \gamma^n_3.
\]
Since $\gamma = t_3^{-1}t_1$, a simple calculation shows that
\[
\varphi(\gamma) = \alpha^k_1 \beta^m_1 \gamma^n_1 \alpha^{-k_2} \alpha^{-k_3} \beta^m_3 \gamma^n_3 \beta^{-m_1} \gamma^n_3 \alpha^{-m_2} \gamma^n_3 \alpha^{-m_3} \gamma^n_3.
\]

Case $\bar{\varphi}(\alpha) = \beta$, $\bar{\varphi}(\beta) = \alpha \beta$.
Then $k_1$ is even and $k_2, m_1, m_2$ are odd. So, we have
\[
\varphi(\gamma) = \alpha^{k_1-k_2} \beta^{-m_1} \gamma^n_1 \alpha^{-m_2} \gamma^n_3 \alpha^{-m_3} \gamma^n_3.
\]
Since $\alpha^2 = t_1, \beta^2 = t_2$ and $\gamma^2 = t_3$, a simple calculation shows that
\[
\alpha^2 = t_1 \Rightarrow \varphi(t_1) = \varphi(\alpha)^2 = (\alpha^k_1 \beta^m_1 \gamma^n_1)^2 = \beta^{2m_1} = t_2^{m_1};
\]
\[ \beta^2 = t_2 \Rightarrow \varphi(t_2) = \varphi(\beta)^2 = (\alpha^{k_2} \beta^{m_2} t_3^{n_2})^2 = t_3^{2n_2-1}; \]
\[ \gamma^2 = t_3 \Rightarrow k_3 = 2(k_1 - k_2 + k_3), m_3 = 0, n_3 = 0. \]

Hence \( \varphi(t_3) = \alpha^{k_3} = t_3^{k_3/2} \), and it follows that \( \varphi(\Gamma) \subset \Gamma \) and so \( \mathbb{D} = \varphi|_{\Gamma} \) and

\[ \mathbb{D}_* = \begin{bmatrix} 0 & 0 & k_2 \\ m_1 & 0 & 0 \\ 0 & 2n_2 - 1 & 0 \end{bmatrix}. \]

Thus \( \det \mathbb{D}_* = \frac{k_2}{2} m_1 (2n_2 - 1) \neq 0, \pm 1 \) if and only if either \( k_3 \neq 0 \) or \( k_3 = \pm 2, m_1 = \pm 1, 2n_2 - 1 = \pm 1 \). If \( \mathbb{D}_* \) has an eigenvalue of modulus 1, then \( \det \mathbb{D}_* = \pm 1 \). This shows that the condition (2) of Corollary 3.5 implies the condition (1). Consequently, when \( k_1, k_3, m_3 \) are even and \( k_2, m_1, m_2 \) are odd, if \( k_3 \neq 0 \) or

\[ (k_3, m_1, n_2) \notin \{(2, 1, 0), (2, -1, 0), (-2, 1, 0), (-2, -1, 0), (2, 1, 1), (2, -1, 1), (-2, 1, 1), (-2, -1, 1)\}, \]
then \( \text{DH}(f) > 0. \)

**Case** \( \varphi(\alpha) = \alpha \beta, \varphi(\beta) = \bar{\alpha}. \)

Then \( k_1, k_2, m_1 \) are odd and \( m_2 \) is even. So, we have

\[ \varphi(\gamma) = \alpha^{k_1 + k_2 - k_3} \beta^{- m_1 + m_2 + m_3} t_3^{- n_1 + n_2 + n_3}. \]

Since \( \alpha^2 = t_1, \beta^2 = t_2 \) and \( \gamma^2 = t_3 \), a simple calculation shows that

\[ \alpha^2 = t_1 \Rightarrow \varphi(t_1) = \varphi(\alpha)^2 = (\alpha^{k_1} \beta^{m_1} t_3^{n_1})^2 = t_3^{2n_1-1}; \]
\[ \beta^2 = t_2 \Rightarrow \varphi(t_2) = \varphi(\beta)^2 = (\alpha^{k_2} \beta^{m_2} t_3^{n_2})^2 = \alpha^{2k_2} = t_4^{k_2}; \]
\[ \gamma^2 = t_3 \Rightarrow k_3 = 0, m_3 = 2(-m_1 + m_2 + m_3), n_3 = 0. \]

Hence \( \varphi(t_3) = \beta^{m_3} = t_2^{m_3/2} \), and it follows that \( \varphi(\Gamma) \subset \Gamma \) and so \( \mathbb{D} = \varphi|_{\Gamma} \) and

\[ \mathbb{D}_* = \begin{bmatrix} 0 & k_2 & 0 \\ 0 & 0 & \frac{m_2}{2} \\ 2n_1 - 1 & 0 & 0 \end{bmatrix}. \]

Thus \( \det \mathbb{D}_* = \frac{k_2}{2} \frac{m_2}{2} (2n_1 - 1) \neq 0, \pm 1 \) if and only if either \( m_3 \neq 0 \) or \( k_2 = \pm 1, m_3 = \pm 2, 2n_1 - 1 = \pm 1 \). If \( \mathbb{D}_* \) has an eigenvalue of modulus 1, then \( \det \mathbb{D}_* = \pm 1 \). This shows that the condition (2) of Corollary 3.5 implies the condition (1). Consequently, when \( k_1, k_2, m_1 \) are odd and \( k_3, m_2, m_3 \) are even, if \( m_3 \neq 0 \) or

\[ (k_3, m_1, n_2) \notin \{(1, 2, 0), (1, -2, 0), (-1, 2, 0), (-1, -2, 0), (1, 2, 1), (1, -2, 1), (-1, 2, 1), (-1, -2, 1)\}, \]
then \( \text{DH}(f) > 0. \)
4.2. An infra-nilmanifold modeled on Nil

We will consider a three-dimensional infra-nilmanifold modeled on the Heisenberg group Nil. Recall that

$$\text{Nil} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \big| x, y, z \in \mathbb{R} \right\}.$$ 

For all integers $k > 0$, we consider the subgroups $\Gamma_k$ of Nil:

$$\Gamma_k = \left\{ \begin{bmatrix} 1 & m & -\frac{z}{k} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \big| \ell, m, n \in \mathbb{Z} \right\}.$$ 

These are lattices of Nil and every lattice of Nil is isomorphic to some $\Gamma_k$. Letting

$$s_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 1 & 0 & -\frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we obtain a presentation of $\Gamma_k$

$$\Gamma_k = \langle s_1, s_2, s_3 \mid [s_3, s_1] = [s_3, s_2] = 1, [s_2, s_1] = s_3^k \rangle.$$ 

Every element of $\Gamma_k$ can be written uniquely as the form

$$s_2^m s_1^n s_3^\ell = \begin{bmatrix} 1 & m & -\frac{\ell}{k} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}.$$ 

Remark that $s_1^n s_2^m = s_2^m s_1^n s_3^{-kmn}$. All possible almost-Bieberbach groups can be found in [3, pp. 799–801] or [5].

Consider an almost Bieberbach group $\Pi$ given by

$$\Pi = \left\{ s_1, s_2, s_3, \alpha \big| [s_3, s_1] = [s_3, s_2] = 1, [s_2, s_1] = s_3^k, \alpha s_1 \alpha^{-1} = s_2, \alpha s_2 \alpha^{-1} = s_1^{-1} s_2^{-1}, \alpha^3 = s_3^2 \right\}.$$ 

This is a 3-dimensional almost Bieberbach group $\pi_{6.2}$ or $\pi_{6.3}$ with Seifert bundle type 6.

Let $\varphi : \Pi \to \Pi$ be a homomorphism. Every element of $\Pi$ is of the form $s_2^m s_1^n s_3^{\ell}$, $s_2^m s_1^n s_3^{\ell} \alpha$ or $s_2^m s_1^n s_3^{\ell} \alpha^2$. In order to have an isomorphism $\bar{\varphi} : \Phi \to \Phi$ such that $\text{fix}(\varphi) = \{ I \}$, we must have that $\bar{\varphi}(\alpha) = \alpha^2$. This implies that $\varphi$ has the form

$$\varphi(s_1) = s_2^n s_1^m s_3^\ell_1, \quad \varphi(s_2) = s_2^n s_1^m s_3^\ell_2, \quad \varphi(\alpha) = s_2^n s_1^m s_3^{3\ell_2+2}.$$ 

Then it can be seen as before that

$$\varphi(\alpha^3) = s_3^{(3\ell_2+2)-\frac{m(3m+1)}{2}k+(m^2+m+1)n^2k-\frac{n(3n+1)k}{2}}.$$

Since $\varphi(s_3) \in \Gamma_k$, $\varphi(s_3)$ is of the form $s_2^m s_1^n s_3^\ell$ and so

$$\varphi(s_3) = (s_2^m s_1^n s_3^\ell)^2 = s_2^{2m} s_1^{2n} s_3^{2\ell-kmn}.$$
Hence $\varphi(s_3) = s_3^6$. Furthermore, the relations $\alpha s_1 \alpha^{-1} = s_2$ and $\alpha s_2 \alpha^{-1} = s_1^{-1} s_2^{-1}$ are preserved by $\varphi$. This induces the conditions $n_1 = n_2 = -m_2$ and $m_1 = -2m_2$. The relation $[s_2, s_1] = s_3^6$ yields that $\ell = m_1 n_2 - m_2 n_1 = 3m_2^2$. Consequently, the integral differential of $\mathbb{D} = \varphi|_{\Gamma}$ with respect to the basis $\{\log(s_1), \log(s_2), \log(s_3)\}$ of $\text{nil}$ is

$$\mathbb{D}_* = \begin{bmatrix} -2m_2 & m_2 & 0 \\ -m_2 & -m_2 & 0 \\ 0 & 0 & 3m_2^2 \end{bmatrix}.$$  

Hence $\det \mathbb{D}_* = (3m_2^2)^2$ and the eigenvalues of $\mathbb{D}_*$ are $\ell$ and $-\frac{3\pm \sqrt{3}}{2} m_2$. No eigenvalues of $\mathbb{D}_*$ are of modulus 1, and $\det \mathbb{D}_* = 0$ (i.e., $m_2 = 0$) or $\det \mathbb{D}_* \geq 9$. Consequently if $m_2 \neq 0$ then $\text{DH}(f) > 0$.

4.3. Infra-solvmanifolds modeled on $\text{Sol}$

Next we will consider a closed 3-manifold with $\text{Sol}$-geometry. Recall that $\text{Sol} = \mathbb{R}^2 \rtimes_{\phi} \mathbb{R}$ where

$$\phi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}. $$

Then $\text{Sol}$ is a connected and simply connected unimodular 2-step solvable Lie group of type (R). It has a faithful representation into $\text{Aff}(\mathbb{R}^3)$ as follows:

$$\text{Sol} = \left\{ \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, t \in \mathbb{R} \right\}.$$

Let $M$ be a closed 3-manifold with $\text{Sol}$-geometry. Then the fundamental group $\Pi$ of $M$ is a Bieberbach group of $\text{Sol}$, and $M = \Pi \setminus \text{Sol}$. Further, $\Pi$ can be embedded into $\text{Aff}(\text{Sol}) = \text{Sol} \ltimes \text{Aut}(\text{Sol})$ so that there is an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \Pi \longrightarrow \Pi/\Gamma \longrightarrow 1,$$

where $\Gamma = \Pi \cap \text{Sol}$ is a lattice of $\text{Sol}$ and $\Phi = \Pi/\Gamma$ is a finite group, called the holonomy group of $\Pi$ or $M$, which sits naturally into $\text{Aut}(\text{Sol})$, see [8]. The lattices $\Gamma$ of $\text{Sol}$ are determined by $2 \times 2$-integer matrices $A$

$$A = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}$$

of determinant 1 and trace $> 2$, see for example [29, Lemma 2.1]. Namely,

$$\Gamma = \Gamma_A = \langle a_1, a_2, \tau \mid [a_1, a_2] = 1, \tau a_i \tau^{-1} = A(a_i) = \mathbb{Z}^2 \times_A \mathbb{Z} \rangle.$$

Let $f$ be a self-map on $\Gamma_A \setminus \text{Sol}$. By [29, Theorem 2.4], the homomorphism $\varphi : \Gamma_A \rightarrow \Gamma_A$ induced by $f$ is determined by

$$\varphi(a_i) = a_i^u, \quad \varphi(\tau) = a^p \tau^f.$$
for some \( u, p \in \mathbb{Z}^2 \) and \( \zeta \in \mathbb{Z} \). Note that \( \varphi \) extends uniquely to a Lie group homomorphism on \( \text{Sol} \). It follows easily that all the possible (integer) matrices \( D_\ast \) are of the form
\[
D_\ast = \begin{bmatrix} u_1 & u_2 & 0 \\ 0 & \ 0 & \zeta \end{bmatrix}.
\]
We say that \( \varphi \) is of type (I) if \( \zeta = 1 \); of type (II) if \( \zeta = -1 \); of type (III) if \( \zeta \neq \pm 1 \). When \( \varphi \) is of type (III), we have \( \varphi(a_i) = 1 \).

Now we consider the conditions of Corollary 3.6. These eliminate \( \varphi \) of type (I) and (III). If \( \varphi \) of type (II) satisfies the conditions of Corollary 3.6, then \( DH(f) > 0 \). In fact, it is shown in [28, Theorem 5.1] that such a map has \( \text{HPer}(f) = N - 2N \), and so \( DH(f) = 1/2 \).

References


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