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RELATIVE ORDER AND RELATIVE TYPE BASED GROWTH PROPERTIES OF ITERATED *P* ADIC ENTIRE FUNCTIONS

Tanmay Biswas

Abstract. Let us suppose that $\mathbb{K}$ be a complete ultrametric algebraically closed field and $\mathcal{A}(\mathbb{K})$ be the $\mathbb{K}$-algebra of entire functions on $\mathbb{K}$. The main aim of this paper is to study some newly developed results related to the growth rates of iterated p-adic entire functions on the basis of their relative orders, relative type and relative weak type.

1. Introduction and Definitions.

Let us consider $\mathbb{K}$ be an algebraically closed field of characteristic 0, complete with respect to a p-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$). For any $\alpha \in \mathbb{K}$ and $R \in]0, +\infty[$, the closed disk $\{x \in \mathbb{K} : |x-\alpha| \leq R\}$ and the open disk $\{x \in \mathbb{K} : |x-\alpha| < R\}$ are denoted by $d(\alpha, R)$ and $d(\alpha, R^-)$ respectively. Also $C(\alpha, r)$ denotes the circle $\{x \in \mathbb{K} : |x-\alpha| = r\}$. Moreover $\mathcal{A}(\mathbb{K})$ represent the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ i.e. the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we refer the reader to the books [12, 13, 18, 20]. During the last several years the ideas of p-adic analysis have been studied

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from different aspects and many important results were gained (see [2] to [11], [14–17, 21]).

Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| \,|\, |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Similarly for the entire function $g \in \mathcal{A}(\mathbb{K})$, $|g|(r)$ is defined. The ratio $\frac{|f|(r)}{|g|(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their multiplicative norm on $\mathcal{A}(\mathbb{K})$.

However, if $f \in \mathcal{A}(\mathbb{K})$ is not a constant, the $|f|(r)$ is strictly increasing function of r and tends to $+\infty$ with r therefore there exists its inverse function $\widehat{|f|} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim_{s\to\infty} \widehat{|f|}(s) = \infty$.

Throughout the paper, log denotes the Neperian logarithm. Taking this into account the order (resp. lower order) of an entire function $f \in \mathcal{A}(\mathbb{K})$ is given by (see [5])

$$\begin{matrix} \rho(f) \\ \lambda(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log(\log|f|(r))}{\log r}.$$

The function $f \in \mathcal{A}(\mathbb{K})$ is said to be of regular growth when order and lower order of f are the same. Functions which are not of regular growth are said to be of irregular growth.

Boussaf et al. [5] also introduce the definition of type (resp. lower type) of an entire function $f \in \mathcal{A}(\mathbb{K})$ which is also another type of growth indicator used for comparing the relative growth of two entire functions defined in $\mathcal{A}(\mathbb{K})$ having same non-zero finite order in the following way:

$$\begin{matrix} \sigma(f) \\ \overline{\sigma}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log|f|(r)}{r^{\rho(f)}} \text{ where } 0 < \rho_f < \infty.$$

Analogously for $0 < \lambda_f < \infty$, one may give the definition of weak type τ_f and the growth indicator $\overline{\tau}_f$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ in the following way:

$$\begin{matrix} \overline{\tau}(f) \\ \tau(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log|f|(r)}{r^{\lambda(f)}}.$$

However the notion of relative order was first introduced by Bernal [1]. In order to make some progress in the study of p-adic analysis, recently Biswas [9] introduce the definition of relative order and relative lower order of entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function

$g \in \mathcal{A}(\mathbb{K})$ in the following way:

$$\begin{matrix} \rho_g(f) \\ \lambda_g(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \widehat{|g|}(|f|(r))}{\log r} .$$

Further the function $f \in \mathcal{A}(\mathbb{K})$, for which relative order and relative lower order with respect to another function $g \in \mathcal{A}(\mathbb{K})$ are the same is called a function of regular relative growth with respect to g. Otherwise, f is said to be irregular relative growth.with respect to g.

Next, we introduce the notion of relative type and relative lower type $f \in \mathcal{A}(\mathbb{K})$ with respect to $g \in \mathcal{A}(\mathbb{K})$ in the following manner in order to compare the relative growth of two p-adic entire functions having same non-zero finite relative order with respect to another p-adic entire function :

Definition 1. Let $f, g \in \mathcal{A}(\mathbb{K})$ be such that $0 < \rho_g(f) < \infty$. Then the relative type $\sigma_g(f)$ and relative lower type $\overline{\sigma}_g(f)$ of f with respect to g are defined as:

$$\begin{matrix} \sigma_g(f) \\ \overline{\sigma}_g(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\widehat{|g|}(|f|(r))}{r^{\rho_g(f)}}.$$

Likewise, to determine the relative growth of two p-adic entire functions having same non-zero finite relative lower order with respect to another p-adic entire function, one may introduce the concept of relative weak type in the following way:

Definition 2. Let $f, g \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_g(f) < \infty$. Then the relative -weak type $\tau_g(f)$ of f with respect to g is defined as:

$$\tau_g(f) = \underline{\lim}_{r+\infty} \frac{\widehat{|g|}(|f|(r))}{r^{\lambda_g(f)}} .$$

Analogously, one can define the growth indicator $\overline{\tau}_g(f)$ of f with respect to g with finite positive relative lower order $\lambda_g(f)$ as

$$\overline{\tau}_g(f) = \overline{\lim}_{r\to+\infty} \frac{\widehat{|g|}(|f|(r))}{r^{\lambda_g(f)}} .$$

Now in the line of Lahiri and Banerjee [19], one may introduce the concept of the iteration of f with respect to g where $f, g \in \mathcal{A}(\mathbb{K})$ in the following manner:

$$
\begin{aligned}
f(x) &= f_1(x) \\
f(g(x)) &= f(g_1(x)) = f_2(x) \\
f(g(f(x))) &= f(g(f_1(x))) = f(g_2(x)) = f_3(x) \\
\ldots\ldots\ldots \quad & \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \\
\ldots\ldots\ldots \quad & \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots
\end{aligned}
$$

$f(g(f.........(f(x) \text{ or } g(x))..........)) = f_n(x)$, according as n is odd or even, and so

$$
\begin{aligned}
g(x) &= g_1(x) \\
g(f(x)) &= g(f_1(x)) = g_2(x) \\
g(f(g(x))) &= g(f(g_1(x))) = g(f_2(x)) = g_3(x) \\
\ldots\ldots\ldots \quad & \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \\
\ldots\ldots\ldots \quad & \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \\
g(f(g_{n-2}(x))) &= g(f_{n-1}(x)) = g_n(x) \quad .
\end{aligned}
$$

Clearly all $f_n \in \mathcal{A}(\mathbb{K})$ and $g_n \in \mathcal{A}(\mathbb{K})$.

Likewise for any two non constant entire functions h, $k \in \mathcal{A}(\mathbb{K})$, we may define the iteration of $\widehat{|h|}(r)$ with respect to $\widehat{|k|}(r)$ in the following manner:

$$
\begin{aligned}
\widehat{|h|}(r) &= \widehat{|h_1|}(r) \\
\widehat{|k|}\left(\widehat{|h|}(r)\right) &= \widehat{|k|}\left(\widehat{|h_1|}(r)\right) = \widehat{|h_2|}(r) \\
\widehat{|h|}\left(\widehat{|k|}\left(\widehat{|h|}(r)\right)\right) &= \widehat{|h|}\left(\widehat{|h_2|}(r)\right) = \widehat{|h_3|}(r) \\
\ldots\ldots\ldots \quad & \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \\
\ldots\ldots\ldots \quad & \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \\
\widehat{|h|}\left(.........\left(\widehat{|h|}\left(\widehat{|k|}\left(\widehat{|h|}(r)\right)\right)\right)\right) &= \widehat{|h_n|}(r),
\end{aligned}
$$

when n is odd,

and

$$
\widehat{|k|}\left(.........\left(\widehat{|h|}\left(\widehat{|k|}\left(\widehat{|h|}(r)\right)\right)\right)\right) = \widehat{|h_n|}(r),
$$

when n is even .

Obviously $\widehat{|h_n|}(r)$ is an increasing functions of r.

The main aim of this paper is to establish some newly developed results related to the growth rates of composite p-adic entire functions on the basis of their relative orders, relative type and relative weak type.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

The following lemma due to A. Escassut [12] which can also be found in [5] or [7].

Lemma 1. [12] *Let* $f, g \in \mathcal{A}(\mathbb{K})$. *Then for all sufficiently large values of* r,

$$|f \circ g|(r) = |f|(|g|(r)) \ .$$

Lemma 2. *Let* $f(x) = \sum_{n=0}^{+\infty} a_n x^n \in \mathcal{A}(\mathbb{K})$. *Then for any positive integer* $m > 1$,

$$(|f|(r))^m \leq |f|(r^m) \ .$$

Proof. Since $f(x) = \sum_{n=0}^{+\infty} a_n x^n \in \mathcal{A}(\mathbb{K})$. Then for all $r > 0$ we have $|f|(r) = \sup_{n\geq 0}|a_n|r^n$. So we obtain that

$$(|f|(r))^m = \sup_{n\geq 0}|a_n|^m r^{mn} \ . \tag{1}$$

Further, we get that

$$|f|(r^m) = \sup_{n\geq 0}|a_n| r^{mn} \ . \tag{2}$$

As we take the supremum value for large r, therefore $n \neq 0$ and $|a_n| \leq 1$. Hence the lemma follows from (1) and (2). □

Lemma 3. *Let* $f,\ g,\ h,\ k \in \mathcal{A}(\mathbb{K})$. *Then for all sufficiently large values of* r,

(i) $\widehat{|h_n|}(|f_n|(r)) \leq \widehat{|k|}\left(|g|\left(r^{\delta}\right)\right)$ *when* n *is even* *where*

$$\begin{cases} (a)\ \ \delta = m^{n-1} \text{ wherever } m \text{ is least positive integer with} \\ 1 \leq \min\left\{\rho_h(f), \rho_k(g)\right\} \leq \max\left\{\rho_h(f), \rho_k(g)\right\} < m; \\ \\ (b)\ \ \delta = m^{\frac{n}{2}-1} \text{ wherever } m \text{ is least positive integer with} \\ 0 < \rho_h(f) < 1 \leq \rho_k(g) < m; \\ \\ (c)\ \ \delta = m^{\frac{n}{2}} \text{ wherever } m \text{ is least positive integer with} \\ 0 < \rho_k(g) < 1 \leq \rho_h(f) < m; \\ \\ (d)\ \ \delta = 1 \quad \text{ while } 0 < \max\left\{\rho_h(f), \rho_k(g)\right\} < 1; \end{cases}$$

and

$$(ii)\ \ \widehat{|h_n|}\left(|f_n|(r)\right) \leq \widehat{|h|}\left(|f|\left(r^{\delta}\right)\right) \text{ when } n\,(n \neq 1) \text{ is odd}$$

where

$$\begin{cases} (a)\ \ \delta = m^{n-1} \text{ wherever } m \text{ is least positive integer with} \\ 1 \leq \min\left\{\rho_h(f), \rho_k(g)\right\} \leq \max\left\{\rho_h(f), \rho_k(g)\right\} < m; \\ \\ (b)\ \ \delta = m^{\frac{n-1}{2}} \text{ wherever } m \text{ is least positive integer with} \\ 0 < \rho_h(f) < 1 \leq \rho_k(g) < m; \\ \\ (c)\ \ \delta = m^{\frac{n-1}{2}} \text{ wherever } m \text{ is least positive integer with} \\ 0 < \rho_k(g) < 1 \leq \rho_h(f) < m; \\ \\ (d)\ \ \delta = 1 \quad \text{ while } 0 < \max\left\{\rho_h(f), \rho_k(g)\right\} < 1\ . \end{cases}$$

Proof. **Case I.** Let $1 \leq \min\left\{\rho_h(f), \rho_k(g)\right\} \leq \max\left\{\rho_h(f), \rho_k(g)\right\} < \infty$. Now we consider that m is least positive integer such that $1 \leq \min\left\{\rho_h(f), \rho_k(g)\right\} \leq \max\left\{\rho_h(f), \rho_k(g)\right\} < m$. Also suppose that $\rho_h(f) + \varepsilon < m$ and $\rho_k(g) + \varepsilon < m$ respectively where $\varepsilon\,(> 0)$ is arbitrary. Now in view of Lemma 1, Lemma 2 and for any even integer n, we get for all sufficiently large values of r that

$$\begin{aligned} |f_n|(r) &= |f|(|g_{n-1}|(r)) \\ i.e.,\ \widehat{|h|}(|f_n|(r)) &= \widehat{|h|}(|f|(|g_{n-1}|(r))) \\ i.e.,\ \widehat{|h|}(|f_n|(r)) &\le (|g_{n-1}|(r))^{(\rho_h(f)+\varepsilon)} \qquad (3) \\ i.e.,\ \widehat{|h|}(|f_n|(r)) &\le (|g_{n-1}|(r))^{m} \\ i.e.,\ \widehat{|h|}(|f_n|(r)) &\le |g|(|f_{n-2}|(r^m)) \end{aligned}$$

$$\begin{aligned} i.e.,\ \widehat{|h_2|}(|f_n|(r)) &\le \widehat{|k|}(|g|(|f_{n-2}|(r^m))) \\ i.e.,\ \widehat{|h_2|}(|f_n|(r)) &\le (|f_{n-2}|(r^m))^{(\rho_k(g)+\varepsilon)} \\ i.e.,\ \widehat{|h_2|}(|f_n|(r)) &\le (|f_{n-2}|(r^m))^{m} \\ i.e.,\ \widehat{|h_2|}(|f_n|(r)) &\le |f_{n-2}|\left(r^{m^2}\right), \end{aligned}$$

....

....

Therefore

$$\widehat{|h_n|}(|f_n|(r)) \le \widehat{|k|}\left(|g|\left(r^{m^{n-1}}\right)\right) \text{ when } n \text{ is even .}$$

Similarly,

$$\widehat{|h_n|}(|f_n|(r)) \le \widehat{|h|}\left(|f|\left(r^{m^{n-1}}\right)\right) \text{ when } n \text{ is odd and } n \ne 1 \text{ .}$$

Thus $(i)(a)$ and $(ii)(a)$ of lemma are proved.

Case II. Let $0 < \min\{\rho_h(f), \rho_k(g)\} < 1 \le \max\{\rho_h(f), \rho_k(g)\} < \infty$.

Sub case (A). Let $0 < \rho_h(f) < 1 \le \rho_k(g) < \infty$. Now we consider that m is least positive integer such that $0 < \rho_h(f) < 1 \le \rho_k(g) < m$. Also suppose that $\rho_h(f) + \varepsilon < 1$ and $\rho_k(g) + \varepsilon < m$ respectively where $\varepsilon (> 0)$ is arbitrary. Now in view of Lemma 1, Lemma 2 and for any even integer n, we get from (3) for all sufficiently large values of r that

$$\begin{aligned} \widehat{|h|}(|f_n|(r)) &\le (|g_{n-1}|(r)) \\ i.e.,\ \widehat{|h|}(|f_n|(r)) &\le |g|(|f_{n-2}|(r)) \end{aligned}$$

$$
\begin{aligned}
i.e.,\ \widehat{|h_2|}\left(|f_n|\,(r)\right) &\leq \widehat{|k|}\left(|g|\left(|f_{n-2}|\,(r)\right)\right)\\
i.e.,\ \widehat{|h_2|}\left(|f_n|\,(r)\right) &\leq \left(|f_{n-2}|\,(r)\right)^{(\rho_k(g)+\varepsilon)}\\
i.e.,\ \widehat{|h_2|}\left(|f_n|\,(r)\right) &\leq \left(|f_{n-2}|\,(r)\right)^{m}\\
i.e.,\ \widehat{|h_2|}\left(|f_n|\,(r)\right) &\leq |f_{n-2}|\,(r^m)\\
\ldots\ldots\quad \ldots\ldots\ldots &\quad \ldots\ldots\ldots\quad \ldots\ldots\ldots\\
\ldots\ldots\quad \ldots\ldots\ldots &\quad \ldots\ldots\ldots\quad \ldots\ldots\ldots
\end{aligned}
$$

Therefore

$$
\widehat{|h_n|}\left(|f_n|\,(r)\right) \leq \widehat{|k|}\left(|g|\left(r^{m^{\frac{n}{2}-1}}\right)\right) \text{ when } n \text{ is even .}
$$

Similarly,

$$
\widehat{|h_n|}\left(|f_n|\,(r)\right) \leq \widehat{|h|}\left(|f|\left(r^{m^{\frac{n-1}{2}}}\right)\right) \text{ when } n \text{ is odd and } n \neq 1 \text{ .}
$$

Thus $(i)\,(b)$ and $(ii)\,(b)$ of lemma are proved.

Sub case (B). Let $0 < \rho_k(g) < 1 \leq \rho_h(f) < \infty$. Now we consider that m is least positive integer such that $0 < \rho_k(g) < 1 \leq \rho_h(f) < m$. Also suppose that $\rho_k(g) + \varepsilon < 1$ and $\rho_h(f) + \varepsilon < m$ respectively where $\varepsilon\,(> 0)$ is arbitrary. Now in view of Lemma 1, Lemma 2 and for any even integer n, we get from (3) for all sufficiently large values of r that

$$
\begin{aligned}
\widehat{|h|}\left(|f_n|\,(r)\right) &\leq \left(|g_{n-1}|\,(r)\right)^{m}\\
i.e.,\ \widehat{|h|}\left(|f_n|\,(r)\right) &\leq |g|\left(|f_{n-2}|\,(r^m)\right)
\end{aligned}
$$

$$
\begin{aligned}
i.e.,\ \widehat{|h_2|}\left(|f_n|\,(r)\right) &\leq \widehat{|k|}\left(|g|\left(|f_{n-2}|\,(r^m)\right)\right)\\
i.e.,\ \widehat{|h_2|}\left(|f_n|\,(r)\right) &\leq \left(|f_{n-2}|\,(r^m)\right)^{(\rho_k(g)+\varepsilon)}\\
i.e.,\ \widehat{|h_2|}\left(|f_n|\,(r)\right) &\leq \left(|f_{n-2}|\,(r^m)\right)\\
\ldots\ldots\quad \ldots\ldots\ldots &\quad \ldots\ldots\ldots\quad \ldots\ldots\ldots\\
\ldots\ldots\quad \ldots\ldots\ldots &\quad \ldots\ldots\ldots\quad \ldots\ldots\ldots
\end{aligned}
$$

Therefore

$$
\widehat{|h_n|}\left(|f_n|\,(r)\right) \leq \widehat{|k|}\left(|g|\left(r^{m^{\frac{n}{2}}}\right)\right) \text{ when } n \text{ is even .}
$$

Similarly,

$$
\widehat{|h_n|}\left(|f_n|\,(r)\right) \leq \widehat{|h|}\left(|f|\left(r^{\frac{n-1}{2}}\right)\right) \text{ when } n \text{ is odd and } n \neq 1 \text{ .}
$$

Hence $(i)\,(c)$ and $(ii)\,(c)$ of lemma are established.

Case III. Let $0 < \max\{\rho_h(f), \rho_k(g)\} < 1$. In this case we can choose an arbitrary $\varepsilon(>0)$ in such a manner so that $\rho_h(f) + \varepsilon < 1$ and $\rho_k(g) + \varepsilon < 1$ hold. Now reasoning similarly as in the proof stated above one can easily deduce the conclusion of $(i)\,(d)$ and $(ii)\,(d)$ of lemma, so its proof is omitted.

This completes the proof of the lemma. □

LEMMA 4. *Let f, g, h, $k \in \mathcal{A}(\mathbb{K})$. Then for all sufficiently large values of r,*

$$(i)\ \ \widehat{|h_n|}\left(|f_n|(r)\right) \geq \widehat{|k|}\left(|g|\left(r^{\frac{1}{\delta}}\right)\right) \text{ when } n \text{ is even,}$$

where

$$\begin{cases} (a)\ \ \delta = m^{n-1} \text{ wherever } m \text{ is least positive integer with} \\ \frac{1}{m} < \min\{\lambda_h(f), \lambda_k(g)\} \leq \max\{\lambda_h(f), \lambda_k(g)\} < 1; \\ \\ (b)\ \ \delta = m^{\frac{n}{2}-1} \text{ wherever } m \text{ is least positive integer with} \\ \frac{1}{m} < \lambda_k(g) \leq 1 < \lambda_h(f) < \infty; \\ \\ (c)\ \ \delta = m^{\frac{n}{2}} \text{ wherever } m \text{ is least positive integer with} \\ \frac{1}{m} < \lambda_h(f) \leq 1 < \lambda_k(g) < \infty; \\ \\ (d)\ \ \delta = 1 \quad \text{while } 1 < \min\{\lambda_h(f), \lambda_k(g)\} < \infty\ . \end{cases}$$

and

$$(ii)\ \ \widehat{|h_n|}\left(|f_n|(r)\right) \geq \widehat{|h|}\left(|f|\left(r^{\frac{1}{\delta}}\right)\right) \text{ when } n\,(n \neq 1) \text{ is odd}$$

where

$$\begin{cases} (a)\ \ \delta = m^{n-1} \text{ wherever } m \text{ is least positive integer with} \\ \frac{1}{m} < \min\{\lambda_h(f), \lambda_k(g)\} \leq \max\{\lambda_h(f), \lambda_k(g)\} < 1; \\ \\ (b)\ \ \delta = m^{\frac{n-1}{2}} \text{ wherever } m \text{ is least positive integer with} \\ \frac{1}{m} < \lambda_k(g) \leq 1 < \lambda_h(f) < \infty; \\ \\ (c)\ \ \delta = m^{\frac{n-1}{2}} \text{ wherever } m \text{ is least positive integer with} \\ \frac{1}{m} < \lambda_h(f) \leq 1 < \lambda_k(g) < \infty; \\ \\ (d)\ \ \delta = 1 \quad \text{while } 1 < \min\{\lambda_h(f), \lambda_k(g)\} < \infty\ . \end{cases}$$

Proof. **Case I.** Let $0 < \min\{\lambda_h(f), \lambda_k(g)\} \leq \max\{\lambda_h(f), \lambda_k(g)\} < 1$. Now we consider that m is least positive integer such that $\frac{1}{m} < \min\{\lambda_h(f), \lambda_k(g)\} \leq \max\{\lambda_h(f), \lambda_k(g)\} < 1$. Also suppose that $\lambda_h(f) - \varepsilon > \frac{1}{m}$ and $\lambda_k(g) - \varepsilon > \frac{1}{m}$ respectively where $\varepsilon(>0)$ is arbitrary. Now in view of Lemma 1, Lemma 2 and for any even integer n, we get for all sufficiently large values of r that

$$\begin{aligned}
\widehat{|h|}(|f_n|(r)) &\geq (|g_{n-1}|(r))^{(\lambda_h(f)-\varepsilon)} \\
i.e.,\ \widehat{|h|}(|f_n|(r)) &\geq (|g_{n-1}|(r))^{\frac{1}{m}} \\
i.e.,\ \widehat{|h|}(|f_n|(r)) &\geq |g_{n-1}|\left(r^{\frac{1}{m}}\right) \\
i.e.,\ \widehat{|h|}(|f_n|(r)) &\geq |g|\left(|f_{n-2}|\left(r^{\frac{1}{m}}\right)\right)
\end{aligned}$$

$$i.e.,\ \widehat{|h_2|}(|f_n|(r)) \geq \widehat{|k|}\left(|g|\left(|f_{n-2}|\left(r^{\frac{1}{m}}\right)\right)\right)$$

$$\begin{aligned}
i.e.,\ \widehat{|h_2|}(|f_n|(r)) &\geq \left(|f_{n-2}|\left(r^{\frac{1}{m}}\right)\right)^{(\lambda_k(g)-\varepsilon)} \\
i.e.,\ \widehat{|h_2|}(|f_n|(r)) &\geq \left(|f_{n-2}|\left(r^{\frac{1}{m}}\right)\right)^{\frac{1}{m}}
\end{aligned}$$

$$i.e.,\ \widehat{|h_2|}(|f_n|(r)) \geq |f_{n-2}|\left(r^{\frac{1}{m^2}}\right)$$

....

....

Therefore,

$$\widehat{|h_n|}(|f_n|(r)) \geq \widehat{|k|}\left(|g|\left(r^{\frac{1}{m^{n-1}}}\right)\right) \quad \text{when } n \text{ is even .}$$

Similarly,

$$\widehat{|h_n|}(|f_n|(r)) \geq \widehat{|h|}\left(|f|\left(r^{\frac{1}{m^{n-1}}}\right)\right) \text{ when } n\,(n \neq 1) \text{ is odd .}$$

Thus $(i)(a)$ and $(ii)(a)$ of lemma are established.

Case II. Let $0 < \min\{\lambda_h(f), \lambda_k(g)\} \leq 1 < \max\{\lambda_h(f), \lambda_k(g)\} < \infty$.

Sub case (A). Let $0 < \lambda_k(g) \leq 1 < \lambda_h(f) < \infty$. Now we consider m is least positive integer such that $\frac{1}{m} < \lambda_k(g) \leq 1 < \lambda_h(f) < \infty$. Also suppose that $\lambda_k(g) - \varepsilon > \frac{1}{m}$ and $\lambda_h(f) - \varepsilon > 1$ respectively where

$\varepsilon\,(>0)$ is arbitrary. Now in view of Lemma 1, Lemma 2 and for any even integer n, we obtain for all sufficiently large values of r that

$$\begin{aligned}
\widehat{|h|}\left(|f_n|\left(r\right)\right) &\geq \left(|g_{n-1}|\left(r\right)\right)^{(\lambda_h(f)-\varepsilon)} \\
i.e.,\ \widehat{|h|}\left(|f_n|\left(r\right)\right) &\geq |g|\left(|f_{n-2}|\left(r\right)\right)
\end{aligned}$$

$$i.e.,\ \widehat{|h_2|}\left(|f_n|\left(r\right)\right) \geq \widehat{|k|}\left(|g|\left(|f_{n-2}|\left(r\right)\right)\right)$$

$$\begin{aligned}
i.e.,\ \widehat{|h_2|}\left(|f_n|\left(r\right)\right) &\geq \left(|f_{n-2}|\left(r\right)\right)^{(\lambda_k(g)-\varepsilon)} \\
i.e.,\ \widehat{|h_2|}\left(|f_n|\left(r\right)\right) &\geq \left(|f_{n-2}|\left(r\right)\right)^{\frac{1}{m}}
\end{aligned}$$

$$i.e.,\ \widehat{|h_2|}\left(|f_n|\left(r\right)\right) \geq |f_{n-2}|\left(r^{\frac{1}{m}}\right)$$

....

....

Therefore,

$$\widehat{|h_n|}\left(|f_n|\left(r\right)\right) \geq \widehat{|k|}\left(|g|\left(r^{\frac{1}{m^{\frac{n}{2}-1}}}\right)\right) \quad \text{when } n \text{ is even}\ .$$

Similarly,

$$\widehat{|h_n|}\left(|f_n|\left(r\right)\right) \geq \widehat{|h|}\left(|f|\left(r^{\frac{1}{m^{\frac{n-1}{2}}}}\right)\right) \text{ when } n\,(n\neq 1) \text{ is odd}\ .$$

Thus $(i)\,(b)$ and $(ii)\,(b)$ of lemma are established.

Subcase (B). Let $0<\lambda_h\left(f\right)\leq 1<\lambda_k\left(g\right)<\infty$. Now we consider m is least positive integer such that $\frac{1}{m}<\lambda_h\left(f\right)\leq 1<\lambda_k\left(g\right)<\infty$. Also suppose that $\lambda_k\left(g\right)-\varepsilon>1$ and $\lambda_h\left(f\right)-\varepsilon>\frac{1}{m}$ respectively where $\varepsilon\,(>0)$ is arbitrary. Now in view of Lemma 1, Lemma 2 and for any even integer n, we get for all sufficiently large values of r that

$$\begin{aligned}
\widehat{|h|}\left(|f_n|\left(r\right)\right) &\geq \left(|g_{n-1}|\left(r\right)\right)^{(\lambda_h(f)-\varepsilon)} \\
i.e.,\ \widehat{|h|}\left(|f_n|\left(r\right)\right) &\geq \left(|g_{n-1}|\left(r\right)\right)^{\frac{1}{m}}
\end{aligned}$$

$$\begin{aligned}
i.e.,\ \widehat{|h|}\left(|f_n|\left(r\right)\right) &\geq |g_{n-1}|\left(r^{\frac{1}{m}}\right) \\
i.e.,\ \widehat{|h|}\left(|f_n|\left(r\right)\right) &\geq |g|\left(|f_{n-2}|\left(r^{\frac{1}{m}}\right)\right)
\end{aligned}$$

$$i.e.,\ \widehat{|h_2|}\left(|f_n|\left(r\right)\right) \geq \widehat{|k|}\left(|g|\left(|f_{n-2}|\left(r^{\frac{1}{m}}\right)\right)\right)$$

$$i.e., \ \widehat{|h_2|}\left(|f_n|(r)\right) \ \geq \ \left(|f_{n-2}|\left(r^{\frac{1}{m}}\right)\right)^{(\lambda_k(g)-\varepsilon)}$$
$$i.e., \ \widehat{|h_2|}\left(|f_n|(r)\right) \ \geq \ \left(|f_{n-2}|\left(r^{\frac{1}{m}}\right)\right)$$
$$i.e., \ \widehat{|h_2|}\left(|f_n|(r)\right) \geq |f_{n-2}|\left(r^{\frac{1}{m}}\right)$$

....

....

Therefore,

$$\widehat{|h_n|}\left(|f_n|(r)\right) \geq \widehat{|k|}\left(|g|\left(r^{\frac{1}{m^{\frac{n}{2}}}}\right)\right) \quad \text{when } n \text{ is even .}$$

Similarly,

$$\widehat{|h_n|}\left(|f_n|(r)\right) \geq \widehat{|h|}\left(|f|\left(r^{\frac{1}{m^{\frac{n-1}{2}}}}\right)\right) \text{ when } n\,(n \neq 1) \text{ is odd .}$$

Thus $(i)\,(c)$ and $(ii)\,(c)$ of lemma are established.

Case III. Let $1 < \min\{\rho_h(f), \rho_k(g)\} < \infty$. In this case we can choose an arbitrary $\varepsilon(>0)$ in such a manner so that $\lambda_k(g) - \varepsilon > 1$ and $\lambda_h(f) - \varepsilon > 1$ hold. Now reasoning similarly as in the proof stated above one can easily deduce the conclusion of $(i)\,(d)$ and $(ii)\,(d)$ of lemma, so its proof is omitted.

This completes the proof of the lemma. □

3. Main Results

In this section we present the main results of the paper.

THEOREM 1. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \lambda_k(g) < \infty$. Then for every positive constant μ and every real number α,*

$$\lim_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|(r)\right)}{\left\{\log \widehat{|h|}\left(|f|(r^{\mu})\right)\right\}^{1+\alpha}} = \infty$$

where n is any integer such that $n > 1$.

Proof. First let us consider n to be an even integer. If α be such that $1+\alpha \leq 0$ then the theorem is trivial. So we suppose that $1+\alpha > 0$. Now it follows from the first part of Lemma 4, for all sufficiently large values of r that

$$\widehat{|h_n|}\left(|f_n|(r)\right) \geq \left(r^{\frac{1}{\delta}}\right)^{\lambda_k(g)-\varepsilon}, \tag{4}$$

where δ satisfies the conditions of Lemma 4.

Again from the definition of $\rho_h(f)$, it follows for all sufficiently large values of r that

$$\left\{\log \widehat{|h|}\left(|f|(r^{\mu})\right)\right\}^{1+\alpha} \leq \left(\rho_h(f)+\varepsilon\right)^{1+\alpha} \mu^{1+\alpha} (\log r)^{1+\alpha} . \tag{5}$$

Now from (4) and (5), it follows for all sufficiently large values of r that

$$\frac{\widehat{|h_n|}\left(|f_n|(r)\right)}{\left\{\log \widehat{|h|}\left(|f|(r^{\mu})\right)\right\}^{1+\alpha}} \geqslant \frac{\left(r^{\frac{1}{\delta}}\right)^{\lambda_k(g)-\varepsilon}}{\left(\rho_h(f)+\varepsilon\right)^{1+\alpha} \mu^{1+\alpha} (\log r)^{1+\alpha}} .$$

Since $\frac{r^{\frac{\lambda_k(g)-\varepsilon}{\delta}}}{(\log r)^{1+\alpha}} \to \infty$ as $r \to \infty$, therefore from above we obtain that

$$\varliminf_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|(r)\right)}{\left\{\log \widehat{|h|}\left(|f|(r^{\mu})\right)\right\}^{1+\alpha}} = \infty \text{ for any even number } n.$$

Similarly, with the help of the second part of Lemma 4 one can easily derive the same conclusion for any odd integer $n\,(\neq 1)$.

Thus the theorem follows from above. □

REMARK 1. Theorem 1 is still valid with "limit superior" instead of " limit " if we replace the condition " $0 < \lambda_h(f) \leq \rho_h(f) < \infty$" by " $0 < \lambda_h(f) < \infty$".

In the line of Theorem 1, one may state the following theorem without its proof:

THEOREM 2. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Then for every positive constant μ and every real number α,*

$$\lim_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|(r)\right)}{\left\{\log \widehat{|k|}\left(|g|(r^{\mu})\right)\right\}^{1+\alpha}} = \infty$$

where n is any integer such that $n > 1$.

REMARK 2. In Theorem 2 if we take the condition $0 < \lambda_k(g) < \infty$ instead of $0 < \lambda_k(g) \leq \rho_k(g) < \infty$, then also Theorem 2 remains true with "limit superior" in place of " limit ".

THEOREM 3. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \rho_k(g) < \infty$. Then for every positive constant μ and each $\alpha \in (-\infty, \infty)$,*

$$(i)\ \lim_{r\to\infty} \frac{\left(\widehat{|h_n|}\left(|f_n|(r)\right)\right)^{1+\alpha}}{\log \widehat{|h|}\left(|f|(\exp r^{\mu})\right)} = 0 \text{ if } \mu > \delta(1+\alpha)\rho_k(g) \text{ when } n \text{ is even,}$$

and

$$(ii)\ \lim_{r\to+\infty} \frac{\left(\widehat{|h_n|}\left(|f_n|(r)\right)\right)^{1+\alpha}}{\log \widehat{|h|}\left(|f|(\exp r^{\mu})\right)} = 0 \text{ if } \mu > \delta(1+\alpha)\rho_h(f) \text{ when } n\,(\neq 1)$$

is any odd integer where δ satisfies the conditions of Lemma 3.

Proof. If $1+\alpha \leq 0$, then the theorem is obvious. We consider that $1+\alpha > 0$. Now it follows from the first part of Lemma 3 for all sufficiently large values of r that

$$\widehat{|h_n|}\left(|f_n|(r)\right) \leq r^{\delta(\rho_k(g)+\varepsilon)}, \tag{6}$$

where δ satisfies the conditions of Lemma 3.

Again for all sufficiently large values of r we get that

$$\log \widehat{|h|}\left(|f|(\exp r^{\mu})\right) \geq (\lambda_h(f) - \varepsilon)\, r^{\mu}\ . \tag{7}$$

Hence for all sufficiently large values of r, we obtain from (6) and (7) that

$$\frac{\left(\widehat{|h_n|}\left(|f_n|(r)\right)\right)^{1+\alpha}}{\log \widehat{|h|}\left(|f|(\exp r^{\mu})\right)} \leq \frac{r^{\delta(\rho_k(g)+\varepsilon)(1+\alpha)}}{(\lambda_h(f) - \varepsilon)\, r^{\mu}}\ , \tag{8}$$

where we choose $0 < \varepsilon < \min\left\{\lambda_h(f), \frac{\mu}{1+\alpha} - \rho_k(g)\right\}$.

So from (8) we obtain that

$$\lim_{r\to+\infty} \frac{\left(\widehat{|h_n|}\left(|f_n|(r)\right)\right)^{1+\alpha}}{\log \widehat{|h|}\left(|f|(\exp r^{\mu})\right)} = 0\ .$$

This proves the first part of the theorem.

Similarly the second part of the theorem follows from the second part of Lemma 3.

This proves the theorem. □

Remark 3. In Theorem 3 if we take the condition $0 < \rho_h(f) < \infty$ instead of $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, the theorem remains true with " limit inferior" in place of "limit ".

In view of Theorem 3, the following theorem can be carried out :

Theorem 4. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \rho_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Then for every positive constant μ and each $\alpha \in (-\infty, \infty)$,*

$$(i)\ \lim_{r\to+\infty} \frac{\left(\widehat{|h_n|}\left(|f_n|(r)\right)\right)^{1+\alpha}}{\log \widehat{|k|}\left(|g|(\exp r^{\mu})\right)} = 0 \text{ if } \mu > \delta(1+\alpha)\rho_k(g) \text{ when } n \text{ is even,}$$

and

$$(ii)\ \lim_{r\to+\infty} \frac{\left(\widehat{|h_n|}\left(|f_n|(r)\right)\right)^{1+\alpha}}{\log \widehat{|k|}\left(|g|(\exp r^{\mu})\right)} = 0 \text{ if } \mu > \delta(1+\alpha)\rho_h(f) \text{ when } n\,(\neq 1)$$

is any odd integer where δ satisfies the conditions of Lemma 3.

The proof is omitted.

Remark 4. In Theorem 4 if we take the condition $0 < \rho_k(g) < \infty$ instead of $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ then the theorem remains true with " limit inferior" in place of "limit ".

Theorem 5. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $\lambda_k(g) < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \rho_k(g) < \infty$. Then for any even number n,*

$$\varliminf_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|(r)\right)}{\widehat{|h|}\left(|f|(r^{\delta})\right)} = 0,$$

where δ satisfies the conditions of Lemma 3.

Proof. From the first part of Lemma 3, we obtain for a sequence of values of r tending to infinity that

$$(9)\qquad \widehat{|h_n|}\left(|f_n|(r)\right) \leq r^{\delta(\lambda_k(g)+\varepsilon)} .$$

Again from the definition of relative order, we obtain for all sufficiently large values of r that

$$\widehat{|h|}\left(|f|\left(r^{\delta}\right)\right) \geqslant r^{\delta(\lambda_h(f)-\varepsilon)}. \tag{10}$$

Now in view of (9) and (10) , we get for a sequence of values of r tending to infinity that

$$\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\widehat{|h|}\left(|f|\left(r^{\delta}\right)\right)} \leq \frac{r^{\delta(\lambda_k(g)+\varepsilon)}}{r^{\delta(\lambda_h(f)-\varepsilon)}}. \tag{11}$$

Now as $\lambda_k\left(g\right) < \lambda_h\left(f\right)$, we can choose $\varepsilon\left(> 0\right)$ in such a way that $\lambda_k\left(g\right) + \varepsilon < \lambda_h\left(f\right) - \varepsilon$ and the theorem follows from (11) . □

Remark 5. If we take $0 < \rho_k\left(g\right) < \lambda_h\left(f\right) \leq \rho_h\left(f\right) < \infty$ instead of " $\lambda_k\left(g\right) < \lambda_h\left(f\right) \leq \rho_h\left(f\right) < \infty$ and $\rho_k\left(g\right) < \infty$ " and the other conditions remain the same, the conclusion of Theorem 5 remains valid with " limit inferior " replaced by " limit ".

Theorem 6. *Let f, g, k, $h \in \mathcal{A}\left(\mathbb{K}\right)$ be such that $\lambda_h\left(f\right) < \lambda_k\left(g\right) \leq \rho_k\left(g\right) < \infty$ and $0 < \rho_h\left(f\right) < \infty$. Then for any odd number $n\left(\neq 1\right)$,*

$$\varliminf_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\widehat{|k|}\left(|g|\left(r^{\delta}\right)\right)} = 0,$$

where δ satisfies the conditions of Lemma 3.

The proof of Theorem 6 is omitted as it can be carried out in the line of Theorem 5 and with the help of the second part of Lemma 3.

Remark 6. If we consider $0 < \rho_h\left(f\right) < \lambda_k\left(g\right) \leq \rho_k\left(g\right) < \infty$ instead of " $\lambda_h\left(f\right) < \lambda_k\left(g\right) \leq \rho_k\left(g\right) < \infty$ and $\rho_h\left(f\right) < \infty$ " and the other conditions remain the same, the conclusion of Theorem 5 remains valid with " limit inferior " replaced by " limit ".

Theorem 7. *Let f, g, k, $h \in \mathcal{A}\left(\mathbb{K}\right)$ be such that $0 < \lambda_h\left(f\right) \leq \rho_h\left(f\right) < \infty$ and $0 < \rho_k\left(g\right) < \infty$. Then*

$$(i)\quad \varlimsup_{r\to+\infty} \frac{\log \widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log \widehat{|h|}\left(|f|\left(r^{\delta}\right)\right)} \leq \frac{\rho_k\left(g\right)}{\lambda_h\left(f\right)} \text{ when } n \text{ is even,}$$

and

$$(ii)\quad \varlimsup_{r\to+\infty} \frac{\log \widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log \widehat{|h|}\left(|f|\left(r^{\delta}\right)\right)} \leq \frac{\rho_h\left(f\right)}{\lambda_h\left(f\right)} \text{ when } n\left(\neq 1\right) \text{ is any odd integer}$$

where δ satisfies the conditions of Lemma 3.

Proof. From the first part of Lemma 3, it follows for all sufficiently large values of r that

$$\frac{\log \widehat{|h_n|}\left(|f_n|(r)\right)}{\log \widehat{|h|}\left(|f|\left(r^\delta\right)\right)} \leq \frac{\log \widehat{|k|}\left(|g|\left(r^\delta\right)\right)}{\log \widehat{|h|}\left(|f|\left(r^\delta\right)\right)}$$

$$i.e., \ \frac{\log \widehat{|h_n|}\left(|f_n|(r)\right)}{\log \widehat{|h|}\left(|f|\left(r^\delta\right)\right)} \leq \frac{\log \widehat{|k|}\left(|g|\left(r^\delta\right)\right)}{\log r^\delta} \cdot \frac{\log r^\delta}{\log \widehat{|h|}\left(|f|\left(r^\delta\right)\right)}$$

$$i.e., \ \overline{\lim_{r\to+\infty}} \frac{\log \widehat{|h_n|}\left(|f_n|(r)\right)}{\log \widehat{|h|}\left(|f|\left(r^\delta\right)\right)} \leq \overline{\lim_{r\to+\infty}} \frac{\log \widehat{|k|}\left(|g|\left(r^\delta\right)\right)}{\log r^\delta} \cdot \overline{\lim_{r\to+\infty}} \frac{\log r^\delta}{\log \widehat{|h|}\left(|f|\left(r^\delta\right)\right)}$$

$$i.e., \ \overline{\lim_{r\to+\infty}} \frac{\log \widehat{|h_n|}\left(|f_n|(r)\right)}{\log \widehat{|h|}\left(|f|\left(r^\delta\right)\right)} \leq \rho_k(g) \cdot \frac{1}{\lambda_h(f)} = \frac{\rho_k(g)}{\lambda_h(f)}\,.$$

Thus the first part of theorem follows from above.

Likewise, with the help of the second part of Lemma 3 one can easily derive conclusion of the second part of theorem.

This proves the theorem. □

Theorem 8. *Let $f, g, k, h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $0 < \rho_h(f) < \infty$. Then*

$$(i)\ \overline{\lim_{r\to+\infty}} \frac{\log \widehat{|h_n|}\left(|f_n|(r)\right)}{\log \widehat{|k|}\left(|g|\left(r^\delta\right)\right)} \leq \frac{\rho_k(g)}{\lambda_k(g)} \text{ when } n \text{ is even,}$$

and

$$(ii)\ \overline{\lim_{r\to+\infty}} \frac{\log \widehat{|h_n|}\left(|f_n|(r)\right)}{\log \widehat{|k|}\left(|g|\left(r^\delta\right)\right)} \leq \frac{\rho_h(f)}{\lambda_k(g)} \text{ when } n\,(\neq 1) \text{ is any odd integer}$$

where δ satisfies the conditions of Lemma 3.

The proof of Theorem 8 is omitted as it can be carried out in the line of Theorem 7.

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 7 and Theorem 8 respectively and with the help of Lemma 3.

THEOREM 9. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Then*

$$(i)\quad \varliminf_{r\to+\infty} \frac{\log \widehat{|h_n|}(|f_n|(r))}{\log \widehat{|h|}(|f|(r^\delta))} \leq \frac{\lambda_k(g)}{\lambda_h(f)} \text{ when } n \text{ is even,}$$

and

$$(ii)\quad \varliminf_{r\to+\infty} \frac{\log \widehat{|h_n|}(|f_n|(r))}{\log \widehat{|h|}(|f|(r^\delta))} \leq 1 \text{ when } n\,(\neq 1) \text{ is any odd integer}$$

where δ satisfies the conditions of Lemma 3.

THEOREM 10. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Then*

$$(i)\quad \varliminf_{r\to+\infty} \frac{\log \widehat{|h_n|}(|f_n|(r))}{\log \widehat{|k|}(|g|(r^\delta))} \leq 1 \text{ when } n \text{ is even,}$$

and

$$(ii)\quad \varliminf_{r\to+\infty} \frac{\log \widehat{|h_n|}(|f_n|(r))}{\log \widehat{|k|}(|g|(r^\delta))} \leq \frac{\lambda_h(f)}{\lambda_k(g)} \text{ when } n\,(\neq 1) \text{ is any odd integer}$$

where δ satisfies the conditions of Lemma 3.

THEOREM 11. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) < \infty$ and $0 < \lambda_k(g) < \infty$. Then for any even number n,*

$$(i)\quad \varliminf_{r\to+\infty} \frac{\log \widehat{|h_n|}\left(|f_n|\left(r^\delta\right)\right)}{\log \widehat{|h|}(|f|(r))} \geq \frac{\lambda_k(g)}{\rho_h(f)} \text{ when } 0 < \rho_h(f) < \infty$$

and

$$(ii)\quad \varliminf_{r\to+\infty} \frac{\log \widehat{|h_n|}\left(|f_n|\left(r^\delta\right)\right)}{\log \widehat{|k|}(|g|(r))} \geq \frac{\lambda_k(g)}{\rho_k(g)} \text{ when } 0 < \rho_k(g) < \infty,$$

where δ satisfies the conditions of Lemma 4.

Proof. From the first part of Lemma 4, we obtain for all sufficiently large values of r that

$$\log \widehat{|h_n|}\left(|f_n|\left(r^\delta\right)\right) \geq (\lambda_k(g) - \varepsilon)\log r\ . \tag{12}$$

Also from the definition of $\rho_h(f)$, we obtain for all sufficiently large values of r that

$$\log \widehat{|h|}(|f|(r)) \leq (\rho_h(f) + \varepsilon) \log r \ . \tag{13}$$

Analogously,from the definition of $\rho_k(g)$, it follows for all sufficiently large values of r that

$$\log \widehat{|k|}(|g|(r)) \leq (\rho_k(g) + \varepsilon) \log r \ . \tag{14}$$

Now from (12) and (13), it follows for all sufficiently large values of r that

$$\frac{\log \widehat{|h_n|}\left(|f_n| r^\delta\right)}{\log \widehat{|h|}(|f|(r))} \geq \frac{(\lambda_k(g) - \varepsilon) \log r}{(\rho_h(f) + \varepsilon) \log r}$$

$$i.e., \ \varliminf_{r \to +\infty} \frac{\log \widehat{|h_n|}\left(|f_n|\left(r^\delta\right)\right)}{\log \widehat{|h|}(|f|(r))} \geq \frac{\lambda_k(g)}{\rho_h(f)} \ . \tag{15}$$

Thus the first part of theorem follows from (15).

Likewise, the conclusion of the second part of theorem can easily be derived from (12) and (14).

Hence the theorem follows. □

Theorem 12. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) < \infty$ and $0 < \lambda_k(g) < \infty$. Then for any odd number $n\,(\neq 1)$,*

$$(i) \ \varliminf_{r \to +\infty} \frac{\log \widehat{|h_n|}\left(|f_n|\left(r^\delta\right)\right)}{\log \widehat{|h|}(|f|(r))} \geq \frac{\lambda_h(f)}{\rho_h(f)} \ \text{when } 0 < \rho_h(f) < \infty$$

and

$$(ii) \ \varliminf_{r \to +\infty} \frac{\log \widehat{|h_n|}\left(|f_n|\left(r^\delta\right)\right)}{\log \widehat{|k|}(|g|(r))} \geq \frac{\lambda_h(f)}{\rho_k(g)} \ \text{when } 0 < \rho_k(g) < \infty,$$

where δ satisfies the conditions of Lemma 4.

The proofs of Theorem 12 is omitted as it can be carried out in the line of Theorem 11 and with the help of the second part of Lemma 4.

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 11 and Theorem 12 respectively and with the help of Lemma 4.

Theorem 13. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) < \infty$ and $0 < \lambda_k(g) \le \rho_k(g) < \infty$. Then for any even number n,*

$$(i)\ \overline{\lim_{r\to+\infty}} \frac{\log \widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(r\right)\right)} \ge \frac{\rho_k(g)}{\rho_h(f)} \text{ when } 0 < \rho_h(f) < \infty$$

and

$$(ii)\ \overline{\lim_{r\to+\infty}} \frac{\log \widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|k|}\left(|g|\left(r\right)\right)} \ge 1,$$

where δ satisfies the conditions of Lemma 4.

Theorem 14. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) \le \rho_h(f) < \infty$ and $0 < \lambda_k(g) < \infty$. Then for any odd number $n\,(\neq 1)$,*

$$(i)\ \overline{\lim_{r\to+\infty}} \frac{\log \widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|k|}\left(|g|\left(r\right)\right)} \ge \frac{\rho_h(f)}{\rho_k(g)} \text{ when } 0 < \rho_k(g) < \infty$$

and

$$(ii)\ \overline{\lim_{r\to+\infty}} \frac{\log \widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(r\right)\right)} \ge 1 \text{ when } \rho_h(f) < \infty,$$

where δ satisfies the conditions of Lemma 4.

Theorem 15. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) \le \rho_h(f) < \infty$, $0 < \lambda_k(g) < \infty$ and $0 < \mu < \rho_k(g) < \infty$. Then for any even number n ,*

$$\overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(\exp r^{\mu}\right)\right)} = \infty,$$

where δ satisfies the conditions of Lemma 4.

Proof. From the first part of Lemma 4, we get for a sequence of values of r tending to infinity that

$$\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right) \ge (r)^{(\rho_k(g)-\varepsilon)} \ . \tag{16}$$

Again from the definition of $\rho_h(f)$, we obtain for all sufficiently large values of r that

$$\log \widehat{|h|}\left(|f|\left(\exp r^{\mu}\right)\right) \le \left(\rho_h(f) + \varepsilon\right) r^{\mu} \ . \tag{17}$$

Now from (16) and (17) , it follows for a sequence of values of r tending to infinity that

$$\frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log\widehat{|h|}\left(|f|\left(\exp r^{\mu}\right)\right)} \geq \frac{r^{(\rho_k(g)-\varepsilon)}}{\left(\rho_h\left(f\right)+\varepsilon\right)r^{\mu}} \ . \tag{18}$$

As $\mu < \rho_k\left(g\right)$, we can choose $\varepsilon(>0)$ in such a way that

$$\mu < \rho_k\left(g\right) - \varepsilon \ . \tag{19}$$

Thus from (18) and (19) we get that

$$\overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log\widehat{|h|}\left(|f|\left(\exp r^{\mu}\right)\right)} = \infty \ . \tag{20}$$

Hence the theorem follows from (20) . □

THEOREM 16. *Let $f,\ g,\ k,\ h \in \mathcal{A}\left(\mathbb{K}\right)$ be such that $0 < \lambda_h\left(f\right) \leq \rho_h\left(f\right) < \infty$, $0 < \lambda_k\left(g\right) < \infty$ and $0 < \mu < \rho_k\left(g\right) < \infty$. Then for any even number n ,*

$$\overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log\widehat{|k|}\left(|g|\left(\exp r^{\mu}\right)\right)} = \infty,$$

where δ satisfies the conditions of Lemma 4.

Proof. Let $0 < \mu < \mu_0 < \rho_k\left(g\right)$. Then from (20), we obtain for a sequence of values of r tending to infinity and $A > 1$ that

$$\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right) > A\log\widehat{|h|}\left(|f|\left(\exp r^{\mu_0}\right)\right)$$

$$i.e.,\ \widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right) > A\left(\lambda_h\left(f\right)-\varepsilon\right)r^{\mu_0} \ . \tag{21}$$

Again from the definition of $\rho_k\left(g\right)$, we obtain for all sufficiently large values of r that

$$\log\widehat{|k|}\left(|g|\left(\exp r^{\mu}\right)\right) \leq \left(\rho_k\left(g\right)+\varepsilon\right)r^{\mu} \ . \tag{22}$$

So combining (21) and (22) , we obtain for a sequence of values of r tending to infinity that

$$\frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log\widehat{|k|}\left(|g|\left(\exp r^{\mu}\right)\right)} > \frac{A\left(\lambda_h\left(f\right)-\varepsilon\right)r^{\mu_0}}{\left(\rho_k\left(g\right)+\varepsilon\right)r^{\mu}}. \tag{23}$$

Since $\mu_0 > \mu$, from (23) it follows that

$$\overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log \widehat{|k|}\left(|g|\left(\exp r^{\delta\mu}\right)\right)} = \infty \ .$$

Thus the theorem follows. □

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 15 and Theorem 16 respectively and with the help of the second part of Lemma 4.

THEOREM 17. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_k(g) \leq \rho_k(g) < \infty$, $0 < \lambda_h(f) < \infty$ and $0 < \mu < \rho_h(f) < \infty$. Then for any odd number $n\,(\neq 1)$,*

$$\overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(\exp r^{\mu}\right)\right)} = \infty,$$

where δ satisfies the conditions of Lemma 4.

THEOREM 18. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_k(g) \leq \rho_k(g) < \infty$, $0 < \lambda_h(f) < \infty$ and $0 < \mu < \rho_h(f) < \infty$. Then for any odd number $n\,(\neq 1)$,*

$$\overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|k|}\left(|g|\left(\exp r^{\mu}\right)\right)} = \infty,$$

where δ satisfies the conditions of Lemma 4.

THEOREM 19. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \rho_k(g) < \infty$ and $\lambda_k(g) < \mu < \infty$. Then for any even number n ,*

$$\underline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log \widehat{|h|}\left(|f|\left(\exp r^{\delta\mu}\right)\right)} = 0,$$

where δ satisfies the conditions of Lemma 3.

Proof. From the first part of Lemma 3, it follows for a sequence of values of r tending to infinity that

$$\widehat{|h_n|}\left(|f_n|\left(r\right)\right) \leq r^{\delta(\lambda_k(g)+\varepsilon)} \ . \tag{24}$$

Again for all sufficiently large values of r we get that

$$\log \widehat{|h|}\left(|f|\left(\exp r^{\delta\mu}\right)\right) \geq \left(\lambda_h(f) - \varepsilon\right) r^{\delta\mu} \ . \tag{25}$$

Now from (24) and (25), it follows for a sequence of values of r tending to infinity that

$$\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log\widehat{|h|}\left(|f|\left(\exp r^{\delta\mu}\right)\right)} \leq \frac{r^{\delta(\lambda_k(g)+\varepsilon)}}{\left(\lambda_h\left(f\right)-\varepsilon\right)r^{\delta\mu}}\ . \tag{26}$$

As $\lambda_k\left(g\right) < \mu$, we can choose $\varepsilon\left(> 0\right)$ in such a way that

$$\lambda_k\left(g\right)+\varepsilon < \mu\ . \tag{27}$$

Thus the theorem follows from (26) and (27). □

In the line of Theorem 19, we may state the following theorem without its proof:

Theorem 20. *Let $f,\ g,\ k,\ h \in \mathcal{A}\left(\mathbb{K}\right)$ be such that $0 < \rho_h\left(f\right) < \infty$, $0 < \rho_k\left(g\right) < \infty$ and $\lambda_k\left(g\right) < \mu < \infty$. Then for any even number n ,*

$$\varliminf_{r\to+\infty}\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log\widehat{|k|}\left(|g|\left(\exp r^{\delta\mu}\right)\right)} = 0,$$

where δ satisfies the conditions of Lemma 3.

Theorem 21. *Let $f,\ g,\ k,\ h \in \mathcal{A}\left(\mathbb{K}\right)$ be such that $0 < \lambda_k\left(g\right) \leq \rho_k\left(g\right) < \infty$, $0 < \rho_h\left(f\right) < \infty$ and $\lambda_h\left(f\right) < \mu < \infty$. Then for any odd number $n\left(\neq 1\right)$,*

$$\varliminf_{r\to+\infty}\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log\widehat{|k|}\left(|g|\left(\exp r^{\delta\mu}\right)\right)} = 0,$$

where δ satisfies the conditions of Lemma 3.

Theorem 22. *Let $f,\ g,\ k,\ h \in \mathcal{A}\left(\mathbb{K}\right)$ be such that $0 < \rho_k\left(g\right) < \infty$, $0 < \rho_h\left(f\right) < \infty$ and $\lambda_h\left(f\right) < \mu < \infty$. Then for any odd number $n\left(\neq 1\right)$,*

$$\varliminf_{r\to+\infty}\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log\widehat{|h|}\left(|f|\left(\exp r^{\delta\mu}\right)\right)} = 0,$$

where δ satisfies the conditions of Lemma 3.

We omit the proofs of Theorem 21 and Theorem 22 as those can be carried out in the line of Theorem 19 and Theorem 20 respectively and with the help of the second part of Lemma 3.

Theorem 23. *Let F, G, H, K, f, g, h, $k \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_H(F) < \infty$, $0 < \lambda_K(G) < \infty$, $0 < \rho_h(f) < \infty$ and $0 < \rho_k(g) < \infty$. Then for any two integers $m(\neq 1)$ and $n(\neq 1)$*

$$(i)\ \lim_{r\to+\infty} \frac{\widehat{|H_m|}(|F_m|(r))}{\widehat{|h_n|}(|f_n|(r)) \cdot \log \widehat{|h|}(|f|(r))} = \infty$$

and

$$(ii)\ \lim_{r\to+\infty} \frac{\widehat{|H_m|}(|F_m|(r))}{\widehat{|h_n|}(|f_n|(r)) \cdot \log \widehat{|k|}(|g|(r))} = \infty,$$

when

$$(28)\quad \begin{cases} \delta^2 \rho_k(g) < \lambda_K(G) & \text{for } m, n \text{ both even} \\ \delta^2 \rho_h(f) < \lambda_H(F) & \text{for } m(\neq 1), n(\neq 1) \text{ both odd} \\ \delta^2 \rho_h(f) < \lambda_K(G) & \text{for } m \text{ even and } n(\neq 1) \text{ odd} \\ \delta^2 \rho_k(g) < \lambda_H(F) & \text{for } m(\neq 1) \text{ odd and } n \text{ even,} \end{cases}$$

where δ satisfies the conditions of Lemma 3.

Proof. We have from the definition of relative order and for all sufficiently large values of r that

$$(29)\quad \log \widehat{|h|}(|f|(r)) \le (\rho_h(f) + \varepsilon) \log r\ .$$

Case I. Let m and n are any two even numbers.

Therefore in view of first part of Lemma 3, we get for all sufficiently large values of r that

$$(30)\quad \widehat{|h_n|}(|f_n|(r)) \le (r)^{\delta(\rho_k(g)+\varepsilon)}\ .$$

So from (29) and (30) it follows for all sufficiently large values of r that

$$(31)\quad \widehat{|h_n|}(|f_n|(r)) \cdot \log \widehat{|h|}(|f|(r)) \le (r)^{\delta(\rho_k(g)+\varepsilon)} \cdot (\rho_h(f) + \varepsilon) \log r\ .$$

Also from first part of Lemma 4, we obtain for all sufficiently large values of r that

$$(32)\quad \widehat{|H_m|}(|F_m|(r)) \ge (r)^{\frac{(\lambda_K(G)-\varepsilon)}{\delta}}\ .$$

Hence combining (31) and (32) we get for all sufficiently large values of r that,

$$\frac{\widehat{|H_m|}\left(|F_m|\left(r\right)\right)}{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)\cdot\log\widehat{|h|}\left(|f|\left(r\right)\right)}\geq\frac{\left(r\right)^{\frac{\left(\lambda_K(G)-\varepsilon\right)}{\delta}}}{\left(r\right)^{\delta\left(\rho_k(g)+\varepsilon\right)}\cdot\left(\rho_h\left(f\right)+\varepsilon\right)\log r}\ . \tag{33}$$

Since $\delta^2\rho_k\left(g\right)<\lambda_K\left(G\right)$, we can choose $\varepsilon(>0)$ in such a manner that

$$\delta^2\left(\rho_k\left(g\right)+\varepsilon\right)\leq\left(\lambda_K\left(G\right)-\varepsilon\right)\ . \tag{34}$$

Thus from (33) and (34) we obtain that

$$\lim_{r\to+\infty}\frac{\widehat{|H_m|}\left(|F_m|\left(r\right)\right)}{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)\cdot\log\widehat{|h|}\left(|f|\left(r\right)\right)}=\infty\ . \tag{35}$$

Case II. Let $m(\neq 1)$ and $n(\neq 1)$ are any two odd numbers .

Now in view of second part of Lemma 3, we get for all sufficiently large values of r that

$$\widehat{|h_n|}\left(|f_n|\left(r\right)\right)\leq\left(r\right)^{\delta\left(\rho_h(f)+\varepsilon\right)}\ . \tag{36}$$

So from (29) and (36) it follows for all sufficiently large values of r that

$$\widehat{|h_n|}\left(|f_n|\left(r\right)\right)\cdot\log\widehat{|h|}\left(|f|\left(r\right)\right)\leq\left(r\right)^{\delta\left(\rho_h(f)+\varepsilon\right)}\cdot\left(\rho_h\left(f\right)+\varepsilon\right)\log r\ . \tag{37}$$

Also from second part of Lemma 4, we obtain for all sufficiently large values of r that

$$\widehat{|H_m|}\left(|F_m|\left(r\right)\right)\geq\left(r\right)^{\frac{\left(\lambda_H(F)-\varepsilon\right)}{\delta}}\ . \tag{38}$$

Hence combining (37) and (38) we get for all sufficiently large values of r that,

$$\frac{\widehat{|H_m|}\left(|F_m|\left(r\right)\right)}{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)\cdot\log\widehat{|h|}\left(|f|\left(r\right)\right)}\geq\frac{\left(r\right)^{\frac{\left(\lambda_H(F)-\varepsilon\right)}{\delta}}}{\left(r\right)^{\delta\left(\rho_h(f)+\varepsilon\right)}\cdot\left(\rho_h\left(f\right)+\varepsilon\right)\log r}\ . \tag{39}$$

As $\delta^2\rho_h\left(f\right)<\lambda_H\left(F\right)$, we can choose $\varepsilon(>0)$ in such a manner that

$$\delta^2\left(\rho_h\left(f\right)+\varepsilon\right)\leq\left(\lambda_H\left(F\right)-\varepsilon\right)\ . \tag{40}$$

Therefore from (39) and (40) it follows that

$$\lim_{r\to+\infty}\frac{\widehat{|H_m|}\left(|F_m|\left(r\right)\right)}{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)\cdot\log\widehat{|h|}\left(|f|\left(r\right)\right)}=\infty\ . \tag{41}$$

Case III. Let m be any even number and $n(\neq 1)$ be any odd number.

Then combining (32) and (37) we get for all sufficiently large values of r that

$$\frac{\widehat{|H_m|}\left(|F_m|\left(r\right)\right)}{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)\cdot\log\widehat{|h|}\left(|f|\left(r\right)\right)}\geq\frac{\left(r\right)^{\frac{\left(\lambda_K(G)-\varepsilon\right)}{\delta}}}{\left(r\right)^{\delta\left(\rho_h(f)+\varepsilon\right)}\cdot\left(\rho_h\left(f\right)+\varepsilon\right)\log r}\ . \tag{42}$$

Since $\delta^2\rho_h\left(f\right)<\lambda_K\left(G\right)$, we can choose $\varepsilon(>0)$ in such a manner that

$$\delta^2\left(\rho_h\left(f\right)+\varepsilon\right)\leq\left(\lambda_K\left(G\right)-\varepsilon\right)\ . \tag{43}$$

So from (42) and (43) we get that

$$\lim_{r\to+\infty}\frac{\widehat{|H_m|}\left(|F_m|\left(r\right)\right)}{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)\cdot\log\widehat{|h|}\left(|f|\left(r\right)\right)}=\infty\ . \tag{44}$$

Case IV. Let $m(\neq 1)$ be any odd number and n be any even number .

Therefore combining (31) and (38) we obtain for all sufficiently large values of r that

$$\frac{\widehat{|H_m|}\left(|F_m|\left(r\right)\right)}{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)\cdot\log\widehat{|h|}\left(|f|\left(r\right)\right)}\geq\frac{\left(r\right)^{\frac{\left(\lambda_H(F)-\varepsilon\right)}{\delta}}}{\left(r\right)^{\delta\left(\rho_k(g)+\varepsilon\right)}\cdot\left(\rho_h\left(f\right)+\varepsilon\right)\log r}\ . \tag{45}$$

As $\delta^2\rho_k\left(g\right)<\lambda_H\left(F\right)$, we can choose $\varepsilon(>0)$ in such a manner that

$$\delta^2\left(\rho_k\left(g\right)+\varepsilon\right)\leq\left(\lambda_H\left(F\right)-\varepsilon\right)\ . \tag{46}$$

Hence from (45) and (46) we have

$$\lim_{r\to+\infty}\frac{\widehat{|H_m|}\left(|F_m|\left(r\right)\right)}{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)\cdot\log\widehat{|h|}\left(|f|\left(r\right)\right)}=\infty\ . \tag{47}$$

Thus the first part of the theorem follows from (35) , (41) , (44) and (47) .

Similarly, from the definition of $\rho_k\left(g\right)$ one can easily derive the conclusion of the second part of the theorem.

Hence the theorem follows. □

Remark 7. If we consider $\rho_K\left(G\right), \rho_H\left(F\right), \rho_K\left(G\right)$ and $\rho_H\left(F\right)$ instead of $\lambda_K\left(G\right), \lambda_H\left(F\right), \lambda_K\left(G\right)$ and $\lambda_H\left(F\right)$ respectively in (28) and the other conditions remain the same, the conclusion of Theorem 23 is remains valid with “limit superior” replaced by “limit”.

Theorem 24. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \rho_h(f) < \infty$, $0 < \rho_k(g) < \infty$ and $\sigma_k(g) < \infty$. Then for any even number n,*

$$(i)\ \overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}(|f_n|(r))}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\delta\rho_k(g)}\right)\right)} \leq \frac{\sigma_k(g)}{\lambda_h(f)} \text{ if } \lambda_h(f) \neq 0$$

and

$$(ii)\ \overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}(|f_n|(r))}{\log \widehat{|k|}\left(|g|\left(\exp(r)^{\delta\rho_k(g)}\right)\right)} \leq \frac{\sigma_k(g)}{\lambda_k(g)} \text{ if } \lambda_k(g) \neq 0,$$

where δ satisfies the conditions of Lemma 3.

Proof. In view of the first part of Lemma 3 we have for all sufficiently large values of r that

$$\frac{\widehat{|h_n|}(|f_n|(r))}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\delta\rho_k(g)}\right)\right)} \leq \frac{\widehat{|k|}\left(|g|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\delta\rho_k(g)}\right)\right)}$$

$$i.e.,\ \frac{\widehat{|h_n|}(|f_n|(r))}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\delta\rho_k(g)}\right)\right)} \leq \frac{\widehat{|k|}\left(|g|\left(r^{\delta}\right)\right)}{(r)^{\delta\rho_k(g)}} \cdot \frac{\log\exp(r)^{\delta\rho_k(g)}}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\delta\rho_k(g)}\right)\right)}$$

$$i.e.,\ \overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}(|f_n|(r))}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\delta\rho_k(g)}\right)\right)} \leq$$

$$\overline{\lim_{r\to+\infty}} \frac{\widehat{|k|}\left(|g|\left(r^{\delta}\right)\right)}{(r)^{\delta\rho_k(g)}} \cdot \overline{\lim_{r\to+\infty}} \frac{\log\exp(r)^{\delta\rho_k(g)}}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\delta\rho_k(g)}\right)\right)}$$

$$i.e.,\ \overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}(|f_n|(r))}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\delta\rho_k(g)}\right)\right)} \leq \sigma_k(g) \cdot \frac{1}{\lambda_h(f)} = \frac{\sigma_k(g)}{\lambda_h(f)}\,.$$

Thus the first part of theorem is established.

Similarly, with the help of the first part of Lemma 3 one can easily derive conclusion of the second part of theorem.

Hence the theorem follows. □

Theorem 25. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\sigma_k(g) < \infty$. Then for any even number n,*

$$(i)\ \ \varliminf_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|(r)\right)}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\delta\rho_k(g)}\right)\right)} \leq \min\left\{\frac{\overline{\sigma}_k(g)}{\lambda_h(f)}, \frac{\sigma_k(g)}{\rho_h(f)}\right\}$$

and

$$(ii)\ \ \varliminf_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|(r)\right)}{\log \widehat{|k|}\left(|g|\left(\exp(r)^{\delta\rho_k(g)}\right)\right)} \leq \min\left\{\frac{\overline{\sigma}_k(g)}{\lambda_k(g)}, \frac{\sigma_k(g)}{\rho_k(g)}\right\},$$

where δ satisfies the conditions of Lemma 3.

Proof of Theorem 25 is omitted as it can be carried out in the line of Theorem 24 and with help of the first part of Lemma 3.

Now we state the following two theorems without their proofs as those can easily be carried out with the help of second part of Lemma 3 and in the line of Theorem 24 and Theorem 25 respectively.

Theorem 26. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \rho_h(f) < \infty$, $0 < \rho_k(g) < \infty$ and $\sigma_h(f) < \infty$. Then for any odd number $n\,(\neq 1)$,*

$$(i)\ \ \varlimsup_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|(r)\right)}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\delta\rho_h(f)}\right)\right)} \leq \frac{\sigma_h(f)}{\lambda_h(f)} \ \text{ if } \lambda_h(f) \neq 0$$

and

$$(ii)\ \ \varlimsup_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|(r)\right)}{\log \widehat{|k|}\left(|g|\left(\exp(r)^{\delta\rho_h(f)}\right)\right)} \leq \frac{\sigma_h(f)}{\lambda_k(g)} \ \text{ if } \lambda_k(g) \neq 0,$$

where δ satisfies the conditions of Lemma 3.

Theorem 27. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\sigma_h(f) < \infty$. Then for any odd number $n\,(\neq 1)$,*

$$(i)\ \ \varliminf_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|(r)\right)}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\delta\rho_h(f)}\right)\right)} \leq \min\left\{\frac{\overline{\sigma}_h(f)}{\lambda_h(f)}, \frac{\sigma_h(f)}{\rho_h(f)}\right\}$$

and

$$(ii)\ \varliminf_{r\to+\infty}\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log\widehat{|k|}\left(|g|\left(\exp\left(r\right)^{\delta\rho_h(f)}\right)\right)}\leq\min\left\{\frac{\overline{\sigma}_h\left(f\right)}{\lambda_k\left(g\right)},\frac{\sigma_h\left(f\right)}{\rho_k\left(g\right)}\right\}\ ,$$

where δ satisfies the conditions of Lemma 3.

Analogously, one may state the following four theorems without their proofs on the basis of relative weak type of p adic entire function with respect to another p adic entire function :

THEOREM 28. *Let f, g, k, $h\in\mathcal{A}(\mathbb{K})$ be such that $0<\rho_h(f)<\infty$, $0<\rho_k(g)<\infty$ and $\overline{\tau}_k(g)<\infty$. Then for any even number n,*

$$(i)\ \varlimsup_{r\to+\infty}\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log\widehat{|h|}\left(|f|\left(\exp\left(r\right)^{\delta\lambda_k(g)}\right)\right)}\leq\frac{\overline{\tau}_k\left(g\right)}{\lambda_h\left(f\right)}\ \textit{if}\ \lambda_h\left(f\right)\neq0$$

and

$$(ii)\ \varlimsup_{r\to+\infty}\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log\widehat{|k|}\left(|g|\left(\exp\left(r\right)^{\delta\lambda_k(g)}\right)\right)}\leq\frac{\overline{\tau}_k\left(g\right)}{\lambda_k\left(g\right)}\ \ \textit{if}\ \lambda_k\left(g\right)\neq0\ ,$$

where δ satisfies the conditions of Lemma 3.

THEOREM 29. *Let f, g, k, $h\in\mathcal{A}(\mathbb{K})$ be such that $0<\lambda_h(f)\leq\rho_h(f)<\infty$, $0<\lambda_k(g)\leq\rho_k(g)<\infty$ and $\overline{\tau}_k(g)<\infty$. Then for any even number n,*

$$(i)\ \varliminf_{r\to+\infty}\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log\widehat{|h|}\left(|f|\left(\exp\left(r\right)^{\delta\lambda_k(g)}\right)\right)}\leq\min\left\{\frac{\tau_k\left(g\right)}{\lambda_h\left(f\right)},\frac{\overline{\tau}_k\left(g\right)}{\rho_h\left(f\right)}\right\}$$

and

$$(ii)\ \varliminf_{r\to+\infty}\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log\widehat{|k|}\left(|g|\left(\exp\left(r\right)^{\delta\lambda_k(g)}\right)\right)}\leq\min\left\{\frac{\tau_k\left(g\right)}{\lambda_k\left(g\right)},\frac{\overline{\tau}_k\left(g\right)}{\rho_k\left(g\right)}\right\}\ ,$$

where δ satisfies the conditions of Lemma 3.

THEOREM 30. *Let f, g, k, $h\in\mathcal{A}(\mathbb{K})$ be such that $0<\rho_h(f)<\infty$, $0<\rho_k(g)<\infty$ and $\overline{\tau}_h(f)<\infty$. Then for any odd number $n\,(\neq1)$,*

$$(i)\ \varlimsup_{r\to+\infty}\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log\widehat{|h|}\left(|f|\left(\exp\left(r\right)^{\delta\lambda_h(f)}\right)\right)}\leq\frac{\overline{\tau}_h\left(f\right)}{\lambda_h\left(f\right)}\ \textit{if}\ \lambda_h\left(f\right)\neq0$$

and

$$(ii)\ \overline{\lim_{r\to+\infty}}\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log\widehat{|k|}\left(|g|\left(\exp\left(r\right)^{\delta\lambda_h(f)}\right)\right)}\leq\frac{\overline{\tau}_h\left(f\right)}{\lambda_k\left(g\right)}\quad\text{if } \lambda_k\left(g\right)\neq 0\ ,$$

where δ satisfies the conditions of Lemma 3.

THEOREM 31. *Let f, g, k, $h\in\mathcal{A}\left(\mathbb{K}\right)$ be such that $0<\lambda_h\left(f\right)\leq\rho_h\left(f\right)<\infty$, $0<\lambda_k\left(g\right)\leq\rho_k\left(g\right)<\infty$ and $\overline{\tau}_h\left(f\right)<\infty$. Then for any odd number $n\left(\neq 1\right)$,*

$$(i)\ \underline{\lim}_{r\to+\infty}\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log\widehat{|h|}\left(|f|\left(\exp\left(r\right)^{\delta\lambda_h(f)}\right)\right)}\leq\min\left\{\frac{\tau_h\left(f\right)}{\lambda_h\left(f\right)},\frac{\overline{\tau}_h\left(f\right)}{\rho_h\left(f\right)}\right\}$$

and

$$(ii)\ \underline{\lim}_{r\to+\infty}\frac{\widehat{|h_n|}\left(|f_n|\left(r\right)\right)}{\log\widehat{|k|}\left(|g|\left(\exp\left(r\right)^{\delta\lambda_h(f)}\right)\right)}\leq\min\left\{\frac{\tau_h\left(f\right)}{\lambda_k\left(g\right)},\frac{\overline{\tau}_h\left(f\right)}{\rho_k\left(g\right)}\right\}\ ,$$

where δ satisfies the conditions of Lemma 3.

THEOREM 32. *Let f, g, k, $h\in\mathcal{A}\left(\mathbb{K}\right)$ be such that $0<\lambda_h\left(f\right)<\infty$, $0<\lambda_k\left(g\right)<\infty$ and $\overline{\sigma}_k\left(g\right)>0$. Then for any even number n,*

$$(i)\ \underline{\lim}_{r\to+\infty}\frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log\widehat{|h|}\left(|f|\left(\exp\left(r\right)^{\rho_k(g)}\right)\right)}\geq\frac{\overline{\sigma}_k\left(g\right)}{\rho_h\left(f\right)}\ \text{if } \rho_h\left(f\right)<\infty$$

and

$$(ii)\ \underline{\lim}_{r\to+\infty}\frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log\widehat{|k|}\left(|g|\left(\exp\left(r\right)^{\rho_k(g)}\right)\right)}\geq\frac{\overline{\sigma}_k\left(g\right)}{\rho_k\left(g\right)}\ \text{if } \rho_k\left(g\right)<\infty,$$

where δ satisfies the conditions of Lemma 4.

Proof. From the first part of Lemma 4, we obtain for all sufficiently large values of r that

$$\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)\geq\left(\overline{\sigma}_k\left(g\right)-\varepsilon\right)r^{\rho_k(g)}\ . \tag{48}$$

Also from the definition of $\rho_h\left(f\right)$, we obtain for all sufficiently large values of r that

$$\log\widehat{|h|}\left(|f|\left(\exp\left(r\right)^{\rho_k(g)}\right)\right)\leq\left(\rho_h\left(f\right)+\varepsilon\right)r^{\rho_k(g)}\ . \tag{49}$$

Analogously,from the definition of $\rho_k(g)$, it follows for all sufficiently large values of r that

$$\log \widehat{|k|}\left(|g|\left(\exp(r)^{\rho_k(g)}\right)\right) \leq (\rho_k(g)+\varepsilon)\, r^{\rho_k(g)} . \tag{50}$$

Now from (48) and (49), it follows for all sufficiently large values of r that

$$\frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\rho_k(g)}\right)\right)} \geq \frac{(\overline{\sigma}_k(g)-\varepsilon)\, r^{\rho_k(g)}}{(\rho_h(f)+\varepsilon)\, r^{\rho_k(g)}}$$

$$i.e., \ \underline{\lim}_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\rho_k(g)}\right)\right)} \geq \frac{\overline{\sigma}_k(g)}{\rho_h(f)} . \tag{51}$$

Thus the first part of theorem follows from (51).

Likewise, the conclusion of the second part of theorem can easily be derived from (48) and (50).

Hence the theorem follows. □

Theorem 33. *Let $f,\ g,\ k,\ h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\overline{\sigma}_k(g) > 0$. Then for any even number n,*

$$(i)\ \overline{\lim}_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\rho_k(g)}\right)\right)} \geq \max\left\{\frac{\sigma_k(g)}{\rho_h(f)}, \frac{\overline{\sigma}_k(g)}{\lambda_h(f)}\right\}$$

and

$$(ii)\ \overline{\lim}_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|k|}\left(|g|\left(\exp(r)^{\rho_k(g)}\right)\right)} \geq \max\left\{\frac{\sigma_k(g)}{\rho_k(g)}, \frac{\overline{\sigma}_k(g)}{\lambda_k(g)}\right\},$$

where δ satisfies the conditions of Lemma 4.

Proof of Theorem 33 is omitted as it can be carried out in the line of Theorem 32 and with help of the first part of Lemma 4.

Similarly, we state the following two theorems without their proofs as those can easily be carried out with the help of second part of Lemma 4 and in the line of Theorem 32 and Theorem 33 respectively.

Theorem 34. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) < \infty$, $0 < \lambda_k(g) < \infty$ and $\overline{\sigma}_h(f) > 0$. Then for any odd number $n\,(\neq 1)$,*

$$(i)\ \underline{\lim}_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\rho_h(f)}\right)\right)} \geq \frac{\overline{\sigma}_h(f)}{\rho_h(f)} \text{ if } \rho_h(f) < \infty$$

and

$$(ii)\ \underline{\lim}_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|k|}\left(|g|\left(\exp(r)^{\rho_h(f)}\right)\right)} \geq \frac{\overline{\sigma}_h(f)}{\rho_k(g)} \text{ if } \rho_k(g) < \infty\ ,$$

where δ satisfies the conditions of Lemma 4.

Theorem 35. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\overline{\sigma}_h(f) > 0$. Then for any odd number $n\,(\neq 1)$,*

$$(i)\ \overline{\lim}_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\rho_h(f)}\right)\right)} \geq \max\left\{\frac{\sigma_h(f)}{\rho_h(f)}, \frac{\overline{\sigma}_h(f)}{\lambda_h(f)}\right\}$$

and

$$(ii)\ \overline{\lim}_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|k|}\left(|g|\left(\exp(r)^{\rho_h(f)}\right)\right)} \geq \max\left\{\frac{\sigma_h(f)}{\rho_k(g)}, \frac{\overline{\sigma}_h(f)}{\lambda_k(g)}\right\},$$

where δ satisfies the conditions of Lemma 4.

Similarly, one may state the following four theorems without their proofs on the basis of relative weak type of p adic entire function with respect to another p adic entire function :

Theorem 36. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) < \infty$, $0 < \lambda_k(g) < \infty$ and $\tau_k(g) > 0$. Then for any even number n,*

$$(i)\ \underline{\lim}_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\lambda_k(g)}\right)\right)} \geq \frac{\tau_k(g)}{\rho_h(f)} \text{ if } \rho_h(f) < \infty$$

and

$$(ii)\ \underline{\lim}_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|k|}\left(|g|\left(\exp(r)^{\lambda_k(g)}\right)\right)} \geq \frac{\tau_k(g)}{\rho_k(g)} \text{ if } \rho_k(g) < \infty,$$

where δ satisfies the conditions of Lemma 4.

THEOREM 37. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\tau_k(g) > 0$. Then for any even number n,*

$$(i)\ \overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\lambda_k(g)}\right)\right)} \geq \max\left\{\frac{\overline{\tau}_k(g)}{\rho_h(f)}, \frac{\tau_k(g)}{\lambda_h(f)}\right\}$$

and

$$(ii)\ \overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|k|}\left(|g|\left(\exp(r)^{\lambda_k(g)}\right)\right)} \geq \max\left\{\frac{\overline{\tau}_k(g)}{\rho_k(g)}, \frac{\tau_k(g)}{\lambda_k(g)}\right\},$$

where δ satisfies the conditions of Lemma 4.

THEOREM 38. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) < \infty$, $0 < \lambda_k(g) < \infty$ and $\tau_h(f) > 0$. Then for any odd number $n\,(\neq 1)$,*

$$(i)\ \underline{\lim}_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\lambda_h(f)}\right)\right)} \geq \frac{\tau_h(f)}{\rho_h(f)} \text{ if } \rho_h(f) < \infty$$

and

$$(ii)\ \underline{\lim}_{r\to+\infty} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|k|}\left(|g|\left(\exp(r)^{\lambda_h(f)}\right)\right)} \geq \frac{\tau_h(f)}{\rho_k(g)} \text{ if } \rho_k(g) < \infty,$$

where δ satisfies the conditions of Lemma 4.

THEOREM 39. *Let f, g, k, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\tau_h(f) > 0$. Then for any odd number $n\,(\neq 1)$,*

$$(i)\ \overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|h|}\left(|f|\left(\exp(r)^{\lambda_h(f)}\right)\right)} \geq \max\left\{\frac{\overline{\tau}_h(f)}{\rho_h(f)}, \frac{\tau_h(f)}{\lambda_h(f)}\right\}$$

and

$$(ii)\ \overline{\lim_{r\to+\infty}} \frac{\widehat{|h_n|}\left(|f_n|\left(r^{\delta}\right)\right)}{\log \widehat{|k|}\left(|g|\left(\exp(r)^{\lambda_h(f)}\right)\right)} \geq \max\left\{\frac{\overline{\tau}_h(f)}{\rho_k(g)}, \frac{\tau_h(f)}{\lambda_k(g)}\right\},$$

where δ satisfies the conditions of Lemma 4.

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Tanmay Biswas
Independent Researcher
Rajbari, Rabindrapalli, R. N. Tagore Road
P.O.-Krishnagar, Dist-Nadia, PIN- 741101, West Bengal, India..
E-mail: tanmaybiswas_math@rediffmail.com