

## A GENERALIZATION OF OSTROWSKI-TYPE INEQUALITY

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ABSTRACT. A generalization of Ostrowski-type inequality involving functions of two independent variables is given.

### 1. Introduction

In 1938, Ostrowski established the following inequality which can be used to estimate the absolute deviation of a function from its integral mean.

**THEOREM 1.1.** *Let  $f : I \rightarrow R$  be a differentiable mapping in the interior  $intI$  of  $I$ , where  $I \subset R$  is an interval, and let  $a, b \in intI$  with  $a < b$ . If  $|f'(t)| \leq M, t \in [a, b]$ , then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)^2 M, x \in [a, b].$$

In [1], Ujević has showed the following Ostrowski-type inequality:

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**THEOREM 1.2.** *Let  $f : [a, b] \rightarrow R$  be a differentiable mapping. If  $\gamma \leq f'(t) \leq \Gamma, \forall t \in [a, b]$ , then*

$$\begin{aligned} & \left| (b-a) \left[ (1-\lambda)f(x) + \lambda f\left(\frac{a+b}{2}\right) - \frac{\Gamma+\gamma}{2}(1-\lambda)\left(x - \frac{a+b}{2}\right) \right] \right. \\ & \left. - \int_a^b f(t)dt \right| \leq \frac{\Gamma-\gamma}{2} \left[ (\lambda^2 + (1-\lambda)^2) \frac{(b-a)^2}{4} + \left(x - \frac{a+b}{2}\right)^2 \right], \end{aligned}$$

where  $\lambda \in [0, 1]$  and  $a + \frac{b-a}{2}\lambda \leq x \leq b - \frac{b-a}{2}\lambda$ .

Recently, many authors have studied Ostrowski-type inequality and obtained some good results [2–6]. Among them, Sarikaya [2] proved the following Ostrowski-type inequality involving functions of two independent variables.

**THEOREM 1.3.** *Let  $f : [a, b] \times [c, d] \rightarrow R$  be an absolutely continuous function such that the partial derivatives of order 2 exists and  $\gamma \leq \frac{\partial^2 f(t,s)}{\partial t \partial s} \leq \Gamma, \forall (t, s) \in [a, b] \times [c, d]$ , then*

$$\begin{aligned} (1) \quad & \left| \frac{1}{4}f(x, y) + \frac{1}{4}G(x, y) - \frac{1}{2(b-a)} \int_a^b f(t, y)dt - \frac{1}{2(d-c)} \int_c^d f(x, s)ds \right. \\ & - \frac{1}{2(b-a)(d-c)} \int_a^b [(y-c)f(t, c) + (d-y)f(t, d)]dt \\ & - \frac{1}{2(b-a)(d-c)} \int_c^d [(x-a)f(a, s) + (b-x)f(b, s)]ds \\ & \left. + \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d f(t, s)dsdt \right| \\ & \leq \frac{[(x-a)^2 + (b-x)^2][(y-c)^2 + (d-y)^2]}{32(b-a)(d-c)} (\Gamma - \gamma) \end{aligned}$$

for all  $(x, y) \in [a, b] \times [c, d]$ , where

$$\begin{aligned}
 G(x, y) = & \frac{(x - a)[(y - c)f(a, c) + (d - y)f(a, d)]}{(b - a)(d - c)} \\
 & + \frac{(b - x)[(y - c)f(b, c) + (d - y)f(b, d)]}{(b - a)(d - c)} \\
 & + \frac{(x - a)f(a, y) + (b - x)f(b, y)}{b - a} \\
 & + \frac{(y - c)f(x, c) + (d - y)f(x, d)}{d - c}.
 \end{aligned}$$

In this paper, our main purpose is to establish a generalization of the inequality (1).

### 2. Main Result

**THEOREM 2.1.** *Let  $f : [a, b] \times [c, d] \rightarrow R$  be an absolutely continuous function such that the partial derivative of order 2 exists and  $\gamma \leq \frac{\partial^2 f(t, s)}{\partial t \partial s} \leq \Gamma, \forall (t, s) \in [a, b] \times [c, d]$ , then*

$$\begin{aligned}
 (2) \quad & |(1 - \frac{\lambda}{2})^2 f(x, y) + \tilde{G}(x, y) \\
 & - (1 - \frac{\lambda}{2})[\frac{1}{(b - a)} \int_a^b f(t, y) dt + \frac{1}{(d - c)} \int_c^d f(x, s) ds] \\
 & - \frac{\lambda}{2(b - a)(d - c)} \int_a^b [(y - c)f(t, c) + (d - y)f(t, d)] dt \\
 & - \frac{\lambda}{2(b - a)(d - c)} \int_c^d [(x - a)f(a, s) + (b - x)f(b, s)] dt \\
 & + \frac{1}{2(b - a)(d - c)} \int_a^b \int_c^d f(t, s) ds dt \\
 & - \frac{\Gamma + \gamma}{8} (1 - \lambda)^2 (a + b - 2x)(c + d - 2y)| \\
 & \leq \frac{\Gamma - \gamma}{8} (1 - \lambda + \frac{\lambda^2}{2})^2 \frac{[(x - a)^2 + (b - x)^2][(y - c)^2 + (d - y)^2]}{(b - a)(d - c)}
 \end{aligned}$$

for all  $(x, y) \in [a, b] \times [c, d]$  and  $\lambda \in [0, 1]$ , where

$$\begin{aligned}\tilde{G}(x, y) &= \frac{\lambda}{2} \left(1 - \frac{\lambda}{2}\right) \frac{(x-a)[(y-c)f(a, c) + (d-y)f(a, d)]}{(b-a)(d-c)} \\ &\quad + \frac{\lambda}{2} \left(1 - \frac{\lambda}{2}\right) \frac{(b-x)[(y-c)f(b, c) + (d-y)f(b, d)]}{(b-a)(d-c)} \\ &\quad + \left(\frac{\lambda}{2}\right)^2 \frac{(x-a)f(a, y) + (b-x)f(b, y)}{b-a} \\ &\quad + \left(\frac{\lambda}{2}\right)^2 \frac{(y-c)f(x, c) + (d-y)f(x, d)}{d-c}.\end{aligned}$$

*Proof.* We define the mappings  $p : [a, b] \times [a, b] \rightarrow R$  and  $q : [c, d] \times [c, d] \rightarrow R$  given by

$$p(x, t) = \begin{cases} t - (a + \lambda \frac{x-a}{2}), & t \in [a, x], \\ t - (b - \lambda \frac{b-x}{2}), & t \in (x, b] \end{cases}$$

and

$$q(y, s) = \begin{cases} s - (c + \lambda \frac{y-c}{2}), & s \in [c, y], \\ s - (d - \lambda \frac{d-y}{2}), & s \in (y, d]. \end{cases}$$

Then, we have

$$\begin{aligned}(3) \quad & \int_a^b \int_c^d p(x, t)q(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\ &= \int_a^x \int_c^y [t - (a + \lambda \frac{x-a}{2})][s - (c + \lambda \frac{y-c}{2})] \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\ &\quad + \int_a^x \int_y^d [t - (a + \lambda \frac{x-a}{2})][s - (d - \lambda \frac{d-y}{2})] \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\ &\quad + \int_x^b \int_c^y [t - (b - \lambda \frac{b-x}{2})][s - (c + \lambda \frac{y-c}{2})] \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\ &\quad + \int_x^b \int_y^d [t - (b - \lambda \frac{b-x}{2})][s - (d - \lambda \frac{d-y}{2})] \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt.\end{aligned}$$

Integrating by parts twice, we have

$$\begin{aligned}
 (4) \quad & \int_a^x \int_c^y [t - (a + \lambda \frac{x-a}{2})][s - (c + \lambda \frac{y-c}{2})] \frac{\partial^2 f(t,s)}{\partial t \partial s} ds dt \\
 = & (1 - \frac{\lambda}{2})(y-c)[(1 - \frac{\lambda}{2})(x-a)f(x,y) + \frac{\lambda}{2}(x-a)f(a,y) - \int_a^x f(t,y) dt] \\
 & + \frac{\lambda}{2}(y-c)[(1 - \frac{\lambda}{2})(x-a)f(x,c) + \frac{\lambda}{2}(x-a)f(a,c) - \int_a^x f(t,c) dt] \\
 & - (x-a)[(1 - \frac{\lambda}{2}) \int_c^y f(x,s) ds + \frac{\lambda}{2} \int_c^y f(a,s) ds] + \int_a^x \int_c^y f(t,s) ds dt.
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & \int_a^x \int_y^d [t - (a + \lambda \frac{x-a}{2})][s - (d - \lambda \frac{d-y}{2})] \frac{\partial^2 f(t,s)}{\partial t \partial s} ds dt \\
 = & \frac{\lambda}{2}(d-y)[(1 - \frac{\lambda}{2})(x-a)f(x,d) + \frac{\lambda}{2}(x-a)f(a,d) - \int_a^x f(t,d) dt] \\
 & + (1 - \frac{\lambda}{2})(d-y)[(1 - \frac{\lambda}{2})(x-a)f(x,y) + \frac{\lambda}{2}(x-a)f(a,y) - \int_a^x f(t,y) dt] \\
 & - (x-a)[(1 - \frac{\lambda}{2}) \int_y^d f(x,s) ds - \frac{\lambda}{2} \int_y^d f(a,s) ds] + \int_a^x \int_y^d f(t,s) ds dt.
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad & \int_x^b \int_c^y [t - (b - \lambda \frac{b-x}{2})][s - (c + \lambda \frac{y-c}{2})] \frac{\partial^2 f(t,s)}{\partial t \partial s} ds dt \\
 = & (1 - \frac{\lambda}{2})(y-c)[\frac{\lambda}{2}(b-x)f(b,y) + (1 - \frac{\lambda}{2})(b-x)f(x,y) - \int_x^b f(t,y) dt] \\
 & + \frac{\lambda}{2}(y-c)[\frac{\lambda}{2}(b-x)f(x,c) + (1 - \frac{\lambda}{2})(b-x)f(x,c) - \int_x^b f(t,c) dt] \\
 & - (b-x)[\frac{\lambda}{2} \int_c^y f(b,s) ds + (1 - \frac{\lambda}{2}) \int_c^y f(x,s) ds] + \int_x^b \int_c^y f(t,s) ds dt.
 \end{aligned}$$

$$\begin{aligned}
(7) \quad & \int_x^b \int_y^d [t - (b - \lambda \frac{b-x}{2})][s - (d - \lambda \frac{d-y}{2})] \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\
& = \frac{\lambda}{2}(d-y) [\frac{\lambda}{2}(b-x)f(b, d) + (1 - \frac{\lambda}{2})(b-x)f(x, d) - \int_x^b f(t, d) dt] \\
& \quad + (1 - \frac{\lambda}{2})(d-y) [\frac{\lambda}{2}(b-x)f(b, y) + (1 - \frac{\lambda}{2})(b-x)f(x, y) - \int_x^b f(t, y) dt] \\
& \quad - (b-x) [\frac{\lambda}{2} \int_y^d f(b, s) ds - (1 - \frac{\lambda}{2}) \int_y^d f(x, s) ds] + \int_x^b \int_y^d f(t, s) ds dt.
\end{aligned}$$

Summing up (4)-(7), we get

$$\begin{aligned}
(8) \quad & \int_a^b \int_c^d p(x, t)q(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\
& = (1 - \frac{\lambda}{2})^2(b-a)(d-c)f(x, y) \\
& \quad + \frac{\lambda}{2}(1 - \frac{\lambda}{2})[(x-a)f(a, y) + (b-x)f(b, y)](d-c) \\
& \quad + \frac{\lambda}{2}(1 - \frac{\lambda}{2})[(y-c)f(x, c) + (d-y)f(x, d)](b-a) \\
& \quad + (\frac{\lambda}{2})^2[(y-c)f(a, c) + (d-y)f(a, d)](x-a) \\
& \quad + (\frac{\lambda}{2})^2[(y-c)f(b, c) + (d-y)f(b, d)](b-x) \\
& \quad - (1 - \frac{\lambda}{2})(d-c) \int_a^b f(t, y) dt - (1 - \frac{\lambda}{2})(b-a) \int_c^d f(x, s) ds \\
& \quad - \frac{\lambda}{2} \int_a^b [(y-c)f(t, c) + (d-y)f(t, d)] dt \\
& \quad - \frac{\lambda}{2} \int_c^d [(x-a)f(a, s) + (b-x)f(b, s)] ds \\
& \quad + \int_a^b \int_c^d f(t, s) ds dt.
\end{aligned}$$

We also have

$$(9) \quad \int_a^b \int_c^d p(x,t)q(y,s)dsdt = \frac{1}{4}(1-\lambda)^2(b-a)(d-c)(a+b-2x)(c+d-2y).$$

Let  $C = \frac{\Gamma+\gamma}{2}$ . Form (9), we get

$$(10) \quad \int_a^b \int_c^d p(x,t)q(y,s)\left(\frac{\partial^2 f(t,s)}{\partial t \partial s} - C\right)dsdt = \int_a^b \int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t \partial s}dsdt - \frac{\Gamma+\gamma}{8}(1-\lambda)^2(b-a)(d-c)(a+b-2x)(c+d-2y).$$

On the other hand, we have

$$(11) \quad \left| \int_a^b \int_c^d p(x,t)q(y,s)\left(\frac{\partial^2 f(t,s)}{\partial t \partial s} - C\right)dsdt \right| \leq \max_{(t,s) \in [a,b] \times [c,d]} \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} - C \right| \int_a^b \int_c^d |p(x,t)q(y,s)|dsdt.$$

We also have

$$(12) \quad \max_{(t,s) \in [a,b] \times [c,d]} \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} - C \right| \leq \frac{\Gamma-\gamma}{2}.$$

and

$$(13) \quad \int_a^b \int_c^d |p(x,t)q(y,s)|dsdt = \frac{1}{4}\left(1-\lambda + \frac{\lambda^2}{2}\right)^2[(x-a)^2 + (b-x)^2][(y-c)^2 + (d-y)^2].$$

By (11) to (13),we have

$$(14) \quad \left| \int_a^b \int_c^d p(x,t)q(y,s)\left(\frac{\partial^2 f(t,s)}{\partial t \partial s} - C\right)dsdt \right| \leq \frac{\Gamma-\gamma}{8}\left(1-\lambda + \frac{\lambda^2}{2}\right)^2[(x-a)^2 + (b-x)^2][(y-c)^2 + (d-y)^2].$$

Form (10) and (14), we get (2). □

REMARK 2.2. We choose  $\lambda = 0$  and  $x = \frac{a+b}{2}, y = \frac{c+d}{2}$  in (2) to obtain [3, Corollary 1] .

REMARK 2.3. We choose  $\lambda = 1$  in (2) to obtain the inequality (1).

COROLLARY 2.4. *Let the assumptions of Theorem 2.1 hold. We have*

$$\begin{aligned}
 (15) \quad & |9f(x, y) + H(x, y) \\
 & - [\frac{12}{(b-a)} \int_a^b f(t, y) dt + \frac{12}{(d-c)} \int_c^d f(x, s) ds] \\
 & - \frac{4}{(b-a)(d-c)} \int_a^b [(y-c)f(t, c) + (d-y)f(t, d)] dt \\
 & - \frac{4}{(b-a)(d-c)} \int_c^d [(x-a)f(a, s) + (b-x)f(b, s)] dt \\
 & + \frac{8}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) ds dt \\
 & - \frac{\Gamma + \gamma}{2} (a+b-2x)(c+d-2y)| \\
 & \leq \frac{25(\Gamma - \gamma)}{32} \frac{[(x-a)^2 + (b-x)^2][(y-c)^2 + (d-y)^2]}{(b-a)(d-c)}
 \end{aligned}$$

for all  $(x, y) \in [a, b] \times [c, d]$  where

$$\begin{aligned}
 H(x, y) = & \frac{3(x-a)[(y-c)f(a, c) + (d-y)f(a, d)]}{(b-a)(d-c)} \\
 & + \frac{3(b-x)[(y-c)f(b, c) + (d-y)f(b, d)]}{(b-a)(d-c)} \\
 & + \frac{(x-a)f(a, y) + (b-x)f(b, y)}{b-a} \\
 & + \frac{(y-c)f(x, c) + (d-y)f(x, d)}{d-c}.
 \end{aligned}$$

and in particular



$$\begin{aligned}
(16) \quad & |36f(\frac{a+b}{2}, \frac{c+d}{2}) + I(\frac{a+b}{2}, \frac{c+d}{2}) \\
& - 48[\frac{1}{(b-a)} \int_a^b f(t, \frac{c+d}{2})dt + \frac{1}{(d-c)} \int_c^d f(\frac{a+b}{2}, s)ds] \\
& - \frac{8}{(b-a)} \int_a^b [f(t, c) + f(t, d)]dt \\
& - \frac{8}{(d-c)} \int_c^d [f(a, s) + f(b, s)]ds \\
& + \frac{32}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s)dsdt| \\
& \leq \frac{25(\Gamma - \gamma)}{32}(b-a)(d-c)
\end{aligned}$$

for all  $(x, y) \in [a, b] \times [c, d]$  where

$$\begin{aligned}
I(\frac{a+b}{2}, \frac{c+d}{2}) &= 3[f(a, c) + f(a, d)] + 3[f(b, c) + f(b, d)] \\
&+ 2[f(a, \frac{c+d}{2}) + f(b, \frac{c+d}{2})] + 2[f(\frac{a+b}{2}, c) + f(\frac{a+b}{2}, d)].
\end{aligned}$$

*Proof.* We choose  $\lambda = \frac{1}{2}$  in (2) to obtain (15) and  $x = \frac{a+b}{2}, y = \frac{c+d}{2}$  in (15) to obtain (16).  $\square$

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### References

- [1] N. Ujević, *A generalization of Ostrowski's inequality and applications in numerical integration*, Appl. Math. lett. **17** (2004), 133–137.
- [2] M. Z. Sarikaya, *On the Ostrowski-type integral inequality*, Acta Math.Univ.Comenianae. **79** (1) (2010), 129–134.
- [3] Q. Xue, J. Zhu, W. Liu, *A new generalization of Ostrowski-type inequality involving functions of two independent variables*, Comput. Math. Appl. **60** (2010), 2219–2224.

- [4] A. Qayyum, M. Shoaib, A. E. Matouk and M. A. Latif, *On new generalized Ostrowski type integral inequalities*, Abstract and Applied Analysis. **2014**, Article ID 275806, 8 p. <http://dx.doi.org/10.1155/2014/275806>
- [5] D. S. Sever, *A companion of Ostrowski's inequality for functions of bounded variation and applications*, Int. J. Nonlinear Anal. Appl. **5** (1) (2014), 89–97.
- [6] M. Z. Sarikaya, H. Budak, *Generalized Ostrowski type inequalities for local fractional integrals*, RGMIA Research Report Collection **18** (2015), Article 62, 11 p.

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