

IHARA ZETA FUNCTION OF DUMBBELL GRAPHS

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ABSTRACT. We study the Ihara zeta function of the dumbbell graph $D_{1,1,n}$ of type $(1, 1, n)$ and $D_{1,2,n}$ of type $(1, 2, n)$. Explicit formulas of the zeta functions of the graphs, their radius of convergence, and the connection with the number of closed cycles are given.

1. Introduction

Let G be a finite connected undirected graph with no degree 1 vertices. Let VG and EG be the set of vertices and the set of edges of G , respectively. In addition, we denote by $E^\pm G$ the set of all oriented edges of G . Thus, we have $|E^\pm G| = 2|EG|$.

Let $P = (e_1, e_2, \dots, e_{l(P)-1}, e_{l(P)})$ be a *primitive closed cycle* without backtracking. That is, $o(e_1) = t(e_{l(P)})$, $e_{i+1} \neq e_i^{-1} \pmod{l(P)}$ for all i and $P \neq D^m$ for any integer $m \geq 2$ and a path D in A . If a closed cycle Q is obtained by changing the cyclic order of P , then we say P and Q are equivalent. A *prime* $[P]$ in G is an equivalence class of primitive closed cycle without backtracking.

The *Ihara zeta function* of G is defined at $u \in \mathbb{C}$, for which $|u|$ is sufficiently small, by

$$Z_G(u) = \prod_{[P]} (1 - u^{l(P)})^{-1}$$

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where $[P]$ runs over the primes of G .

The Ihara determinant formula [1] gives that $Z_G(u)$ is a rational function, given by

$$Z_G(u) = \frac{1}{(1 - u^2)^{\chi(X)-1} \det(I - A(u) + Bu^2)}$$

where $\chi = |EG| - |VG| + 1$, A is the vertex adjacency matrix of G and B is diagonal matrix whose j -th diagonal entry is $\deg(v_j) - 1$. Let R_G be the radius of convergence of $Z_G(u)$.

We denote by $D_{a,b,n}$ the *dumbbell* graph of type (a, b, n) defined as a graph consisting of two vertex-disjoint cycles C_a, C_b and a path P_n ($a, b \geq 1, n \geq 2$) joining them having only its end-vertices in common with the two cycles. It has $a + b + n - 2$ number of vertices and $a + b + n - 1$ number of edges. Below is the figure of the graph $D_{a,b,n}$.



FIGURE 1. Dumbbell graph $D_{a,b,n}$

In this article, we study the Ihara-zeta function of the graph $D_{1,1,n}$ and $D_{1,2,n}$. If $G = D_{1,1,n}$, then we have $|VG| = n$ and $|EG| = n + 1$. If $G = D_{1,2,n}$, then we have $|VG| = n + 1$ and $|EG| = n + 2$.

The following two theorems are main results of the paper.

THEOREM 1.1. *Let G be the dumbbell graph $D_{1,1,n}$ of type $(1, 1, n)$. If n is odd, then*

$$Z_G(u) = \frac{1}{(1 - u^2)(u - 1)(2u^{n-1} - 2u^{n-2} + \dots - 2u + 1)(2u^n + u - 1)}$$

and if n is even, then

$$Z_G(u) = \frac{1}{(1 - u^2)(u - 1)(2u^{n-1} - 2u^{n-2} + \dots + 2u - 1)(2u^n - u + 1)}.$$

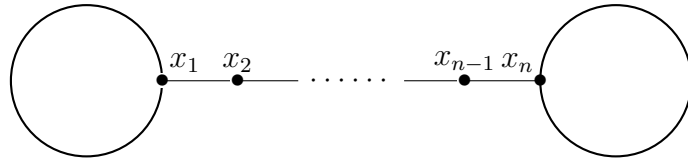


FIGURE 2. Dumbbell graph $D_{1,1,n}$

THEOREM 1.2. *Let G be the dumbbell graph $D_{1,2,n}$ of type $(1, 2, n)$. Then, we have*

$$Z_G(u) = \frac{1}{(1 - u^2)(u - 1)(4u^{2n-1} - u^3 + u^2 + u - 1)}.$$

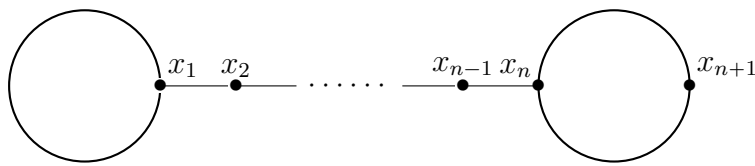


FIGURE 3. Dumbbell graph $D_{1,2,n}$

From the formula of the zeta function $Z_G(u)$ and the Perron-Frobenius theory, we get the irreducible polynomial of the radius of convergence of $Z_G(u)$.

COROLLARY 1.3. *Let G be the dumbbell graph $D_{1,1,n}$. The radius R_G of convergence of the rational function $Z_G(u)$ the unique real root of $P(u)$ where $P(u)$ is given by*

$$P(u) = \begin{cases} 2u^n + u - 1, & n \text{ is odd} \\ 2u^{n-1} - 2u^{n-2} + \dots - 2u^2 + 2u - 1, & n \text{ is even.} \end{cases}$$

Finally, the prime geodesic theorem gives the following.

COROLLARY 1.4. *Let G be the dumbbell graph $D_{1,1,n}$. If we denote by $\pi(m)$ the number of prime cycles of length m , then we have*

$$\lim_{m \rightarrow \infty} \frac{m\pi(m)}{\lambda_G^m} = 1.$$

Here, λ_G is the unique real root of $Q(u)$ where $Q(u)$ is given by

$$Q(u) = \begin{cases} u^n - u^{n-1} - 2, & n \text{ is odd} \\ u^{n-1} - 2u^{n-2} + \dots - 2u^2 + 2u - 2, & n \text{ is even.} \end{cases}$$

The analogous results for $D_{1,2,n}$ hold.

2. Proof of the results

In this section, we prove Theorem 1.1 and Theorem 1.2. First, we consider the case when the graph G is $D_{1,1,n}$.

Proof of Theorem 1.1. By the Ihara determinant formula, we have

$$Z_G(u)^{-1} = (1 - u^2) \det(B)$$

where B is an $n \times n$ tri-diagonal matrix given by

$$B = \begin{pmatrix} 1 - 2u + 2u^2 & -u & & & & \\ & -u & 1 + u^2 & -u & & \\ & & -u & \cdots & -u & \\ & & & -u & 1 + u^2 & -u \\ & & & & -u & 1 - 2u + 2u^2 \end{pmatrix}.$$

Let us denote by $f(k)$ the determinant of the $k \times k$ matrix given by

$$\begin{pmatrix} 1 + u^2 & -u & & & \\ & -u & 1 + u^2 & -u & \\ & & -u & \cdots & -u \\ & & & -u & 1 + u^2 \end{pmatrix}.$$

Then $f(k) = (1 + u^2)f(k-1) - u^2f(k-2)$. Since $f(1) = 1 + u^2$ and $f(2) = 1 + u^2 + u^4$, it follows that $f(k) = 1 + u^2 + \dots + u^{2k}$. Hence, this yields

$$\begin{aligned} \det(B) &= (1 - 2u + 2u^2)[(1 - 2u + 2u^2)f(n-2) - u^2f(n-3)] \\ &\quad - u^2[(1 - 2u + 2u^2)f(n-3) - u^2f(n-4)] \\ &= (1 - 2u + 2u^2)^2 f(n-2) - 2(1 - 2u + 2u^2)u^2 f(n-3) + u^4 f(n-4) \\ &= 4u^{2n} - 8u^{2n-1} + 8u^{2n-2} - \dots - 8u^3 + 7u^2 - 4u + 1 \\ &= (u-1)(2u^{n-1} - 2u^{n-2} + 2u^{n-3} - \dots \pm 1)(2u^n \pm u \mp 1). \end{aligned}$$

Therefore, we have

$$Z_G(u)^{-1} = (1 - u^2)(u - 1)(2u^{n-1} - 2u^{n-2} + \dots \pm 1)(2u^n \pm u \mp 1)$$

(Section 4 of [4]) implies that the Perron-Frobenius eigenvalue λ_G of $L(G)$ is simple and real. It follows that $0 < R_G < 1$ and $\lambda_G = R_G^{-1}$ is an algebraic integer. This gives us that R_G is the unique real root of $P(u)$ where

$$P(u) = \begin{cases} 2u^n + u - 1, & n \text{ is odd} \\ 2u^{n-1} - 2u^{n-2} + \dots - 2u^2 + 2u - 1, & n \text{ is even.} \end{cases}$$

Corollary 1.4 directly follows from the prime geodesic theorem in graphs (Theorem 2.10 of [2]) since $\Delta_G = 1$.

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