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A MAXIMUM PRINCIPLE FOR NON-NEGATIVE ZEROTH ORDER COEFFICIENT IN SOME UNBOUNDED DOMAINS

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Abstract. We study a maximum principle for a uniformly elliptic second order differential operator in nondivergence form. We allow a bounded positive zeroth order coefficient in a certain type of unbounded domains. The results extend a result by J. Busca in a sense of domains, and we present a simple proof based on local maximum principle by Gilbarg and Trudinger with iterations.

1. Introduction

We study a linear differential operator of the second order in nondivergence form. Formally, it can be written in the following form:

$$L = \sum_{1\le i,j\le n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{1\le i\le n} b_i(x)\frac{\partial}{\partial x_i} + c(x). \tag{1}$$

Here, $\frac{\partial}{\partial x_i}$ means the derivative with respect to x_i direction, and

$$\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}(\frac{\partial}{\partial x_j}).$$

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We impose a restriction on its second order coefficients a_{ij} as follows: for some positive constant $\nu > 0$,

$$\nu|\xi|^2 \leq \sum_{1\leq i,j\leq n} a_{ij}(x)\xi_i\xi_j \leq \nu^{-1}|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n, x \in \overline{\Omega}. \tag{2}$$

Here, Ω is a given domain, open and connected set in $\mathbb{R}^n$, and $\overline{\Omega}$ denotes a topological closure of Ω. The operator L satisfying (2), is called uniformly elliptic and ν (uniformly) ellipticity constant.

Unless it is stated otherwise, throughout the paper, we assume that the operator L acts on functions in $W^{2,n}(\Omega)$, which have the weak derivative up to the second order and its up to n-th power functions are integrable in a given domain in a Lebesgue sense.

Many researchers studied the operator, and there are a lot of known results, including regularities, a priori estimates, the existence of the solution to the corresponding boundary value problems. One may consult details in [5–8].

Among various available results, we are interested in a maximum (minimum) principle.

THEOREM 1.1. *Let Ω be a given domain, which is bounded in $\mathbb{R}^n$, and $u \in W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ such that $Lu \geq 0$ in Ω. Here, L has a form of (1) satisfying (2) with $c \leq 0$.*

Then, the positive maximum of u is attained on $\partial\Omega$.

Similarly, if $Lu \leq 0$ in Ω, then, the negative minimum of u is attained on $\partial\Omega$.

For the proof, one may refer to the references above. As an easy application, the uniqueness of the solution the corresponding Dirichlet boundary value problem is immediate from Theorem 1.1 above if $Lu = 0$ and $u = 0$ on its boundary.

Here, the bounded condition of the given domain and the sign condition of the zeroth order coefficient c are necessary. To illustrate these facts, we present two examples.

EXAMPLE 1.1. Consider the domain of the upper half space; $\mathbb{R}^+ := \{x \in \mathbb{R}^n \,|\, x_n > 0, x = (x_1, x_2, ..., x_n)\}$. In addition, we let $L = \Delta$, where Δ is the Laplace operator:

$$\Delta := \sum_{1\leq i\leq n} \frac{\partial^2}{\partial x_i \partial x_i}.$$

Note that L is reduced to Δ when $a_{ij} = \delta_{ij}$, $b_i = 0$, $c = 0$, and δ_{ij} denotes a Kronecker delta function.

Then $u^{\pm}(x) = \pm x_n$ satisfies $\Delta u^{\pm} = 0$, and $u^{\pm} = 0$ on $\partial\Omega = \{x_n = 0\}$. But neither the positive maximum of u^+ nor the negative minimum of u^- is attained on $\partial\Omega$, respectively.

Example 1.2. Let $L := \frac{d^2}{dx^2} + 1$ be defined in one dimensional real space $\mathbb{R}$, and $u^{\pm}(x) = \pm \sin x$. Then $Lu^{\pm} = 0$ in $(0, \pi)$, $u^{\pm}(0) = u^{\pm}(\pi) = 0$. But neither the positive maximum of u^+ nor the negative minimum of u^- is attained on $\partial\Omega$, respectively.

In this paper, we are interested in a maximum principle in some unbounded domains, or with a possibly positive condition for the zeroth order coefficient. In this direction, J. Busca obtained some results. To present his results, we need to introduce a condition for the domain:

Definition 1.3. [2, p. 2027] We shall call a domain Ω in $\mathbb{R}^n$ narrow with constants (ρ, δ), $\rho > 0$, $0 < \delta < 1$, if for any $y \in \Omega$, there exists a ball $B_{\rho_y}(y)$ centered at y with radius $\rho_y \le \rho$, such that

$$|B_{\rho_y}(y) \cap \Omega^c| \ge \delta |B_{\rho_y}(y)|.$$

For an example, an infinite cylinder of radius r is narrow with $(2r, \frac{1}{2})$. Now we are ready to state Busca's result:

Theorem 1.2. [2, Theorem 1.3] *Let Ω be a domain in $\mathbb{R}^n$ and L satisfy* (1), (2), *and the following conditions on its coefficients:*

$$a_{ij} \in C(\overline{\Omega}), b_i, c \in L^{\infty}(\Omega) \quad \text{with } |b_i(x)|, c(x) \le b. \tag{3}$$

Let $u \in W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ be such that

$$\begin{cases} Lu \le 0 & \text{in } \Omega, \\ u \ge 0 & \text{on } \partial\Omega, \\ u(x) \ge -Ce^{\mu|x|} & \text{in } \Omega. \end{cases} \tag{4}$$

for some positive constants C and μ. Then, for any δ, $0 < \delta < 1$, there exists $\rho = \rho(n, b, \nu, \mu, \delta) > 0$ such that if Ω is narrow with constants (ρ, δ), then $u \ge 0$ in Ω.

Remark 1.4. Under the same conditions to Theorem 1.2, if we apply $-u$ instead of u from (4), we obtain the following: if

$$\begin{cases} Lu \geq 0 & \text{in } \Omega, \\ u \leq 0 & \text{on } \partial\Omega, \\ u(x) \leq Ce^{\mu|x|} & \text{in } \Omega, \end{cases} \tag{5}$$

then $u \leq 0$. It is immediate to see that this fact is equivalent to Theorem 1.2.

In the same paper, Busca also treat the case of $c \leq 0$:

Theorem 1.3. [2, Proposition 6.1] *Let L and u satisfy the conditions in Theorem 1.3 and assume $c \leq 0$. Then there exists $\mu_0 = \mu_0(n, b, \nu) > 0$ such that if u satisfies* (4) *with $\mu < \mu_0$, then $u \geq 0$ in Ω.*

For the proof of Theorem 1.2, he used the so-called "local Maximum Principle" [6, Theorem 9.26]. We state it below, Theorem 2.1.

Note that he assumed $u \geq -Ce^{\mu|x|}$, thus u is possibly unbounded. To treat an unboundedness of a supersolution, the following function is used. He set

$$g(x) = \cosh(\beta x_1) \cdots \cosh(\beta x_n),$$

where $\beta > \mu$. He used the fact $\tilde{L}(\frac{u}{g}) \leq 0$ for some $\tilde{L}$.

In this paper, we also use the "local Maximum Principle" [6, Theorem 9.26], but with its iteration, we are able to present a simple proof without a comparison function, Theorem 1.4. Note the necessary conditions and differences of the dependences of constants in Theorem 1.2 and Theorem 1.4 below. Also, we generalized the domain used by Busca. To state the result specifically, we need to introduce another definition for the domains.

Definition 1.5. Let δ be a fixed number between 0 and 1, R and R_0 be a positive number, and Ω be a given domain. We say that Ω satisfies condition (M) (or simply (M)-domain) relative to δ, R outside of $B_{R_0}(0)$ if for any $x \in \Omega$ and $|x| > R_0$, there exists a ball $B_{R_x}(x_0)$ of radius R_x centered at x_0 such that $R_x \leq R$ and

$$x \in B_{R_x}(x_0) \quad \text{and} \quad |B_{2R_x}(x_0) \setminus \Omega| \geq \delta |B_{2R_x}|, \tag{6}$$

where $|\cdot|$ denotes a Lebesgue measure.

Note that Definition 1.5 is more general than Definition 1.3. In fact, if $R_0 = 0$ and $x = x_0$ for each $x \in \Omega$, then (G)-domain is a narrow domain.

We remark here that the definition is a modified one to our purpose from [4, Definition 1.2], which originally used in [1, Theorem 2.5].

Note also that $x \in B_{R_x}(x_0)$ but $B_{2R_x}(x_0)$ is used in the inequality. One can replace $B_{2R_x}(x_0)$ by $B_{R_x}(x_0)$ with a different constant δ. See Remark 2.1 below.

In this paper, we improve the results: the coefficients are assumed to satisfy:

$$(7) \qquad a_{ij} = a_{ji}, a_{ij}, b_i, c \in L^\infty(\Omega) \quad \text{with } |b_i(x)| \le b_0, |c(x)| \le c_0.$$

Compare (3) and (7), and see Remark 3.1. Now we state the main result of the paper.

THEOREM 1.4. *Let the operator L satisfy* (1), (2), (7) *in Ω, and $u \in W^{2,n}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ satisfy* (5) *for some $\mu > 0$. Furthermore, we assume that there exist positive constants p_0, δ, R, R_0 such that $C(1-\delta)^{1/p_0} < 1$, where C is the constant from* (8) *of Theorem 2.1 corresponding to $n, \nu, (\frac{b_0 R}{\nu})^2, \frac{c_0 R^2}{\nu}, p_0$, and Ω is (M)-domain relative to δ, R outside of $B_{R_0}(0)$. Then, there exists μ_0 depending on C, δ, p_0, R such that $\mu_0 > \mu$, then $u \le 0$ in Ω.*

An example showing that the existence of subsolution having exponential decay is illustrated. See Example 3.2.

We also have to mention Cabre's result [4, Corollary 7.1], which is already appeared in [1]. See also [3]. X. Cabre obtained a result for bounded subsolutions.

THEOREM 1.5. [4, Corollary 7.1] *Let $0 < \delta < 1$ be a fixed constant and assume that $c \le c_0$ in Ω, for a positive constant c_0. Then there exists a positive constant R^*, which depends only on n, ν, b_0, c_0, and δ such that if the domain Ω satisfies condition (M) relative to δ, R^*, the following form of maximum principle holds: for $u \in W^{2,n}_{loc}(\Omega)$,*

$$\sup_\Omega u < \infty, Lu \ge 0 \text{ in } \Omega, \text{ and } \lim sup_{x \to \partial\Omega} u(x) \le 0$$

imply $u \le 0$ in Ω.

In fact, the condition for the domain is more general. See [4, Definition 1.2].

We prepare some preliminaries in Section 2, where several remarks regarding the properties of (M)-domain and a key lemma, Theorem 2.1, are stated. The main result, Theorem 1.4 is proved in Section 3 along with Theorem 3.1. To finish the introductory section, we list some notations which will be used in the paper.

For the point $x, y \in \mathbb{R}^n$, $|x-y|$ denote the standard distance.

For $B_r(x_0)$, we denote the open ball of radius r centered at x_0 in $\mathbb{R}^n$. Namely, $B_r(x_0) := \{x \in \mathbb{R}^n \,|\, |x-x_0| < r\}$. For the ball centered at the origin, we will sometimes omit the center, like B_r.

For the set Ω, it is domain if the set Ω is open and connected. We denote its topological boundary by $\partial\Omega$, its topological closure by $\overline{\Omega}$, the Lebesgue measure by $|\Omega|$.

For u^+, we denote the positive part of the given function u. Namely, $u^+(x) = \max\{u, 0\}$.

2. Preliminaries

First, we make some remarks regarding the properties of (M)-domain.

Remark 2.1. One can replace $2R_x$ to R_x from the inequality (6) of Definition 1.5. For this, note that

$$|B_{2R_x}(x_0) \setminus \Omega| \geq |B_{R_x}(x_0) \setminus \Omega| \geq \delta |B_{R_x}| \geq \frac{\delta}{2^n} |B_{2R_x}|.$$

Thus in all, if

$$|B_{R_x}(x_0) \setminus \Omega| \geq \delta |B_{R_x}|,$$

then

$$|B_{2R_x}(x_0) \setminus \Omega| \geq \delta' |B_{2R_x}|$$

for some $\delta' = \frac{\delta}{2^n}$.

Remark 2.2. Note that Definition 1.5 holds for any subdomain of (M)-domain of Ω with the same constants δ, R. Namely, if Ω is (M)-domain relative to δ, R outside of $B_{R_0}(0)$, then any subdomain $\Omega'(\subset \Omega)$ is (M)-domain relative to δ, R outside of $B_{R_0}(0)$.

Remark 2.3. Note also that if Ω is (M)-domain relative δ, R outside of $B_{R_0}(0)$, then Ω is (M)-domain relative δ, R outside of $B_{R_1}(0)$ for any $R_1 \geq R_0$.

The following is the local boundary maximum principle by Gilbarg and Trudinger [6, Theorem 9.26], which is a key ingredient of the results of the paper.

THEOREM 2.1. *Let the operator* L *satisfy* (1), (2), (7) *in* Ω, *and* $u \in W^{2,n}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ *satisfy* $Lu \geq f$ *in* Ω, $u \leq 0$ *on* $B_{2R}(y) \cap \partial\Omega$ *where* $f \in L^n(\Omega)$ *and* $B_{2R}(y)$ *is a ball in* $\mathbb{R}^n$. *Then, for any* $p > 0$, *we have*

$$\sup_{\Omega\cap B_R(y)} u \leq C\left\{\left(\frac{1}{|B_{2R}|}\int_{B_{2R}(y)\cap\Omega}(u^+)^p\right)^{1/p} + \frac{R}{\nu}\|f\|_{L^n(B_{2R}(y)\cap\Omega)}\right\}, \tag{8}$$

where $C = C(n, \nu, (\frac{b_0 R}{\nu})^2, \frac{c_0 R^2}{\nu}, p)$.

3. Main results

Proof of Theorem 1.4. Without loss of generality, we may assume that $u \geq 0$ in Ω. If not, we consider $\Omega \cap \{u > 0\}$, which will be denoted by Ω from no on. If this set is empty, we have nothing to prove. By Remark 2.2, Ω is a (M)-domain relative to δ, R outside of $B_{R_0}(0)$. Let

$$M(r) = \sup_{\partial B_r(0)\cap\Omega} u.$$

By the maximum principle for a bounded domain, Theorem 1.1, M is a monotone increasing function. We may assume that $M(R_0) > 0$. If not, we can choose $R_1 > R_0$ such that $m(R_1) > 0$. If R_1 does not exist, it implies that $u \leq 0$ in Ω. Note that Ω is (M)-domain relative to δ, R outside of $B_{R_1}(0)$ by Remark 2.3. We still denote R_1 by R_0 from now on.

Choose any $x \in \Omega$ and $|x| = R_0 + 3kR$ for each positive integer k. Then, by Definition 1.5, we have for some x_0, $R_x < R$,

$$x \in B_{R_x}(x_0) \quad \text{and} \quad |B_{2R_x}(x_0) \setminus \Omega| \geq \delta |B_{2R_x}|. \tag{9}$$

Using (9), Theorem 2.1 with $p = p_0$, $|x - x_0| < R_x$, $R_x < R$.

$$\begin{aligned} u(x) \leq \sup_{\Omega\cap B_{R_x}(x_0)} u &\leq C\left(\frac{|B_{2R_x}(x_0)\cap\Omega|}{|B_{2R_x}|}\right)^{1/p_0} \sup_{\Omega\cap B_{2R_x}(x_0)} u \\ &\leq C(1-\delta)^{1/p_0} \sup_{\Omega\cap B_{3R}(x)} u. \end{aligned}$$

By the given condition, $\beta := C(1-\delta)^{1/p_0} < 1$. Thus,

$$u(x) \leq \beta \sup_{\Omega \cap B_{3R}(x)} u.$$

Since x was an arbitrary on $\Omega \cap \partial B_{R_0+3kR}(0)$, we have

$$M(R_0 + 3kR) \leq \beta M(R_0 + 3(k+1)R). \tag{10}$$

For an arbitrary $x \in \Omega \setminus B_{R_0}(0)$, let $|x| = r$ and $R_0 + 3kR \leq r < R_0 + 3(k+1)R$ for some positive integer k. Using the maximum principle for the bounded domain, Theorem 1.1, (10),

$$M(r) \geq M(R_0 + 3kR) \geq (\frac{1}{\beta})^k M(R_0) \geq (\frac{1}{\beta})^{\frac{r-R_0-3R}{3R}} M(R_0)$$

$$= C(R, R_0, C, p_0, \delta)(e^{\ln \frac{1}{\beta}})^{\frac{|x|}{3R}} M(R_0).$$

By (5), $M(R_0) > 0$, we have

$$M(r) \leq C' e^{\mu r},$$

which leads to a contradiction when we choose $\mu_0 < \frac{1}{3R} \ln \frac{1}{\beta}$. □

Remark 3.1. For Theorem 1.4, we replace the condition of $|c(x)| \leq c_0$ from (7) by $c(x) \leq c_0$. For this, note that $c^+(x) \geq c(x)$ where $c^+(x) := \max\{c(x), 0\}$. Thus $Lu \geq f$ implies that $L^+ u \geq f$ in $\Omega \cap \{u > 0\}$, where

$$L^+ = \sum_{1 \leq i,j \leq n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{1 \leq i \leq n} b_i(x) \frac{\partial}{\partial x_i} + c^+(x).$$

Also note that $|c^+(x)| \leq c_0$ if $c(x) \leq c_0$. Hence, we can apply Theorem 2.1.

Is is possible to have more than one solution having exponential decay, which is illustrated in the following:

Example 3.2. Let $\Omega := \{(x, y) \in \mathbb{R}^2 \,|\, x \in \mathbb{R}, y \in (0, \frac{\pi}{\mu})\}$ and $u(x, y) = e^{\mu x} \sin \mu y$. Then $\Delta u = 0$, $u \geq 0$ in Ω, $u = 0$ on $\partial\Omega$. But $M(r) \sim e^{\mu r}$.

If $c_0 \cdot R^2$ is sufficiently small, we have the following:

Theorem 3.1. *Let the operator L satisfy (1), (2), (7) in Ω, and $u \in W^{2,n}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ satisfy (5) for some $\mu > 0$. Furthermore, we assume that there exist positive constants p_0, δ, R, R_0 such that $C(1-\delta)^{1/p_0} + \frac{2c_0 R^2}{\nu}|B_1| < 1$, where C is the constant from (8) of Theorem 2.1* *corresponding to* $n, \nu, (\frac{b_0 R}{\nu})^2, 0, p_0$, *and* Ω *is (M)-domain relative to* δ, R *outside of* $B_{R_0}(0)$. *Then, there exists* μ_0 *depending on* $C, \delta, p_0, R, c_0, \nu, n$ *such that* $\mu_0 > \mu$, *then* $u \leq 0$ *in* Ω.

Proof of Theorem 3.1. The proof is similar to Theorem 1.4. We will be brief. Without loss of generality, we assume that $u > 0$ in Ω. We define a monotone increasing function M by

$$M(r) = \sup_{\partial B_r(0) \cap \Omega} u,$$

and we assume that $M(R_0) > 0$. Choose any $x \in \Omega$ and $|x| = R_0 + 3kR$ for each positive integer k. Note the fact that

$$Lu \geq 0 \text{ iff and only if } \sum_{1 \leq i,j \leq n} a_{ij}(x) \frac{\partial^2}{\partial_{x_i} \partial_{x_j}} u + \sum_{1 \leq i \leq n} b_i(x) \frac{\partial}{\partial_{x_i}} u \geq -cu.$$

Thus, by (7), we have $L'u \geq -cu \geq -c_0 u$ where

$$L' = \sum_{1 \leq i,j \leq n} a_{ij}(x) \frac{\partial^2}{\partial_{x_i} \partial_{x_j}} + \sum_{1 \leq i \leq n} b_i(x) \frac{\partial}{\partial_{x_i}}.$$

Fix $p = p_0$, and using $|x - x_0| < R_x$, Theorem 2.1 with L', $R_x < R$.

$$\begin{aligned} u(x) \leq \sup_{\Omega \cap B_{R_x}(x_0)} u &\leq C \left(\frac{|B_{2R_x}(x_0) \cap \Omega|}{|B_{2R_x}|} \right)^{1/p_0} \sup_{\Omega \cap B_{2R_x}(x_0)} u \\ &\quad + \frac{R}{\nu} \cdot c_0 \cdot \|u\|_{L^n(\Omega \cap B_{2R_x}(x_0))} \\ &\leq C(1-\delta)^{1/p_0} \sup_{\Omega \cap B_{3R}(x)} u + \frac{2c_0 R^2}{\nu} |B_1| \sup_{\Omega \cap B_{3R}(x)} u. \end{aligned}$$

Thus, we have that

$$u(x) \leq \beta \sup_{\Omega \cap B_{3R}(x)} u,$$

where $\beta := C(1-\delta)^{1/p_0} + \frac{2c_0 R^2}{\nu} |B_1| < 1$. Now we follow the proof of Theorem 1.4. □

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