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Tinkering with Number Lines

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While the utility of the number line is considerable, articulating its conceptual foundation is often neglected in school mathematics. We suggest that it is important to build up strong conceptual foundations in the earlier grades so that number lines can be used in a more meaningful way and that any misconceptions associated with the number line can be prevented or intervened. This paper addresses unit, direction, and origin as the key elements of number lines and presents activities from Davydov's curriculum for early grades that promote exploration of those key elements and may resolve some students' misconceptions. As shown in sample activities from Davydov's curriculum, this paper suggests that students can broaden their perspectives on the number line and use it versatilely in various areas of mathematics learning when they deeply engage in the construction of a number line and have flexibility in interpreting the relationships between key number line elements.

Keywords: number line, Davydov's elementary curriculum, unit, direction, origin
ZDM classification: D42
2000 Mathematics Subject Classification: 97D40

I. INTRODUCTION

The number line is a frequently utilized representation for the conceptual development of numbers in primary school mathematics lessons. In the context of the United States, the Common Core State Standards (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010), which are the academic standards that are widely adopted in many states in the United States indicate that the number line model is expected for students to use across multiple grades. For example, after some foundational experiences in length measurement are introduced in Kindergarten and first grade, one of the second grade standards indicates that students

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are expected to “represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to numbers 0, 1, 2, …, and represent whole-number sums and differences within 100 on a number line diagram” (CCSS 2. MD.6). In third grade, students should be able to “understand a fraction as a number on the number line; [and] represent fractions on a number line diagram” (CCSS 3. NF.2). Later a coordinate system, which uses a pair of perpendicular number lines called axes, is introduced. Later on, understanding the number line is even related to a vector, a quantity with a magnitude and direction, which provides some mathematical foundation for introductory physics (Kanopka, 2016).

While its utility is considerable, the number line is one of the most overlooked tools in mathematics classrooms (Frykholm, 2010) in that its conceptual foundation is rarely articulated. Therefore, it is important to give in-depth consideration to how students are first introduced to the concept of the number line because their early experiences with, and understanding of, the number line can either provide firm foundations for future learning or lead to misconceptions. This paper outlines an introductory approach that can be used in primary school in which the conceptual aspects of the number line are explicitly taught.

II. MISCONCEPTIONS AND MISUSES OF THE NUMBER LINE

Despite their frequent appearance in math instruction from early grades, number lines are not often explored in depth. This situation frequently leads to common misuses such as focusing on tick marks or numerals on a number line instead of the intervals and misunderstanding the nature of a unit (Cramer et al., 2016; van de Walle, Karp, & Bay-Williams, 2013). Similar misconceptions may be found among adults as well.

In a mathematics methods class for future elementary school teachers, which was taught by one of the authors in a Midwestern University in the United States, 52 pre-service teachers (PSTs) were engaged in eliciting and interpreting students’ thinking. To investigate fourth-grade students’ thinking about representing fractions on a number line diagram, the PSTs examined the state and district standards and developed assessment questions. PSTs posed questions they had prepared or conceived on the spot in one-on-one task-based interviews. Later, PSTs analyzed and interpreted students’ thinking based on the students’ responses and work samples.

It was notable that more than 90 % of the PSTs exclusively used horizontal number lines with pre-marked line segments and labels (see Figure 1) when they initially prepared their assessment questions, which structured the students’ thinking.

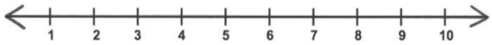
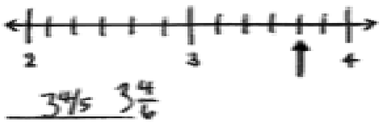
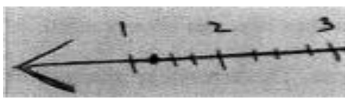


Figure 1. An example of a structured number line

In the very few cases in which PSTs either presented an empty number line or asked students to draw their own number line diagrams, several misconceptions were revealed in the students' work as well as PSTs' interpretations of the students' work. For more details, refer to the following examples in Table 1.

Table 1. Examples of misconceptions

Misconceptions	Example and description
1. We should count tick marks to place numbers on the number line. (Confusion in recognizing the unit)	Example: What fraction is the arrow on the numberline pointing to? 
	Description: Concerning a student's solution: A student was asked to determine the fraction value for the point indicated by the arrow. The student counted tick marks instead of intervals to determine the denominator. The student's answer was $3\frac{4}{5}$. A pre-service teacher corrected it to $3\frac{4}{6}$ while grading it.
2. A number line should always start with zero. (Stereotype about the origin)	Example: 
	Description: Concerning a pre-service teachers' interpretations: A student constructed a number line in order to locate 5/4. The PST conducting the interview interpreted that this student incorrectly used a number line because the origin (zero) is not noted, and thus the indicated point cannot represent 5/4.
3. A number line should always be presented horizontally and marked off. (Stereotype about the direction and structuring)	Example: 
	Description: Concerning a student's solution: A pre-service teacher asked a student to

	show 6/5 on an open number line that was presented vertically. The student attempted to write a couple of fractional numbers but did not complete the task, stating that it did not make sense. Two reasons were provided: (a) the number line should be presented horizontally, and (b) there should be lines or dots on the number line (i.e., pre-structured intervals and numerals).

These examples may imply an unstable understanding of the number line in both students' solutions and PSTs' interpretations. As educators have noted, when the structured number line is overused without deeper examination, students may come to expect a particular visual presentation and overlook key elements (Bartolini Bussi, 2015). Thus, in order to provide some insights on addressing these misconceptions, this paper examines specific key elements of a number line along with examples of activities that can be introduced in the earlier grades in the following sections.

III. WHAT CONSTITUTES A NUMBER LINE?

Gullberg (1997) describes a number line a line that extends "from negative infinity to positive infinity" and is "graduated in unit distance on both sides of the origin, which symbolizes zero" (p. 71). For example, to construct the number line for the natural numbers as shown in Figure 2, Tapia et al. (as cited in Herbst 1997) state:

> On the line R, one marks a point o and chooses a segment U as a unit. The segment U is translated consecutively from o to each point of division on matches sequentially a natural number (p. 36).

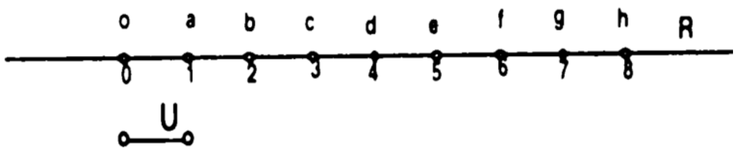


Figure 2. The number line for the natural numbers (Tapia et al. as cited in Herbst, 1997)

As noted, in order to construct a number line, an *origin* and *unit* should be selected as well as the measurement principle, which delineates the relationship between the unit and the quantity to be measured, should be considered for interpretation (i.e., the number labeling a point on the number line represents the distance between that point and the point of origin as measured in number of units). Also, in this way, the number line can be extended infinitely in both *directions* to represent various number systems. The arrows on the number line usually indicate this infinity, but the use of arrows differs by different cultures and curriculums (Suh, Shin, & Na, 2005).

All in all, the key elements that constitute a number line include an *origin* (starting point), a choice of *unit* (a selected line segment), and *direction* (polar orientation). With an absence of any of these elements, it is not feasible to place numbers on a number line. Thus, it is worthwhile to explore the construction of the number line focusing on these elements, rather than just providing fully structured number lines and treating them as universal. Although it is rare, some curriculum materials provide explicit exploration opportunities for these key elements. One such example is the mathematics curriculum developed by Davydov and his colleagues (Davydov, Gorbov, Mikulina, & Savel'eva, 1999), which is known for the application of Vygotskian theory to school mathematics. His curriculum develops number concept from the measurement of quantities rather than counting by considering the essence of mathematics as the science of quantity and relation (Lee, 2002, 2006; Schmittau 2003, 2010). An emblematic approach in this curriculum can be seen by the way of conceptualizing and utilizing the number line. Schmittau (2010) describes the use of number line in Davydov's curriculum:

> The number line arises in Davydov's curriculum from a consideration of simple medicinal dosage calculation using a graduated cylinder which is then tipped ninety degrees to create a horizontal gradient. The elements that are essential to the creation of a number line are also explored thoroughly, including direction, a starting point and choice of a unit. If any one of these is unspecified, it is impossible to determine where specific numbers will be located even if the other two elements are provided. When the number line is presented as a ready-made representation with these elements in place a priori, as is typically the case in the U.S., the arbitrary nature of these determinations remains undetected, since they are usually never explored (p. 273).

With a strong emphasis on resolving the separation between empirical and theoretical thinking as well as arithmetic and algebra, it is common to deal with the problems that highlight the relationship between quantities rather than numeric answers and the use of number line as a representational tool. For example, in Lee's (2006) implementation of Davydov's curriculum in a U.S. school, a story problem from the textbook (Davydov et al., 1999) was presented, and children were asked to use a number line to show their answer. The word problem asked for the number that was 2 more than the given number, a: "Peter has a candies. Lena has 2 candies more than Peter. How many candies does Lena have?" Along with this story problem, a number line marked a was provided. This problem resulted in three 1-hour discussion sessions over 3 days for an in-depth discussion on the relationship between two quantities, a and $a + 2$, on the number line representation. Lee (2006) asserts that students' learning engagement in the deeper exploration of quantitative relationship using the number line paid off later.

Similarly, some researchers have used number lines with primary school students to allow them to explore several examples of relationships between quantities before representing generalizations with equations (Carraher & Schliemann, 2007; Carraher, Schliemann, & Brizuela, 2006; Carraher, Schliemann, & Schwartz, 2008). For examples, Carraher, Schliemann, and Schwartz (2008) utilized a number line with variables to develop elementary students' concepts of unknowns and variables. Carraher and colleagues introduced a problem in which one student had a certain amount of money and the other student had 8 dollars more than the amount of money the former student had. During the lesson, students discussed different possible values for the two amounts of money by using a number line. That is, students were asked to indicate the other quantity on the number line after marking a known number on a number line to show the money that the former student has. After exploring the quantitative relationship with several examples, the students finally asked to use a letter to represent the indeterminate amounts of money. Through this experience, 62% of primary school students were able to understand the concept of unknowns and variables.

In the following section, we introduce some examples from Davydov's curriculum that promote students' exploration of the number line concepts in the early grades. They may shed light on the design of meaningful early exploration of the number line.

IV. DEEPER EXPLORATION OF KEY ELEMENTS OF THE NUMBER LINE

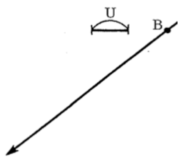
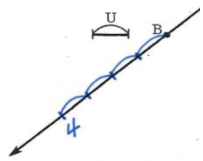
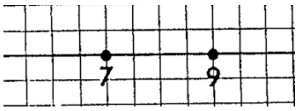
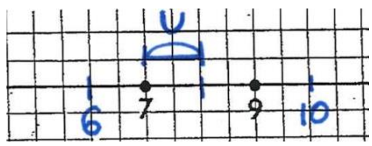
Typically, a number line is presented as a horizontal straight line with even demarcations neatly labeled with numbers that increase from left to right. This typical presentation is convenient; but should it always be the way a number line is shown? Its convenience may prevent students from perceiving the role of key elements when all of these are given. Below are examples of alternative presentations that promote understanding of the key elements of the number line by providing activities that purposefully omit one or two of these key elements. These activities, besides fostering exploration of key elements of the number line, will also help address the three misconceptions and errors that were mentioned in the previous section (see Table 1).

Activities 1, 2, and 3 presented below are from Grade 1 of Davydov's curriculum, indicating that an in-depth exploration of and generalizations about the number line may start at the very beginning of formal mathematics instruction. In these number line diagrams, one arrow is used to indicate the direction for the greater numbers as opposed to the two arrows that are commonly used in the U.S. curriculum. The origin is noted as B [beginning] for easier communication with young students, and the size of unit is noted as U.

1. MISCONCEPTION: WE SHOULD COUNT TICK MARKS TO PLACE NUMBERS ON THE NUMBER LINE

This is a well-known misconception in which students tend to count the tick marks rather than the intervals between tick marks (Bright, Behr, Post, & Wachsmuth, 1988). Through the following activity, students are asked to build their own number lines using the clues of size of unit (U) and direction.

Activity 1. Place the given numbers on the number line**.**

Problem	Clues	Expected solution
1-(a) Where is the number 4?  	Although it is not a horizontal line, all other relationships stay the same. Having the origin, direction, and the size of unit, the given number can be placed.	
1-(b) Is it possible to locate 6 and 10 on the number line?  	No explicit information is provided for all three key elements (origin, direction, and unit). However, the order of two numbers (7 and 9) and the distance between these two numbers provide clues for the direction and the size of unit.	

In Activity 1-(a), students can focus on the role of the unit in creating tick marks and labels by providing an empty number line. Activity 1-(b) asks students to identify the size of unit by using the implicitly given information; it is determined by the relationship between other key elements. If this understanding is not clear, students may solve Activity 1-(b) as shown in Figure 3. Instead of finding the size of unit based on the places of 7 and 9, students may assume that the unit is the width of one grid section.

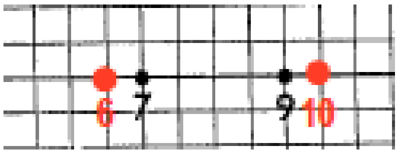
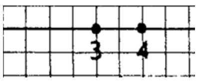
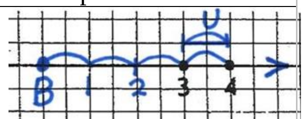
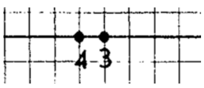
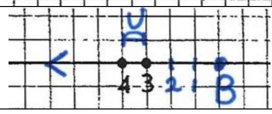
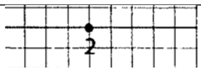


Figure 3. Example of incorrect identification of the unit

2. MISCONCEPTION: A NUMBER LINE ALWAYS STARTS WITH ZERO AND BIGGER NUMBERS SHOULD BE ALWAYS ON THE RIGHT SIDE

As noted, because the typical number line is presented as a line with numbers increasing from left to right, students may think any other configuration is wrong. Through the following activity, students need to use the clues on the partially constructed number lines (i.e., without identification of direction and origin) to find out the missing elements.

Activity 2. Mark the direction and the beginning on the number line

Problem		Clues	Expected solution
2-(a)		The size of unit is given (i.e., the distance between 3 and 4). The bigger number (4) is placed on the right side.	
2-(b)		The size of unit is given (i.e., the distance between 3 and 4). The bigger number (4) is placed on the left side.	
2-(c)		There are no clues to identify the size of unit and the direction.	Unsolvable

This activity guides students to pay attention to the flexible relationship between the three key elements: origin, unit, and direction. When two elements are given, the third element can be found. For example, in Activities 2-(a) and 2-(b), we can identify the origin because the given information provides clues for direction and unit. However, in Activity 2- (c), both the origin and direction are uncertain. If students are unable to notice this, they may solve the problem by assuming the unit as 1 section as typical as shown in Figure 4.

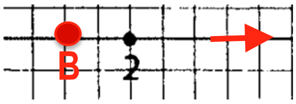
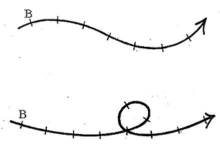
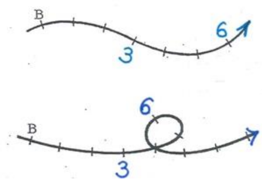


Figure 4. Example of unjustifiable identification of the unit and direction

3. MISCONCEPTION: A NUMBER LINE SHOULD ALWAYS BE PRESENTED HORIZONTALLY

Using a straight horizontal number line is undoubtedly convenient, for example, by making it easy to seriate numbers and present the size of unit. However, this stereotypical presentation is not necessarily applicable when students use the coordinate system where two number lines are used simultaneously. The following activity shows different ways of presenting number lines other than as a straight horizontal line. If students have a rigid idea of a number line as straight and horizontal, they may say that these problems are unsolvable.

Activity 3. Place the given numbers on the number line**.**

Problem	Clues	Expected solution
Show 3 and 6 on the different number lines. 	Given number lines are not straight lines. However, all other information on the origin, direction, and the size of unit is provided.	

Early exposure to the key elements of the number line would help students flexibly use and apply this understanding in various contexts. Although after this exploration students may eventually still be primarily exposed to the typical, structured, horizontal number lines, the time spent on this concept-building experience will pay off later on. The next section presents examples of activities in primary grades in which students can benefit from earlier in-depth exploration of the key elements of the number line.

V. IMPLICATIONS: BROADENING PERSPECTIVES ON THE NUMBER LINE

As shown in activities taken from Davydov's curriculum in the previous section, when students deeply engage in the construction of a number line and have flexibility in interpreting the relationships between key number line elements, they can broaden their perspective on the number line and use it versatilely in subsequent mathematical learning.

For example, knowing that the number line does not have to be straight and that a number line can be used to sort different types of numbers. A zigzag-shaped line is still a valid form of a number line with an origin, direction, and size of unit. At the same time, this presentation contains two different sets of number lines (see Figure 5 for odd-number line and even-number line). Other types of number sorting activities can be used in a similar way.

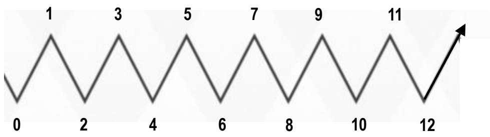


Figure 5. A number line for sorting even and odd numbers

When teaching rounding numbers to the nearest place, various strategies are used. Some treat this as a rule (e.g., when the number is 5 or more, round up, for the others round down). Others use metaphorical visual contexts like hills and valleys or camel-back to determine when and how to round numbers to the nearest places. However, when the alternative representations of number lines are explored, it can be easier to use the same explanation by providing a curved number line as shown in Figure 6.

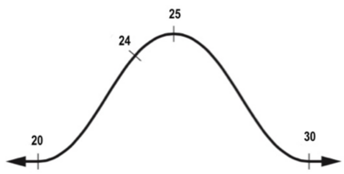


Figure 6. A curved number line showing that 24 rounded to the nearest ten is 20

Double number line diagrams are used when the quantities have different units, especially when dealing with the concept of ratio in the later grades. However, double number line diagrams can be used when discussing money-related concepts in early grades. For example, the non-proportional nature of coins often confuses students. A dime is smaller than a nickel, but the value of a dime is greater than that of a nickel. To proportionate the value of coins, the number lines in Figure 7 can be utilized to compare different units in each number line and the relationship between the number lines.

Figure 7. Coin lines
(Source: Money-in-Mind® : Coinline, (S)HE LEANRING UNLIMITED®)

While there are reasons why the typical number line is a straight line from left to right, when the completed number line is always given, students may not pay attention to the key elements of a number line and the relationship between the visual representation and symbolic labels on it. As a result, the student may be limited to ordering numbers from his/her original perspective. We are not suggesting that all kinds of idiosyncratic number lines should be used all the time. Rather, we emphasize that the mathematical essence of the number line should be clearly explored through various activities in the early grades before students learn to use it only as a mechanical tool. Procedural fluency should be developed on a foundation of conceptual understanding. Such an approach is applicable to teaching and using the number line.

VI. CLOSING REMARKS

When considering the usefulness of the number line in learning mathematics, building up strong conceptual foundations for the number line in the earlier grades is important. In accordance with this need, this paper looked closely into the key components of number lines such as unit, direction, and origin and presented useful activities from Davydov's curriculum. However, to confirm the impact of the activities suggested in this article on early elementary grade students' conceptual development of number line, empirical research targeting early elementary students will be required in the future.

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