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SERIAL EXECUTION JOSEPHUS PROBLEM

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Abstract. In this paper, we will study a generalized version of Josephus where a serial execution occurs at each iteration and give a non-recursive formula for the initial positions of survivors.

1. Introduction

Josephus problem is named after Flavius Josephus, a Jewish historian in the first century [4]. The problem statement is as follows: There are n people, numbered from 0 to $n-1$ in a circle. In the first iteration, the person 0 is skipped and the person 1 is executed. In the second iteration, the person 2 is skipped and the person 3 is executed. Then the iteration is repeated until there is only one person remaining. A question that may arise is where to stand to be the last survivor. The Josephus problem is to find the initial position of the survivor in the scenario described above. Popular generalization of the problem is to allow skipping more than one person followed by executing one person. This version has been extensively studied in [2], [3], [5], [6], [8], [11], and [12]. In [7], Ruskey and Williams introduced an interesting variant of the problem in which each person has multiple lives, hence the name is a "feline Josephus problem". Sharma et al [9] studied a version where

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the number of persons that are skipped in each iteration changes. The original Josephus problem is also a main idea behind a famous magic trick [1] and [10] provided the mathematical generalization of the magic trick.

2. Generalized Josephus Problem

In this section, we generalize Josephus Problem. Suppose there are n people, numbered from 0 to $n-1$ in a circle. Starting from the person 0, the following iteration is applied: skip s people and execute k people after. The iteration is repeated until there are at most k people left. We exclude the case where there is no survivor. Without loss of generality, we assume that n, k, and s are positive integers and $n \geq k$.

PROPOSITION 2.1. *Let I be the total number of iterations and P the number of survivors at the end. Then*

$$I = \begin{cases} \lfloor \frac{n}{k} \rfloor & \text{if } n \not\equiv 0 \pmod{k} \\ \lfloor \frac{n}{k} \rfloor - 1 & \text{if } n \equiv 0 \pmod{k}. \end{cases}$$

and

$$P = \begin{cases} n \bmod k & \text{if } n \not\equiv 0 \pmod{k} \\ k & \text{if } n \equiv 0 \pmod{k}. \end{cases}$$

Proof. It is obvious since after each iteration, the number of people will decrease by k and we repeat until the number of people is less than or equal to k. □

DEFINITION 2.2. Suppose that we have a situation described above. A generalized Josephus problem $J(n, s, k; i)$ is to find the initial position of the i_{th} survivor after I iterations.

With this definition, the original Josephus problem can be viewed as $J(n, 1, 1; 1)$. We are interested in finding the initial positions of all survivors without performing the procedure. To understand the problem better, we view skipping s people as shifting the starting point of the positions of people in the sequence and executing k people as deleting them from the sequence. For example, in the beginning of $J(10, 2, 3; 1)$, we have

⓪①②③④⑤⑥⑦⑧⑨

Then, after skipping 2 people, we have

②③④⑤⑥⑦⑧⑨⓪①

Then, we execute 3 people and we have

⑤⑥⑦⑧⑨⓪①

After another iteration is applied,

⓪①⑤⑥

Then, after the final iteration,

①

Thus, ① is the survivor. To be consistent with modulo operation, the positions of people start at 0. As we see, the starting point of the positions changes after each iteration. The starting point in the beginning is ⓪ and the position of ① is 1. After the first iteration, the starting point is ⑤ and the position of ① is 6. After the second iteration, ⓪ is the starting point and the position of ① is 1. Then, after the final iteration, the position of ① is 0. So, it is easy to see that, in $J(n, s, k; i)$, the position of the i_{th} survivor at the end is simply $i - 1$. To find the explicit formulas, we will take a backward approach.

Definition 2.3. For $0 \leq j \leq I$, $S_i(j)$ denotes the position of the i_{th} survivor and $N(j)$ represents the number of people at the $(I - j)_{th}$ iteration.

Note that the position of the i_{th} survivor at the end is $S_i(0) = i-1$ for $1 \leq i \leq P$ and the number of people at the end is $N(0) = P$. Using this notation, we can have a following recurrence relation: For $1 \leq i \leq P$ and $0 \leq j \leq I - 1$,

$$N(j+1) = N(j) + k$$
$$S_i(j+1) = S_i(j) + (k+s) \pmod{N(j+1)}$$

This recurrence relations suggests that while $N(j+1) > S_i(j) + (k+s)$, $S_i(j+1) = S_i(j) + (k+s)$ and since $S_i(\cdot)$ increases faster than $N(\cdot)$, $S_i(\cdot)$ will eventually catch up $N(\cdot)$ and move to the front of the sequence. To find out when this would happen, we introduce the following term:

Definition 2.4. The m_{th} crossover of the i_{th} survivor, denoted by $Cr_i(m)$, is the number of iterations, j, such that $0 \leq S_i(j) < s$ for the m_{th} time.

In the rest of the paper, we focus on $J(n,1,k;i)$ where k consecutive people are executed in each iteration.

LEMMA 2.5. *In $J(n,1,k;i)$, $S_i(Cr_i(m)) = 0$ for all m.*

Proof. When the m_{th} crossover happens, $0 \leq S_i(Cr_i(m)) < s$ by the definition. Since $s = 1$ in $J(n,1,k:i)$, $S_i(Cr_i(m)) = 0$. □

LEMMA 2.6. *In $J(n,1,k;i)$ with P survivors,*

$$N(Cr_i(1)) = P + (P - i + 1)k.$$

Proof. Consider the i_{th} survivor where $1 \leq i \leq P$. We already know that $S_i(0) = i - 1$. Then $Cr_i(1) = j$ where j is the smallest integer satisfying

$$N(j) \leq S_i(j-1) + (k+1).$$

Then $j = \lceil l \rceil$ such that

$$P + l \cdot k = (i-1) + l \cdot (k+1).$$

Since $l = P - i + 1$, which is an integer, $j = l$, i.e. $Cr_i(1) = (P - i + 1)$. Thus, $N(Cr_i(1)) = P + (P - i + 1)k$. □

LEMMA 2.7. *For all positive integer m,*

$$N(Cr_i(m)) = (P + (P - i + 1)k)(k+1)^{m-1}.$$

Proof. We will prove this using the mathematical induction. Lemma 2.6 proves the base case. Assume that for any $m \geq 1$, $N(Cr_i(m)) = (P + (P - i + 1)k)(k+1)^{m-1}$. Then $Cr_i(m+1) = Cr_i(m) + j$ where j is the smallest positive integer satisfying

$$N(Cr_i(m)) + k \cdot j \leq S_i(Cr_i(m)) + (k+1) \cdot j.$$

Since $S_i(Cr_i(m)) = 0$ by Lemma 2.5, $Cr_i(m+1) = Cr_i(m) + \lceil l \rceil$ where l is the solution of the following equation:

$$(P + (P - i + 1)k)(k+1)^{m-1} + k \cdot l = (k+1) \cdot l.$$

Thus, by solving the equation, we can find the number of iterations required to reach from $Cr_i(m)$ to $Cr_i(m+1)$. In this case, the value of l is an integer and

$$l = (P + (P - i + 1)k)(k+1)^{m-1}.$$

So,

$$
\begin{aligned}
&N(Cr_i(m+1)) \\
&= N(Cr_i(m)) + l \cdot k \\
&= (P + (P-i+1)k)(k+1)^{m-1} + (P + (P-i+1)k)(k+1)^{m-1} \cdot k \\
&= (P + (P-i+1)k)(k+1)^{m-1}(1+k) \\
&= (P + (P-i+1)k)(k+1)^{m}.
\end{aligned}
$$

This completes the proof. □

THEOREM 2.8. *In $J(n,1,k;i)$ with P survivors, for $1 \leq i \leq P$, n can be decomposed as*

$$n = (P + (P-i+1)k) \cdot (k+1)^{\alpha} + \beta \cdot k$$

where α and β are nonnegative integers and α is the highest such power. Consequently, the initial position of the i_{th} survivor, $S_i(I)$, is

$$S_i(I) = \beta \cdot (k+1).$$

Proof. For $1 \leq i \leq P$, suppose there are α crossovers. Then

$$n = N(Cr_i(\alpha)) + \beta \cdot k$$

for some nonnegative integer β. So, by Lemma 2.7,

$$n = (P + (P-i+1)k) \cdot (k+1)^{\alpha} + \beta \cdot k.$$

Now consider the initial position of the i_{th} survivor. By Lemma 2.6 and Lemma 2.7, when $N(I-\beta) = (P + (P-i+1)k) \cdot (k+1)^{\alpha}$, the α_{th} crossover occurs. This implies

$$S_i\,(I-\beta) = 0.$$

So, by applying β iterations,

$$S_i(I) = \beta \cdot (k+1).$$

□

Here is an example.

EXAMPLE 2.9. Consider $J(207,1,4;i)$. Since $207 \bmod 4 = 3$ and $\left\lfloor \frac{207}{4} \right\rfloor = 51$, $P = 3$ and $I = 51$.

- For $J(207, 1, 4; 1)$, $(P + (P - i + 1) \cdot k) = 15$. So
$$207 = 15 \cdot 5^1 + 132 = 15 \cdot 5^1 + 33 \cdot 4.$$
Thus, the initial position, $S_1(51) = 33 \cdot 5 = 165$, i.e. So the 166_{th} person will be the first survivor.
- For $J(207, 1, 4; 2)$, $(P + (P - i + 1) \cdot k) = 11$.
$$207 = 11 \cdot 5^1 + 152 = 15 \cdot 5^1 + 38 \cdot 4.$$
Thus, the initial position, $S_2(51) = 38 \cdot 5 = 190$. Thus, the 191_{st} person will be the second survivor.
- For $J(207, 1, 4; 3)$, $(P + (P - i + 1) \cdot k) = 7$. So
$$207 = 7 \cdot 5^2 + 32 = 7 \cdot 5^2 + 8 \cdot 4.$$
So, the initial position, $S_3(51) = 8 \cdot 5 = 40$. Thus, the 41_{st} person will be the third survivor.

3. Conclusion

In this paper, we provided the explicit formulas of the positions of the survivors in $J(n, 1, k; i)$ for any positive integer k. The general case of the Josephus problem, $J(n, s, k; i)$, is still open. We will investigate other cases to which the same approach can be applied. We will continue studying the behavior of the crossovers for general case as we have seen that their behaviors were closely related to the explicit formulas. It is also very interesting to consider the serial execution case with multiple lives similar to the case studied in [7].

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