ON m-ISOMETRIC TOEPLITZ OPERATORS

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Abstract. In this paper, we study m-isometric Toeplitz operators $T_\varphi$ with rational symbols. We characterize m-isometric Toeplitz operators $T_\varphi$ by properties of the rational symbols $\varphi$. In addition, we give a necessary and sufficient condition for Toeplitz operators $T_\varphi$ with analytic symbols $\varphi$ to be m-expansive or m-contractive. Finally, we give some results for m-expansive and m-contractive Toeplitz operators $T_\varphi$ with trigonometric polynomial symbols $\varphi$.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. In 1990’s, Agler and Stankus [2] intensively studied the following operators; for a fixed positive integer $m$, we denote

$$B_m(T) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j}T^j$$

for an operator $T \in \mathcal{L}(\mathcal{H})$. We say that $T \in \mathcal{L}(\mathcal{H})$ is m-expansive if $B_m(T) \leq 0$ for some positive integer $m$. In particular, if $B_m(T) = 0$, then $T$ is said to be m-isometric. When $B_m(T) \geq 0$, we say that $T$ is m-contractive.

The class of m-isometric operators has been widely investigated in latest years. In [1], J. Agler characterized subnormality with the positivity of $B_m(T)$ and also extended his results to the concept of m-isometric operators. The theory of these operators was investigated especially by Agler and Stankus [2–4]. In these papers, they developed a theory for the m-isometric operators with rich connections to Toeplitz operators and function theory. Recently, there has been worked on products of m-isometries [6] and m-isometric composition operators [17]. Many researchers have extensively studied the isometric Toeplitz operators in various ways; see [11–13] and the references therein. Based on these papers, we are studying the m-isometric Toeplitz operators.

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A function $\theta \in H^\infty$ satisfies $|\theta| = 1$ a.e. on $\mathbb{T}$ is an inner function. If $\theta$ is an inner function, the degree of $\theta$, denoted by $\deg \theta$, is defined as $n + s$ if $\theta$ is a finite Blaschke product of the form

$$
\theta(z) = e^{i\xi} z^n \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \overline{\alpha}_j z} \quad (|\alpha_j| < 1 \text{ for } j = 1, 2, \ldots, n),
$$

otherwise the degree of $\theta$ is infinite. For an inner function $\theta$, write

$$
\mathcal{H}(\theta) := H^2 \ominus \theta H^2.
$$

In [14], it was shown that if $f \in H^\infty$ is a rational function, then we can write

$$
f = \theta a,
$$

where $\theta$ is a finite Blaschke product and $a \in H^\infty$ satisfies that the inner parts of $a$ and $\theta$ are coprime. For $\varphi$ in $L^\infty(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial \mathbb{D}$, the Toeplitz operator $T_\varphi$ with symbol $\varphi$ on the Hardy space $H^2(\mathbb{T})$ is given by

$$
T_\varphi f := P(\varphi f) \quad (f \in H^2(\mathbb{T})),
$$

where $P$ denotes the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

If $\varphi, \psi$ are in $L^\infty(\mathbb{T})$, then it is well-known that

1) $T_{\varphi + \psi} = T_\varphi + T_\psi$,
2) $T_{\varphi}^* = T_{\overline{\varphi}}$,
3) $T_{\varphi} T_{\psi} = T_{\varphi \psi}$ if $\varphi$ or $\psi$ is analytic.

These properties enable us to establish several consequences of $m$-isometric operators.

This paper is organized as follows. In Section 2, we study some properties of $m$-isometric Toeplitz operators. In particular, we give several results for the $m$-isometric Toeplitz operators with rational symbols. In Section 3, we establish some results for the $m$-expansive and $m$-contractive Toeplitz operators.

### 2. $m$-isometric operators

First, we briefly recall the definitions and some elementary properties of Toeplitz operators and $m$-isometric operators. We refer the reader to [2–4, 8] for further references.

Given a positive integer $m$, it follows from definition that an operator $T \in \mathcal{L}(\mathcal{H})$ is an $m$-isometry if and only if

$$
\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} |T^j x|^2 = 0 \text{ for all } x \in \mathcal{H}.
$$

The above formulation was used to define $m$-isometries on a Banach space by Sid Ahmed [5] and by Botelho [7] on $l_p$ spaces and general function spaces.
Using the identity (2.1) and the Toeplitz operator with symbol $\varphi$, we consider the following equation
\[(2.2) \quad \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} |T_{\varphi}^j k|^2 = 0\]
for all $k \in H^2(T)$.

In [8], A. Brown and P. R. Halmos characterize isometric Toeplitz operators $T_{\varphi}$ by properties of the symbols $\varphi$.

**Lemma 2.1** ([8]). A Toeplitz operator $T_{\varphi}$ is an isometric operator if and only if $\varphi$ is inner.

We recapture the following lemma for the convenience of the readers.

**Lemma 2.2** ([2]). If $T$ is an $m$-isometry, then it is an $m + 1$-isometry.

**Proof.** If $T$ is an $m$-isometry, then $B_m(T) = 0$ from (1.1). Since $B_{m+1}(T) = T^* T_m(T) T - B_m(T)$, $B_{m+1}(T) = 0$. Hence $T$ is an $m + 1$-isometry. This completes the proof. □

Next, we give several results of $m$-isometric Toeplitz operators. The following results are the consequences of $m$-isometric Toeplitz operators with rational symbols.

**Lemma 2.3.** If a Toeplitz operator $T_{\varphi}$ with rational symbols $\varphi$ is an $m$-isometry, then $\varphi$ is analytic.

**Proof.** Suppose that $\varphi(z) = f + \overline{g}$ is a rational function. Then we can write
\[f(z) = \theta_1 \overline{a} \quad \text{and} \quad g(z) = \theta_2 \overline{b}\]
for some finite Blaschke products $\theta_1$ and $\theta_2$, where $a \in H(\theta_1)$ and $b \in H(\theta_2)$.

Since $T_{\varphi}$ is an $m$-isometry,
\[\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_{\varphi}^j T_{\varphi}^j k = 0\]
holds for all $k \in H^2(T)$. Put $k(z) = c$ for some nonzero constant $c$. Then
\[\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_{\varphi}^j T_{\varphi}^j c = 0\]

\[= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_{\overline{f} + g}^j T_{\overline{f} + g}^j c\]

\[= T_{\overline{f} + g}^m T_{\overline{f} + g}^m c - m T_{\overline{f} + g}^{m-1} T_{\overline{f} + g}^{-1} c + \cdots + (-1)^{m-1} T_{\overline{f} + g} T_{\overline{f} + g} c + (-1)^m c.\]

Since the maximal degree term of the above relation is included only in $T_{\overline{f} + g}^m T_{\overline{f} + g}^m c$ term and the maximal degree term $c \theta_1^m \theta_2^m a^m \overline{b}^m$ must be a zero,
we have either $f$ or $g$ is zero. If $f = 0$, i.e., $\varphi = \overline{g}$, then for some nonzero constant $c \in H^2(T)$,

$$0 = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^*_j T_j^j c$$

$$= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^*_j T_j^j g c$$

$$= (-1)^{m} c,$$

we have a contradiction. Therefore $g = 0$ and hence $\varphi$ is analytic. This completes the proof. □

Recall that $T \in \mathcal{L}(\mathcal{H})$ is said to be subnormal if $T$ has a normal extension, i.e., $T = N|_{\mathcal{H}}$, where $N$ is a normal operator on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\mathcal{H}$ is invariant for $N$. From Lemma 2.3, we get the following corollary immediately.

**Corollary 2.4.** Every $m$-isometric Toeplitz operators $T_\varphi$ with rational symbols $\varphi$ is subnormal.

We next show that every $m$-isometric Toeplitz operators with rational symbol is an isometry.

**Theorem 2.5.** Let $\varphi$ be a rational function. A Toeplitz operator $T_\varphi$ is an $m$-isometry if and only if $T_\varphi$ is an isometry.

**Proof.** If $T_\varphi$ is an $m$-isometry, Lemma 2.3 ensures that $\varphi$ is analytic. Put $\varphi = f$ where $f \in H^\infty$. Then

$$T_\varphi^m T_\varphi = T_f^m T_f = T_f^m f_{m-j}.$$

Hence

$$0 = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_f^j T^*_f$$

$$= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^*_f f_j$$

$$= T \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f_{m-j}$$

$$= T f_{(m-1)^{m}}.$$

Thus $\overline{f} f = 1$, or equivalently, $\varphi \overline{\varphi} = 1$ and by Lemma 2.1, $T_\varphi$ is an isometry. The converse implication is trivial by Lemma 2.2. □

From Theorem 2.5, we get the following results.
Corollary 2.6. Suppose that $\varphi$ is a rational function. If $T_\varphi$ and $T^*_\varphi$ are $m$-isometric operators, then $T_\varphi$ is unitary and $\sigma(T_\varphi) \subset \partial \mathbb{D}$.

Corollary 2.7. Suppose that $\varphi$ is a rational function. Then $T_\varphi$ is an $m$-isometry if and only if $\varphi$ is a finite Blaschke product.

Proof. If $T_\varphi$ is an $m$-isometry, then it is a symmetry from Theorem 2.5. Hence it follows from Lemma 2.1 that $\varphi$ is inner. Since $\varphi$ is rational, $\varphi$ is a finite Blaschke product. □

As some applications of $m$-isometric Toeplitz operators, we talk about the hyponormal Toeplitz operators. Hyponormal operators are closely connected to $m$-isometric operators; see [9,18]. Recall that $T \in \mathcal{L}(\mathcal{H})$ is said to be hyponormal if its self-commutator $[T^*, T] := T^*T - TT^* \geq 0$. As considering Toeplitz operators with symbol $\varphi \in L^\infty(T)$, the relationship between the positivity of the self-commutator $[T^*_\varphi, T_\varphi]$ and the symbol $\varphi$ was solved by C. Cowen [10] in 1988.

Lemma 2.8 (Cowen’s Theorem [10]). For $\varphi \in L^\infty(T)$, write
\[ \mathcal{E}(\varphi) := \{ k \in H^\infty(T) : \|k\|_\infty \leq 1 \text{ and } \varphi - k\varphi \in H^\infty(T) \}. \]

Then $T_\varphi$ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

The following lemma is a result on hyponormal Toeplitz operators with a finite rank self-commutator.

Lemma 2.9 (Nakazi-Takahashi Theorem [16]). A Toeplitz operator $T_\varphi$ is hyponormal and $[T^*_\varphi, T_\varphi]$ is a finite rank operator if and only if there exists a finite Blaschke product $k$ in $\mathcal{E}(\varphi)$. In this case, we can choose $k$ such that $\deg(k) = \text{rank}[T^*_\varphi, T_\varphi]$.

Using Lemma 2.9, authors in [14] characterized the rank of self-commutator as follows.

Lemma 2.10 ([14]). Let $\varphi = \bar{g} + f \in L^\infty$, where $f$ and $g$ are in $H^2$. If $\varphi$ is of bounded type and $T_\varphi$ is hyponormal, then
\[ \text{rank}[T^*_\varphi, T_\varphi] = \min\{\deg(k) : k \text{ is an inner function in } \mathcal{E}(\varphi)\}. \]

Next, we deduced the rank of self-commutator of $m$-isometric Toeplitz operators.

Theorem 2.11. Suppose that $T_\varphi$ is an $m$-isometric Toeplitz operator with rational symbols. Then $\deg(\varphi) = \text{rank}[T^*_\varphi, T_\varphi]$.

Proof. By Corollary 2.4, $T_\varphi$ is subnormal and hence hyponormal. Since $T_\varphi$ is $m$-isometric, from Corollary 2.7, $\varphi$ is a finite Blaschke product. Put $\varphi = \theta$ where $\theta$ is a finite Blaschke product. Thus if $k \in \mathcal{E}(\varphi)$ is inner, then $k = \varphi(0)h$ for some $h \in H^\infty (0 \in \mathcal{E}(\varphi))$. By Lemma 2.10, $\text{rank}[T^*_\varphi, T_\varphi] = \deg(\theta) = \deg(\varphi)$. This completes the proof. □
Example 2.12. Suppose that \( \varphi \) is a finite Blaschke product of the form \( \varphi(z) = \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \) (\(|\alpha_j| < 1\) for \( j = 1, 2, \ldots, n \)). Then \( [T_\varphi^*, T_\varphi] = T_\varphi^* T_\varphi - T_\varphi T_\varphi^* = I - T_\varphi T_\varphi^* \). By Theorem 2.11, \( \text{rank}(I - T_\varphi T_\varphi^*) = \deg(\varphi) = n \).

3. Expansive and contractive operators

In this section, we study the \( m \)-expansive and \( m \)-contractive Toeplitz operators with trigonometric polynomial symbols.

It follows from definition that for \( \varphi \in L^\infty(\mathbb{T}) \), a Toeplitz operator \( T_\varphi \) is \( m \)-expansive if and only if

\[
(3.1) \quad \sum_{j=0}^{m} (-1)^{m-j} \left( \begin{array}{c} m \\ j \end{array} \right) \| T_\varphi^j k \|^2 \leq 0 \quad \text{for all } k \in H^2(\mathbb{T})
\]

and \( m \)-contractive if and only if

\[
(3.2) \quad \sum_{j=0}^{m} (-1)^{m-j} \left( \begin{array}{c} m \\ j \end{array} \right) \| T_\varphi^j k \|^2 \geq 0 \quad \text{for all } k \in H^2(\mathbb{T})
\]

By definition of expansive operators and properties of Toeplitz operators, the following lemma is easily checked.

Lemma 3.1. For \( \varphi \in L^\infty(\mathbb{T}) \),

(i) \( T_\varphi \) is expansive if and only if \( \| \varphi \|_\infty \leq 1 \);

(ii) \( T_\varphi \) is contractive if and only if \( \| \varphi \|_\infty \geq 1 \).

Proof. (i) Suppose that \( \| \varphi \|_\infty \leq 1 \). Let \( \varphi = f + \overline{g} \) with \( f, g \in H^\infty \). Then

\[
\| T_\varphi^j k \|^2 = \| P(\varphi k) \|^2 \leq \| \varphi k \|^2 \leq \| \varphi \|_\infty \| k \|^2 \leq \| k \|^2.
\]

Hence \( T_\varphi \) is expansive. Conversely, suppose that \( T_\varphi \) is expansive, i.e., \( \| T_\varphi^j k \|^2 \leq \| k \|^2 \) for all \( k \in H^2 \). Then \( \| T_\varphi \| \leq 1 \). Since \( \| T_\varphi \| = \| \varphi \|_\infty \), \( \| \varphi \|_\infty \leq 1 \).

(ii) Since \( T_\varphi \) is contractive if and only if \( B(T) \geq 0 \), we get the result with the same method. This complete the proof.

Theorem 3.2. Suppose that \( T_\varphi \) is a Toeplitz operator with trigonometric polynomial symbol \( \varphi = f + \overline{g} \) where \( f, g \in H^\infty(\mathbb{T}) \). If \( T_\varphi \) is 2-expansive, then \( |f| = 1 \) and \( P(\overline{g} f) = 0 \).

Proof. Suppose that \( \varphi(z) = f + \overline{g} \) where \( f, g \in H^\infty(\mathbb{T}) \). Put \( k(z) = \sum_{i=0}^{\infty} c_i z^i \). Then we have

\[
\| T_\varphi k \|^2 = \| P(f k + \overline{g} k) \|^2 = \| f k + P(\overline{g} k) \|^2
\]

and

\[
\| T_\varphi^2 k \|^2 = \| f^2 k + f P(\overline{g} k) + P(\overline{g} f k) + P(\overline{g} P(\overline{g} k)) \|^2.
\]

From the relation (3.1), \( T_\varphi \) is 2-expansive if and only if

\[
(3.3) \quad \| f^2 k + f P(\overline{g} k) + P(\overline{g} f k) + P(\overline{g} P(\overline{g} k)) \|^2 - 2\| f k + P(\overline{g} k) \|^2 + |k|^2 \leq 0
\]
for all $k \in H^2(T)$. Put $k(z) = c$ for some nonzero constant $c$. Then from (3.3) we have
\[ |cf^2 + cP(\overline{gf})|^2 - 2|cf|^2 + |c|^2 \leq 0 \]
or equivalently,
\[ |cf^2|^2 + |cP(\overline{gf})|^2 + 2\text{Re}(cf^2, cP(\overline{gf})) - 2|cf|^2 + |c|^2 \leq 0. \]
Since $\text{Re}(cf^2, cP(\overline{gf})) \leq |c|^2 |f|^2 ||P(\overline{gf})||$, we have
\[ |c|^2 ((|f|^2 - 1)^2 + |P(\overline{gf})|^2 + 2|f|^2 ||P(\overline{gf})||) \leq 0. \]
Hence if $T_\varphi$ is 2-expansive, then $||f|| = 1$ and $P(\overline{gf}) = 0$. \(\square\)

In the following example, we show that the converse of Theorem 3.2 does not hold.

**Example 3.3.** Suppose that $\varphi(z) = f + \overline{g} = 1 + z$. Then $||f|| = 1$ and $P(\overline{gf}) = 0$. But for $k(z) = 1 + z$, a straightforward calculation shows that
\[ |T_\varphi k|^2 - 2||T_\varphi k||^2 + |k|^2 = 2 > 0. \]
Therefore $T_\varphi$ is not 2-expansive. Hence the converse of Theorem 3.2 does not hold.

**Corollary 3.4.** Suppose that $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ with $a_N \neq 0$ where $||\varphi||_{\infty} \leq 1$. Then $T_\varphi$ is expansive but not 2-expansive.

**Proof.** From Lemma 3.1, it is obvious that $T_\varphi$ is expansive. Set $f(z) = \sum_{n=0}^{N} a_n z^n$ and $g(z) = \sum_{n=1}^{N} a_{-n} z^n$. Since $P(\overline{gf}) \neq 0$, $T_\varphi$ is not 2-expansive from Theorem 3.2. \(\square\)

**Corollary 3.5.** Suppose that $T_\varphi$ is hyponormal with polynomial symbols and 2-expansive if and only if $\varphi \in H^\infty(T)$ with $||\varphi|| = 1$.

**Proof.** For $\varphi = f + \overline{g}$, if $T_\varphi$ is hyponormal, then $\deg f \geq \deg g$. From Theorem 3.2, if $T_\varphi$ is 2-expansive, then $||f|| = 1$ and $P(\overline{gf}) = 0$. Set $f(z) = \sum_{n=0}^{N} a_n z^n$ and $g(z) = \sum_{n=1}^{N} a_{-n} z^n$ with $N \geq m$ and $a_N a_{-m} \neq 0$. Since
\[ P(\overline{gf}) = P \left( \sum_{n=0}^{N} a_n z^n \cdot \sum_{n=1}^{m} a_{-n} z^n \right) = P \left( \sum_{i=0}^{N} \sum_{j=0}^{m} a_i \overline{a_j} z^i \overline{z^j} \right) \]
$P(\overline{gf}) = 0$ if and only if $a_N \overline{a_j} = 0$ for some nonzero $a_j$, which is a contradiction. Hence we conclude that $g = 0$, and so $\varphi = f$ with $||\varphi|| = 1$. Conversely, suppose that $\varphi \in H^\infty$ with $||\varphi|| = 1$. Then $T_\varphi$ is hyponormal. And from Lemmas 2.1 and 2.2, $T_\varphi$ is a 2-isometry and hence 2-expansive. This completes the proof. \(\square\)

Next, we consider the $m$-expansive Toeplitz operators with analytic symbols.

**Theorem 3.6.** Suppose that $\varphi$ is analytic. Then

(i) If $m$ is even, then $T_\varphi$ is $m$-expansive if and only if $T_\varphi$ is an isometry;
(ii) If $m$ is odd, then $T_{\varphi}$ is $m$-expansive if and only if $|\varphi| \leq 1$. In particular, if $|\varphi| = 1$, then $T_{\varphi}$ is an isometry.

Proof. (i) Clearly $T_{\varphi}$ is an $m$-isometry implies $T_{\varphi}$ is $m$-expansive. If $\varphi(z)$ is an analytic, then from the relation (1.1), we have

$$
\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_{\psi}^j T_{\varphi}^j = \sum_{j=0}^{m} \binom{m}{j} T_{\varphi}^j
$$

$$
= T_{\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \overline{\varphi}^j}
$$

$$
= T_{(\overline{\varphi}^j-1)^m}.
$$

Since $T_{\varphi}$ is $m$-expansive, we have

$$
\langle P((\overline{\varphi}^j-1)^m k), k \rangle \leq 0
$$

for all $k \in H^2(\mathbb{T})$. Hence

$$
\langle P((\overline{\varphi}^j-1)^m k), k \rangle = \langle (\overline{\varphi}^j-1)^m k, k \rangle = ||(\overline{\varphi}^j-1)^m k||^2 \leq 0.
$$

Hence we have that $||(\overline{\varphi}^j-1)^m k||^2 \leq 0$ if and only if $|\varphi| = 1$. Moreover, by Lemma 2.1, $T_{\varphi}$ is an isometry.

(ii) From the inequality (3.4), $T_{\varphi}$ is $m$-expansive if and only if

$$
\langle P((\overline{\varphi}^j-1)^m k), k \rangle \leq 0.
$$

Applying the H"older-McCarthy inequality introduced in [15], we have

$$
\langle (\overline{\varphi}^j-1)^m k, k \rangle \leq ||k||^{2(m-1)}(\overline{\varphi}^j-1)^m k, k).
$$

Hence $T_{\varphi}$ is $m$-expansive if and only if $|\varphi| \leq 1$. This complete the proof. □

Corollary 3.7. Every subnormal and $m$-expansive Toeplitz operator is an isometry where $m$ is a positive even number.

Corollary 3.8. Suppose that $\varphi$ is analytic. If $T_{\varphi}$ is $m$-expansive, then $\|T_{\varphi}\| \leq 1$.

Proof. Since $\|T_{\varphi}\| = |\varphi|_\infty$, the proof follows from Theorem 3.6. □

Example 3.9. Suppose that $\varphi(z) = (\frac{a-z}{1-\overline{a}z})$ where $\alpha, \overline{\alpha} \in \mathbb{C}$. By Theorem 3.6, for every positive even number $m$, $T_{\varphi}$ is $m$-expansive, and $T_{\varphi}$ is an $m$-isometry if and only if $|\alpha| = 1$. But from Lemma 3.1 and Theorem 3.6, $T_{\varphi}$ is expansive (contractive) if and only if $|\alpha| \leq 1$ ($|\alpha| \geq 1$), respectively.

Now, we study $m$-contractive Toeplitz operators with trigonometric polynomial symbols. The following result is necessary and sufficient conditions for the 2-contractive Toeplitz operators with coanalytic inner symbol.

Proposition 3.10. Suppose that $\varphi(z) = a_k z^k$. Then $T_{\varphi}$ is 2-contractive if and only if $|a_k| \leq \frac{1}{\sqrt{2}}$. 
Proof. From the relation (3.2) for $m = 2$, put $k(z) = \sum_{i=0}^{\infty} c_i z^i$ ($c_i \in \mathbb{C}$ $(i = 0, 1, 2, \ldots)$). Then
\[
|T_\varphi k|^2 = \left\| a_{-k} \sum_{i=k}^{\infty} c_i z^{i+k} \right\|^2 = |a_{-k}|^2 \sum_{i=k}^{\infty} |c_i|^2
\]
and
\[
|T_\varphi^2 k|^2 = \left\| P \left( a_{-k} \sum_{i=2k}^{\infty} c_i z^{i+2k} \right) \right\|^2 = |a_{-k}|^4 \sum_{i=2k}^{\infty} |c_i|^2.
\]
Hence, $T_\varphi$ is $2$-contractive if and only if
\[
\|T_\varphi^2 k\|^2 - 2|T_\varphi k|^2 + |k|^2
\]
\[
= |a_{-k}|^4 \sum_{i=2k}^{\infty} |c_i|^2 - 2|a_{-k}|^2 \sum_{i=k}^{\infty} |c_i|^2 + \sum_{i=0}^{\infty} |c_i|^2
\]
\[
= \sum_{i=2k}^{\infty} |c_i|^2 (|a_{-k}|^2 - 1)^2 + \sum_{i=k}^{\infty} |c_i|^2 (-2|a_{-k}|^2 + 1) + \sum_{i=0}^{\infty} |c_i|^2 \geq 0
\]
for all $c_i \in \mathbb{C}$ $(i = 0, 1, 2, \ldots)$, or equivalently,
\[
|a_{-k}| \leq \frac{1}{\sqrt{2}}.
\]
This completes the proof. \(\square\)

**Corollary 3.11.** Suppose that $\varphi(z) = a_{-k}z^k$. Then $T_\varphi$ is never $2$-expansive.

**Proof.** We argue by contradiction. Suppose that $T_\varphi$ is $2$-expansive. Put $k(z) = c_0 (c_0 \neq 0)$. Then from the same arguments in proof of Proposition 3.10,
\[
\|T_\varphi^2 k\|^2 - 2|T_\varphi k|^2 + |k|^2 = |c_0|^2 > 0.
\]
Therefore we can conclude a contradiction. \(\square\)

The following result is a consequence of $2$-contractive Toeplitz operators with trigonometric polynomial symbols.

**Proposition 3.12.** Suppose that $\varphi(z) = a_1 z + \overline{a_{-1} z}$ ($a_{-1} \neq 0$). If $T_\varphi$ is $2$-contractive, then
\[
(|a_1|^2 - 1)^2 + |a_{-1}|^2 (3|a_1|^2 - 2) \geq 0.
\]

**Proof.** Put $k(z) = \sum_{i=0}^{\infty} c_i z^i$. Then from (3.2), we have
\[
|T_\varphi k|^2 = \left\| a_1 \sum_{i=0}^{\infty} c_i z^{i+1} + \overline{a_{-1}} \sum_{i=1}^{\infty} c_i z^{i-1} \right\|^2
\]
\[
= |a_1|^2 \sum_{i=0}^{\infty} |c_i|^2 + |a_{-1}|^2 \sum_{i=1}^{\infty} |c_i|^2 + 2 \Re \left\{ a_1 \overline{a_{-1}} \sum_{i=1}^{\infty} c_i \overline{c}_{i+2} \right\},
\]
\[
= |a_1|^2 \sum_{i=0}^{\infty} |c_i|^2 + |a_{-1}|^2 \sum_{i=1}^{\infty} |c_i|^2 + 2 |a_1|^2 \sum_{i=1}^{\infty} |c_i|^2 + 2 |a_{-1}|^2 \sum_{i=1}^{\infty} |c_i|^2 + 4 \Re \left\{ a_1 \overline{a_{-1}} \sum_{i=1}^{\infty} c_i \overline{c}_{i+2} \right\}.
\]
and
\[ |T_\varphi^2 k|^2 = \left\| P((a_1 z + a_{-1}) \left( a_1 \sum_{i=0}^{\infty} c_i z^{i+1} + \overline{a}_{-1} \sum_{i=1}^{\infty} c_i z^{i-1} \right) \right\|^2 \]
\[ = \left\| a_1^2 \sum_{i=0}^{\infty} c_i z^{i+2} + a_1 a_{-1} \sum_{i=1}^{\infty} c_i z^t + a_1 \overline{a}_{-1} \sum_{i=0}^{\infty} c_i z^t + \overline{a}_{-1}^2 \sum_{i=2}^{\infty} c_i z^{i-2} \right\|^2 \]
\[ = |a_1|^4 \sum_{i=0}^{\infty} |c_i|^2 + |a_1 a_{-1}|^2 \sum_{i=1}^{\infty} |c_i|^2 + |a_1 a_{-1}|^2 \sum_{i=1}^{\infty} |c_i|^2 \]
\[ + |a_{-1}|^4 \sum_{i=2}^{\infty} |c_i|^2 + 4 \text{Re} \left\{ |a_1|^2 a_1 a_{-1} \sum_{i=0}^{\infty} c_i \overline{c}_{i+2} \right\} \]
\[ + 2 \text{Re} \left\{ a_1 a_{-1}^2 c_1 \overline{c}_{i+1} \right\} + |a_1 a_{-1}|^2 \sum_{i=1}^{\infty} |c_i|^2 \]
\[ + 2 \text{Re} \left\{ a_1 |a_{-1}|^2 a_{-1} \sum_{i=1}^{\infty} c_i \overline{c}_{i+2} \right\} + 2 \text{Re} \left\{ a_1 |a_{-1}|^2 a_{-1} \sum_{i=0}^{\infty} c_i \overline{c}_{i+2} \right\}. \]

Put \( c_1 = 1 \) and \( c_i = 0 \) for all \( i \geq 0, \ i \neq 1 \), then
\[ |T_\varphi^2 k|^2 - 2 |T_\varphi k|^2 + \| k \|^2 = |a_1|^4 - 2 |a_1|^2 + 3 |a_1 a_{-1}|^2 - 2 |a_{-1}|^2 + 1 \]
\[ = (|a_1|^2 - 1)^2 + |a_{-1}|^2 (3 |a_1|^2 - 2) \geq 0. \]

This completes the proof. \( \square \)

**Example 3.13.** Suppose that \( \varphi(z) = \frac{1}{2} z + \overline{z} \). Then from Proposition 3.12, \( T_\varphi \) is not 2-contractive but by Lemma 3.1, it is contractive.

Next, we consider the \( m \)-contractive Toeplitz operators with analytic symbols.

**Theorem 3.14.** If \( \varphi \) is analytic, then \( T_\varphi \) is \( m \)-contractive where \( m \) is a positive even number.

**Proof.** For a positive even number \( m \), if \( \varphi(z) \) is an analytic, then from (1.1), we have
\[ \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_\varphi^j T_\overline{\varphi} = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_{\varphi^j \overline{\varphi}} \]
\[ = T_{\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \overline{\varphi}^j} = T_{\overline{\varphi}^{m-1}}. \]

Hence, for a positive even number \( m \), \( T_\varphi \) is \( m \)-contractive if and only if
\[ \langle P((\overline{\varphi} - 1)^m k), k \rangle \geq 0 \]
for all $k \in H^2(\mathbb{T})$. Hence
\[
\langle P((\overline{\varphi} - 1)^m k), k \rangle = \langle (\overline{\varphi} - 1)^m k, k \rangle = \langle (\overline{\varphi} - 1)^m k, (\overline{\varphi} - 1)^m k \rangle = \left\| (\overline{\varphi} - 1)^m k \right\|^2 \geq 0.
\]
This completes the proof. □

**Remark 3.15.** It is easy to confirm that Toeplitz operator $T_\varphi$ with analytic symbols of the form $\varphi(z) = \sum_{n=0}^{N} a_n z^n$ with $\sum_{n=0}^{N} |a_n|^2 < 1$ is not contractive. Indeed, from Lemma 3.1, $a_{-k} = 0$ for all $k = 1, 2, \ldots, m$, $T_\varphi$ is contractive if and only if $\sum_{n=0}^{N} |a_n|^2 \geq 1$. So we conclude that there exists a Toeplitz operator $T_\varphi$ with analytic symbols that is not contractive.

**Example 3.16.** Consider the trigonometric polynomial
\[
\varphi(z) = z + z^2.
\]
From Lemma 3.1, $T_\varphi$ is not contractive, but by Theorem 3.14, it is 2-contractive.

**References**


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