A SUFFICIENT CONDITION FOR ACYCLIC 5-CHOOSABILITY OF PLANAR GRAPHS WITHOUT 5-CYCLES

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Abstract. A proper vertex coloring of a graph $G$ is acyclic if $G$ contains no bicolored cycle. A graph $G$ is acyclically $L$-list colorable if for a given list assignment $L = \{L(v) : v \in V(G)\}$, there exists an acyclic coloring $\phi$ of $G$ such that $\phi(v) \in L(v)$ for all $v \in V(G)$. A graph $G$ is acyclically $k$-choosable if $G$ is acyclically $L$-list colorable for any list assignment with $L(v) \geq k$ for all $v \in V(G)$. Let $G$ be a planar graph without 5-cycles and adjacent 4-cycles. In this article, we prove that $G$ is acyclically 5-choosable if every vertex $v$ in $G$ is incident with at most one $i$-cycle, $i \in \{6, 7\}$.

1. Introduction

For a given graph $G = (V(G), E(G))$, a proper $k$-coloring of $G$ is an assignment of $k$ colors to the vertices of $G$ such that no two adjacent vertices are assigned the same color. A proper vertex coloring of $G$ is acyclic if every cycle in $G$ uses at least three colors [12]. For a given list assignment $L = \{L(v) : v \in V(G)\}$, a graph $G$ is acyclically $L$-list colorable if $G$ has an acyclic coloring $\phi$ such that $\phi(v) \in L(v)$ for each vertex $v \in V(G)$, and we say that $\phi$ is an acyclic $L$-coloring of $G$. A graph $G$ is acyclically $k$-choosable if $G$ is acyclically $L$-list colorable for any list assignment $L$ with $L(v) \geq k$ for each $v \in V(G)$. The acyclic list chromatic number of $G$, denoted by $\chi_a^l(G)$, is the smallest integer $k$ such that $G$ is acyclically $k$-choosable. In this paper, we only consider finite, simple and undirected graphs.

Borodin et al. [2] proved that every planar graph is acyclically 7-choosable and put forward to a challenging conjecture as follows:

**Conjecture 1.** Every planar graph is acyclically 5-choosable.

So far, Conjecture 1 remains open, and it has been verified only for some restricted planar graphs. It is proved that Conjecture 1 holds for planar graphs...
with girth at least 5 by Montassier and Ochemin [15], without 4-cycles by Borodin and Ivanova [3]. Particularly, Borodin and Inanova [4] verified that a planar graph is acyclically 5-choosable if it does not contain an i-cycle adjacent to a j-cycle, where $3 \leq j \leq 5$ if $i = 3$ and $4 \leq j \leq 6$ if $i = 4$. This result absorbs most of the previous work in this direction. For the acyclic 6-choosability, Wang and Chen [17] proved that every planar graph without 4-cycles is acyclically 6-choosable. Later, Wang et al. [18] proved that every planar graph $G$ is acyclic 6-choosable if $G$ does not contain 4-cycles adjacent to $i$-cycles for each $i \in \{3, 4, 5, 6\}$. For the acyclic 4-choosability of a planar graph $G$, it is proved that $G$ is acyclic 4-choosable if $g(G) \geq 5$ [14], or $G$ does not contain 4-cycles and triangular 6-cycles [7], or $G$ does not contain an $i$-cycle adjacent to a $j$-cycle, where $3 \leq j \leq 6$ if $i = 3$ and $4 \leq j \leq 7$ if $i = 4$ [5], or $G$ does not contain 4-, 7- and 8-cycles [10], or $G$ does not contain 4- and 5-cycles [6, 9].

For the graphs embedded on a special surface, Hou and Liu [13] proved that every toroidal graph is acyclically 8-choosable.

The purpose of this paper is to give a sufficient conditions for the acyclic 5-choosable planar graphs in which the number of adjacent 4'-cycles is increased to a certain extent.

**Theorem 2.** Let $G$ be a planar graph without 5-cycles and adjacent 4-cycles. Then $G$ is acyclically 5-choosable if every vertex in $G$ is incident with at most one $i$-cycle, $i \in \{6, 7\}$.

A plane graph is a planar embedding of a planar graph in the Euclidean plane. For a plane graph $G$, let $F(G)$ denote its face set. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of vertices adjacent to $v$, $d_G(v) = |N_G(v)|$ denotes the degree of $v$. The degree of a face $f$ of $G$, denoted by $d_G(f)$, is the number of edges incident with $f$ where each cut edge is counted twice. A $k$, $k^+$- and $k^-$-vertex (face) is a vertex (face) of degree $k$, at least $k$ and at most $k$, respectively. A vertex $u$ is called a $k$-neighbor (resp. $k^-$-neighbor, $k^+$-neighbor) of a vertex $v$ if $uw \in E(G)$ and $d_G(u) = k$ (resp. $d_G(u) \leq k$, $d_G(u) \geq k$). A $k$-cycle is a cycle of length $k$. Two cycles (or faces) are said to be intersecting if they have at least one common vertex, and adjacent if they have at least one common edge. An edge $uv$ is a $(b_1, b_2)$-edge if $d_G(u) = b_1$ and $d_G(v) = b_2$. A face $f \in F(G)$ is usually denoted by $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the boundary vertices of $f$ in a cyclic order. For convenience, a face $f = [u_1u_2 \cdots u_n]$ is called an $(a_1, a_2, \ldots, a_n)$-face if the degree of the vertex $u_i$ is $a_i$ for $i = 1, 2, \ldots, n$. For $x \in V(G) \cup F(G)$, let $n_i(x)$ denote the number of $i$-vertices adjacent to or incident with $x$ and $f_k(v)$ denote the number of $k$-faces incident with a vertex $v \in V(G)$. For a vertex $v \in V(G)$, if $v$ is incident with two adjacent 3-faces with a common edge $vu$, then the two adjacent 3-faces are called bad faces of $v$. Let $f_3(v)$ denote the number of bad 3-faces of $v$. For a 3-face $[uvw]$, if each of the two edges $uv$ and $vu$ is not incident with other 3-face except the face $[uvw]$, then $[uvw]$ is called an independent face of $v$, and let $s(v)$ denote the number of independent faces of $v$. If there is no confusion in the context, we
write $d_G(v)$, $N_G(v)$ as $d(v)$ and $N(v)$ for short. For other undefined notations, we refer the readers to [1].

2. Structural properties of the minimum counterexample to Theorem 2

Suppose that $G$ is a counterexample to Theorem 2 with the minimum number of vertices embedded in the plane. Then $G$ is connected. Let $L$ be a list assignment of $G$ with $|L(v)| = 5$ for all $v \in V(G)$, and $G$ is not acyclically $L$-list colorable, but any proper subgraph $G'$ with $|V(G')| < |V(G)|$ is acyclically $L$-list colorable.

Firstly, we give the following known lemmas, the proofs of which were provided in [8, 11, 16].

Lemma 3.

(C1) [16] There are no 1-vertices.
(C2) [16] No 2-vertex is adjacent to a 4$^-$-vertex.
(C3) [16] Let $v$ be a 3-vertex. If $v$ is adjacent to a 3-vertex, then $v$ is not adjacent to other 4$^-$-vertex;
(C4) Let $v$ be a 5-vertex. Then
(C4.1) [16] $n_2(v) \leq 1$;
(C4.2) [11] If $n_2(v) = 1$ and $v$ is incident with a 3-face $f$, then $n_3(f) = 0$.
(C5) Let $v$ be a 6-vertex. Then
(C5.1) [16] $n_2(v) \leq 4$;
(C5.2) [16] If $n_2(v) = 4$, then $v$ is not adjacent to any 3-vertex;
(C5.3) [11] If $n_2(v) = 2$ and $v$ is incident with a $(3, 3, 6)$-face, then $n_3(v) \leq 2$;
(C5.4) [11] If $n_2(v) = 3$, then $n_3(v) \leq 1$;
(C5.5) [11] If $n_2(v) = 4$, then $f_3(v) = 0$;
(C5.6) [8] If $v$ is incident to a $(3, 4, 6)$-face, then $n_2(v) \leq 2$.
(C6) Let $v$ be a 7-vertex. Then
(C6.1) [16] $n_2(v) \leq 5$;
(C6.2) [11] If $n_2(v) = 4$, then $n_3(v) \leq 2$;
(C6.3) [11] If $n_2(v) = 5$, then $n_3(v) = 0$ and $f_3(v) = 0$.
(C7) [16] No 3-face $[xyz]$ with $d(x) \leq d(y) \leq d(z)$ satisfies one of the following:
(C7.1) $d(x) = 2$;
(C7.2) $d(x) = d(y) = 3$ and $d(z) \leq 5$;
(C7.3) $d(x) = 3$ and $d(y) = d(z) = 4$.
(C8) [8] Let $v$ be a 8-vertex. Then
(C8.1) $n_2(v) \leq 6$;
(C8.2) If $f_3(v) = 1$, then $n_2(v) \leq 5$.

In all figures of this paper, a vertex $v$ is drawn in black if $v$ has no other neighbors besides the ones already depicted and in white otherwise.
Lemma 4. If a 3-vertex \( v \in V(G) \) is incident to two adjacent 3-faces \([vv_1v_2]\) and \([vv_2v_3]\), then \( \min\{d(v_1), d(v_3)\} \geq 5 \).

Proof. Suppose to the contrary that one of \( v_1 \) and \( v_3 \) is a 4-vertex. Without loss of generality, assume that \( d(v_1) \leq 4 \). Let \( G' = G - v \), then by the minimality of \( G \), \( G' \) admits an acyclic \( L \)-coloring \( \phi \). Note that \( \phi(v_2) \neq \phi(v_1) \) and \( \phi(v_2) \neq \phi(v_3) \). If \( \phi(v_1) \neq \phi(v_3) \), then we color \( v \) with a color in \( L(v) \setminus \{\phi(v_1), \phi(v_2), \phi(v_3)\} \) to get an acyclic \( L \)-coloring of \( G \), a contradiction.

Otherwise we color \( v \) with a color in \( L(v) \setminus \{\phi(v_1), \phi(v_3)\} \cup \{\phi(x) : (x \in N(v_1)) \land (x \notin \{v, v_2\})\} \). So we extend \( \phi \) to an acyclic \( L \)-coloring of \( G \), a contradiction. \( \square \)

Lemma 5. Let \( v \) be 5-vertex. Then \( G \) can not contain the configurations shown in Figure 1.

![Figure 1. A 5-vertex v.](image)

Proof. (1) Suppose to the contrary that \( G \) contains the configuration in Fig 1(1). Let \( G' = G - v_1 \). By the minimality of \( G \), \( G' \) admits an acyclic \( L \)-coloring \( \phi \). Note that \( \phi(v) \neq \phi(v_2) \). If \( \phi(x_1) \notin \{\phi(v), \phi(v_2)\} \), then we color \( v_1 \) with a color in \( L(v_1) \setminus \{\phi(x_1), \phi(v_2)\} \) to get an acyclic \( L \)-coloring of \( G \), a contradiction. So we have \( \phi(x_1) \in \{\phi(v), \phi(v_2)\} \). If \( \phi(x_1) = \phi(v_2) \), then we color \( v_1 \) with a color in \( L(v_1) \setminus \{\phi(x_1), \phi(v), \phi(x_2), \phi(y_2)\} \). Otherwise \( \phi(x_1) = \phi(v) \). Thus, each color \( c \) in \( L(v_1) \) appears in \( \{\phi(v), \phi(v_2), \phi(v_3), \phi(v_4), \phi(v_5)\} \) for otherwise we can properly color \( v_1 \) with \( d \in L(v_1) \setminus \{\phi(v), \phi(v_2), \phi(v_3), \phi(v_4), \phi(v_5)\} \) to get an acyclic \( L \)-coloring of \( G \), a contradiction. Without loss of generality, we assume that \( L(v_1) = \{1, 2, 3, 4, 5\} \), \( \phi(x_1) = \phi(v) = 1 \) and \( \phi(v_i) = i \) for \( i = 2, 3, 4, 5 \).

Firstly, we have \( \phi(x_3) = 1 \) for otherwise we can properly color \( v_1 \) with 3. Secondly, we have \( L(v) = \{1, 2, 3, 4, 5\} \) for otherwise we can recolor \( v \) with a color \( \alpha \in L(v) \setminus L(v_1) \) and color \( v_1 \) with a color in \( L(v_1) \setminus \{1, 2\} \) to get an acyclic \( L \)-coloring of \( G \), a contradiction. Hence, we recolor \( v \) with 3, recolor \( v_3 \) with a color in \( L(v_3) \setminus \{1, 3, 4\} \), and color \( v_1 \) with a color in \( L(v_1) \setminus \{1, 2, 3\} \). Hence we extend \( \phi \) to an acyclic \( L \)-coloring of \( G \), a contradiction.
The proof of the forbidden configuration as shown in Fig. 1(2) is similar to the above.

Lemma 6. \( G \) cannot contain the configurations as shown in Figure 2.

\[ \text{Figure 2. A 6-vertex v.} \]

Proof. (1) Suppose that \( G \) contains the configuration as shown in Figure 2(1). By the minimality of \( G \), \( G' = G - \{v, v_1, v_2, v_3\} \) has an acyclic \( L \)-coloring \( \phi \). Obviously, \( \phi(v_5) \neq \phi(v_6) \). We divide this problem into three cases.

Case 1. Suppose that \( \phi(v_4), \phi(v_5) \) and \( \phi(v_6) \) are pairwise different.

Then there exists a color \( c \in L(v) \setminus \{\phi(v_1), \phi(v_2), \phi(v_6)\} \) appearing at most once in \( \{x_1, x_2, x_3\} \). We color \( v \) with \( c \). Assume that \( \phi(x_i) = c, i \in \{1, 2, 3\} \), then we color \( v_1 \) with a color in \( L(v_1) \setminus \{c, \phi(v_4), \phi(v_5), \phi(v_6)\} \). Now, we color the vertices in \( U = \{v_1, v_2, v_3\} \setminus \{v_1\} \). Let \( v_2 \in U \). If \( d(v_2) = 2 \), then we color \( v_2 \) with a color in \( L(v_2) \setminus \{c, \phi(x_k)\} \). If \( d(v_2) = 3 \), then let \( v_3 \in N(v) \cap N(v_2) \). If \( v_3 \) is not assigned any color, then we color \( v_3 \) with a color in \( L(v_3) \setminus \{c, \phi(x_k)\} \). Otherwise, assume that \( v_3 \) is colored with \( \alpha \). If \( \phi(x_k) = \alpha \), then we color \( v_3 \) with a color in \( L(v_3) \setminus \{c, \alpha, \phi(x_1)\} \). Otherwise, we color \( v_3 \) with a color in \( L(v_3) \setminus \{c, \phi(x_k), \alpha\} \). Hence, we extend \( \phi \) to an acyclic \( L \)-coloring of \( G \), a contradiction.

Case 2. Suppose that \( \phi(v_4) = \phi(v_6) \).

If \( \phi(x_6) = \phi(v_5) \), we recolor \( v_6 \) with a color in \( L(v_6) \setminus \{\phi(v_6), \phi(v_5), \phi(x_5), \phi(y_5)\} \) and then go back to the previous case. Otherwise, we recolor \( v_6 \) with a color in \( L(v_6) \setminus \{\phi(v_6), \phi(v_5), \phi(x_1)\} \) and then go back to Case 1.

Case 3. Suppose that \( \phi(x_5) = \phi(y_5) \).

If \( \phi(x_5) = \phi(y_5) \), then there exists a color \( c' \in L(v) \setminus \{\phi(v_5), \phi(v_6), \phi(x_5)\} \) appearing at most once in \( \{x_1, x_2, x_3\} \). Then we color \( v \) with \( c \) and the left is the same to Case 1. Otherwise, \( \phi(x_5) \neq \phi(y_5) \). If \( \phi(v_6) \in \{\phi(x_5), \phi(y_5)\} \), then we recolor \( v_5 \) with a color in \( L(v_5) \setminus \{\phi(v_5), \phi(x_5), \phi(y_5), \phi(x_6)\} \) and then go back to Case 1. Otherwise, we recolor \( v_6 \) with a color in \( L(v_5) \setminus \{\phi(v_5), \phi(y_5), \phi(x_6)\} \) and then go back to Case 1.
Proof. Suppose to the contrary that we always denote \( v \) with \( \{ \phi(v_5), \phi(x_3), \phi(y_3), \phi(v_b) \} \) and then go back to Case 1. Hence, in any case, we can extend \( \phi \) to an acyclic \( L \)-coloring of \( G \), a contradiction.

(2) Suppose that \( G \) contains the configuration as shown in Figure 2(2). By the minimality of \( G \), \( G' = G - \{ v, v_1, v_2, v_3 \} \) has an acyclic \( L \)-coloring \( \phi \). By the same argument as in (1), there always exists a color \( c \in L(v) \) \( \{ \phi(v_4), \phi(v_5), \phi(v_b) \} \) which appears at most once in \( \{ x_1, x_2, x_3 \} \). Then we color \( v \) with \( c \) and the colorings of \( v_1 \) and \( v_2 \) are in the same way as in (1). In the following, we mainly discuss the coloring of \( v_3 \).

If \( \phi(x_3) = c \), then \( \phi(x_3) \neq \phi(v_4) \). Then we color \( v_3 \) with a color in \( L(v_3) \setminus \{ c, \phi(v_4), \phi(v_5), \phi(v_b) \} \). If \( \phi(x_3) \neq c \), then we color \( v_3 \) with a color in \( L(v_3) \setminus \{ c, \phi(x_3), \phi(v_4), \phi(y_4) \} \) if \( \phi(x_3) = \phi(v_4) \) and color \( v_3 \) with a color in \( L(v_3) \setminus \{ c, \phi(x_3), \phi(v_b) \} \) otherwise. Hence we extend \( \phi \) to an acyclic \( L \)-coloring of \( G \), a contradiction.

(3) The proof of (3) is similar to those of (1) and (2). \( \Box \)

**Lemma 7.** \( G \) does not contain a \( d \)-vertex \( v \) (\( 6 \leq d \leq 10 \)) such that except at most one neighbor or two adjacent neighbors of \( v \), the other neighbors of \( v \) are \( 4^- \)-vertices (these vertices are called to be small) and these \( 4^- \)-vertices satisfy the following conditions (see Figure 3):

1. For every \( 3 \)-neighbor \( u \) of \( v \), \( uv \) is incident with exact one \( 3 \)-face. Furthermore, if \( u \) is adjacent to another \( 3 \)-neighbor \( w \) of \( v \), then \( u, w, v \) forms a \( (3, 3, d) \)-face;
2. For every \( 4 \)-neighbor \( w \) of \( v \), \( vw \) is incident with two \( (3, 4, d) \)-faces.

**Proof.** Suppose to the contrary that \( G \) contains such a \( d \)-vertex with \( 6 \leq d \leq 10 \). We just settle the case as shown in Figure 3(1) (the case as shown in Figure 3(2) can be settled similarly). We denote all small neighbors of \( v \) to be \( v_3, v_4, \ldots, v_d \). For any \( v_i \) (\( 3 \leq i \leq d \)), we have \( |N(v_i) \setminus (N(v) \cup \{ v \})| = 1 \), so we always denote \( v'_i \) to be the neighbor of \( v_i \) which is not adjacent to \( v \). Let \( S = \{ v'_i \mid 3 \leq i \leq d \} \).

![Figure 3. A configuration in Lemma 7.](image-url)
By the minimality of $G$, $G' = G - \{v, v_2, \ldots, v_d\}$ has an acyclic $L$-coloring $\phi$. Let $L' = L(v) \setminus \{\phi(v_1), \phi(v_2)\}$. Then $|L'| \geq 3$. Since $d \leq 10$, $|S| \leq 8$, and it follows that there exists a color $c \in L'$ which appears at most twice in $S$. Firstly, we color $v$ with $c$. Without loss of generality, assume that $\phi(v'_i) = \phi(v'_j) = c$ for $3 \leq i < j \leq d$, and we color $v_i$ with a color in $L(v_i) \setminus \{c, \phi(v_1), \phi(v_2)\}$ and color $v_j$ with a color in $L(v_j) \setminus \{c, \phi(v_1), \phi(v_2)\}$.

Then we color the vertices in $U = \{v_3, \ldots, v_d\} \setminus \{v_i, v_j\}$. Let $v_k \in U$. Suppose that $d(v_k) = 2$. Then we color $v_k$ with a color in $L(v_k) \setminus \{c, \phi(v'_i)\}$. Suppose that $d(v_k) = 3$. Let $v_t \in N(v) \cap N(v_k)$. Note that $d(v_t) \leq 4$. If $v_t$ is not colored, then we color $v_k$ with a color in $L(v_k) \setminus \{c, \phi(v'_i)\}$. If $v_t$ is colored with $\alpha$, then we discuss this problem in two cases. The first case is that $\phi(v'_i) \neq \alpha$, then we color $v_k$ with a color in $L(v_k) \setminus \{c, \alpha, \phi(v'_i)\}$). The second case is that $\phi(v'_i) = \alpha$, then we color $v_k$ with a color in $L(v_k) \setminus \{c, \phi(v'_i)\}$. Then there exist one $3$-vertex $v_p \in \{v, v_t\}$ such that the color of $v_p$ appears twice in $\eta, \theta, \phi(v'_i)$. Then $|T| = \{\eta, \theta, \phi(v'_i)\} \leq 4$ and we color $v_k$ with a color in $L(v_k) \setminus T$. If $|\eta, \theta, \phi(v'_i)| > 1$, then $|T'| = \{\eta, \theta, \phi(v'_i), \phi(v'_i)\} \leq 4$ and we color $v_k$ with a color in $L(v_k) \setminus T'$.

It is easy to check that the coloring obtained as above is an acyclic $L$-coloring of $G$, a contradiction. □

**Lemma 8.** $G$ can not contain the configurations as shown in Figure 4, where $v_4$ is a $4^-$-vertex in each of Figure 4(1), (2) and (3).

![Figure 4](image-url)

**Proof.** Suppose to the contrary that $G$ contains the structure in Figure 4(1). We only prove the case that $v_4$ is a $4^-$-vertex (the proof of the case that $v_4$ is a $3$-vertex is much easier). Let $G' = G - \{v, v_1, v_2, v_3\}$, then $G'$ admits an acyclic $L$-coloring $\phi$. It is easy to see that $\phi(v_1) \neq \phi(v_6)$.


Case 1. Suppose that $\phi(v_4) \neq \phi(v_6)$.
Then $\phi(v_4), \phi(v_5)$ and $\phi(v_6)$ are pairwise different. Hence, there exists a color $c \in L(v) \setminus \{\phi(v_4), \phi(v_5), \phi(v_6)\}$ appearing at most once in $\{x_1, x_2, x_3\}$. Then we color $v$ with $c$. For the coloring process of the vertices $v_1, v_2, v_3$ is the same as that of Lemma 7.

Case 2. Suppose that $\phi(v_4) = \phi(v_6)$.
Let $T = \{\phi(v_4), \phi(v_5), \phi(v_6), \phi(x_4), \phi(y_4)\}$. Firstly, we assume that $\phi(x_4) \neq \phi(y_4)$. If $\phi(v_5) \notin \{\phi(x_4), \phi(y_4)\}$, then we color $v_4$ with a color in $L(v_4) \setminus \{\phi(v_4), \phi(v_5), \phi(x_4), \phi(y_4)\}$ and go back to Case 1. If $\phi(v_5) \in \{\phi(x_4), \phi(y_4)\}$, then $|T| = 3$. Hence, there exists a color $c \in L(v) \setminus T$ which appears at most once in $\{x_1, x_2, x_3\}$. Then we color $v$ with $c$. For the coloring process of the vertices $v_1, v_2, v_3$ is the same as that of Lemma 7. Secondly, assume that $\phi(x_4) = \phi(y_4)$. Then we have $|T| \leq 3$. It follows that there exists a color in $c \in L(v) \setminus T$ which appears at most once in $\{x_1, x_2, x_3\}$ and go back to the previous case.

The proofs of the other two forbidden configuration as shown in Figure 4(2), (3) are the same as the above.

By Lemma 7, we have the following two results:

**Corollary 9.** Let $v$ be 9-vertex. Then

(C9.1) $n_2(v) \leq 7$.
(C9.2) If $n_2(v) = 7$, then $f_3(v) = 0$.

3. The proof of Theorem 2

Let $G$ be a minimal counterexample to Theorem 2 (i.e., with the least number of vertices). From Euler’s formula $|V(G)| - |E(G)| + |F(G)| = 2$ and the relation that $\sum_{v \in V(G)} d_G(v) = \sum_{f \in F(G)} d_G(f) = 2|E(G)|$, we have

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12.$$  

Then we assign an initial charge $c(v) = 2d(v) - 6$ to $v \in V(G)$ and $c(f) = d(f) - 6$ to $f \in F(G)$. By applying a set of discharging rules and the structural properties of $G$, we can obtain the final charge $c'(x)$ such that $c'(x) \geq 0$ for every element $x \in V(G) \cup F(G)$, which is a contradiction to equation (1) in final and completes our proof.

Suppose that $f = [v_1v_2\cdots v_k]$ is a $k$-face. We use $(d_G(v_1), \ldots, d_G(v_k)) \rightarrow (c_1, \ldots, c_k)$ to denote that the vertex $v_i$ gives $f$ the charge $c_i$ for $i = 1, \ldots, k$.

The discharging rules are given as follows.

**R1.** Let $f = [v_1v_2v_3]$ be a 3-face with $d_G(v_1) \leq d_G(v_2) \leq d_G(v_3)$. Then

$(3, 3, 6^+) \rightarrow (0, 0, 3)$;
$(3, 4, 5^+) \rightarrow (0, 1, 2)$;
$(3, 5^+, 5^+) \rightarrow (0, \frac{5}{2}, \frac{5}{2})$;
$(4^+, 4^+, 4^+) \rightarrow (1, 1, 1)$.
R2. Let \( f = [v_1v_2v_3v_4] \) be a 4-face. Then
\[
(2, 5^+, 3^-, 5^+) \rightarrow (0, 1, 0, 1); \\
(3, 3, 5^+, 5^+) \rightarrow (0, 0, 1, 1); \\
(3, 4^+, 3, 4^+) \rightarrow (0, 1, 0, 1); \\
(3^-, 4^+, 4^+, 4^+) \rightarrow (0, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}); \\
(4^+, 4^+, 4^+, 4^+) \rightarrow (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}).
\]

R3. Let \( v \) be a 2-vertex. Then \( v \) receives 1 from each of its neighbors.

R4. Let \( f \) be a 7-face. Then \( f \) sends \( \frac{1}{2} \) to each vertex incident with \( f \).

R5. Let \( f \) be a 8\(^+\)-face. Then \( f \) sends \( \frac{1}{4} \) to each vertex incident with \( f \).

R6. Let \( v \) be a 4\(^+\)-vertex and \( \sigma(v) \) denote the charges obtained from its incident 7\(^+\)-faces. Then \( v \) sends \( \sigma(v) \) uniformly among its 5\(^+\)-neighbors.

By the assumption of Theorem 2, we have the following result.

**Fact 1.** For any vertex \( v \in V(G) \), we have
\[
(1) \ f_3(v) \leq \left\lfloor \frac{2}{3} d(v) \right\rfloor \text{ if } n_2(v) = 0 \text{ and } f_3(v) \leq \left\lfloor \frac{2}{3} (d(v) - n_2(v) - 1) \right\rfloor \text{ if } n_2(v) > 0. \\
(2) \text{ There is no 4-cycle adjacent to a 4-cycle.} \\
(3) \text{ Let } f \text{ be a 3-face in } G. \text{ Then } f \text{ is not adjacent to 4-, 5- and 6-faces.} \\
(4) f_6(v) + f_7(v) \leq 1. \\
(5) \text{ If } v \text{ is a 2-vertex with } f_4(v) \geq 1 \text{ in } G, \text{ then } f_4(v) = 1 \text{ and } f_5(v) = f_6(v) = f_7(v) = 0. \\

Let \( f \in F(G) \). Suppose that \( f \) is a 3-face \( f = [v_1v_2v_3] \) with \( d(v_1) \leq d(v_2) \leq d(v_3) \). By Lemma 3 and (C7.1), \( d(v_1) \geq 3 \). If \( d(v_1) = d(v_2) = 3 \), then by Lemma 3 (C7.2), \( d(v_3) \geq 6 \) and it follows that \( c'(f) \geq 3 - 6 + 3 = 0 \) by R1. If \( d(v_1) = 3 \) and \( d(v_2) = 4 \), then by Lemma 3 (C7.3), \( d(v_3) \geq 5 \) and it follows that \( c'(f) \geq 3 - 6 + 2 \times \frac{2}{3} = 0 \) by R1. If \( d(v_1) \geq 4 \), then \( c'(f) \geq 3 - 6 + 3 \times 1 = 0 \) by R1.

Suppose that \( f \) is a 4-face. By Lemma 3(C1)-(C3) and R2, we can easily check that \( c'(f) \geq 0 \). There are no 5-face in \( G \). Suppose that \( f \) is a 6-face. Then by discharging rules, \( c'(v) = c(v) = 6 - 6 = 0 \). Suppose that \( f \) is a 7-face. Then \( c'(f) \geq d(f) - 6 - 7 \times \frac{1}{7} \geq 0 \) by R4. Suppose that \( f \) is a 8\(^+\)-face. Then \( c'(f) \geq d(f) - 6 - \frac{1}{2}d(f) \geq 0 \) by R5.

Let \( v \in V(G) \). By Lemma 3(C1), \( d(v) \geq 2 \). Suppose that \( d(v) = 2 \). By Lemma 3(C7.1), \( f_3(v) = 0 \). Hence, by R2-R6, \( c'(v) \geq 2 \times 2 - 6 + 2 \times 1 = 0 \). Suppose that \( d(v) = 3 \). By R1-R2 and R4-R6, \( c'(v) \geq d(v) \geq 2 \times 3 - 6 = 0 \). Suppose that \( d(v) = 4 \). By Fact 1(2), \( f_3(v) + f_4(v) \leq 2 \). By Lemma 3(C2), \( n_2(v) = 0 \). It follows that \( c'(v) \geq 2 \times 4 - 1 \times 2 = 0 \) by R1-R2 and R4-R6.

For a vertex \( v \in V(G) \), let \( v_1, \ldots, v_{d(v)} \) denote the neighbors of \( v \) in a clockwise order and let \( f_i \) be a face incident with \( v v_i \) and \( v v_{i+1} \), where the sum of the subscription is taken modular \( d(v) \).
We denote by \( \rho \) the charge which \( v \) sends to a \( i \)-diamond of \( v \), and \( f_{i,diam}(v) \) the number of \( i \)-diamonds of \( v \), where \( i = 1, 2, 3, 4 \). By R1, we have \( \rho_1 = 4 \), \( \rho_2 = 3 \), \( \rho_3 = \frac{2}{7} \), \( \rho_4 = 2 \), and \( f_3(v) = 2 \sum_{i=1}^4 f_{i,diam}(v) \).

Suppose that \( d(v) = 5 \). Then \( n_2(v) \leq 1 \) by Lemma 3(C4.1). Let \( n_2(v) = 0 \), then \( f_3(v) + f_4(v) \leq 3 \) by Fact 1(1)-(3). If \( f_3(v) \leq 2 \) and \( v \) is not incident with a \((3, 4, 5)\)-face, then \( c'(v) \geq 2 \times 5 - 6 - 2 - \frac{3}{2} - 1 = 0 \) by R1-R2. If \( f_3(v) \leq 2 \) and \( v \) is incident with a \((3, 4, 5)\)-face, then by Lemma 5, any other 3-face incident with \( v \) is not incident with 3-vertices. If follows that \( c'(v) \geq 2 \times 5 - 6 - (2 + 1) - 1 = 0 \) by R1-R2. If \( f_3(v) = 3 \), then \( f_{i,diam}(v) = 0 \) by Lemma 5 (Figure 1(2)), \( f_4(v) = 0 \) and \( \sum_{i=2}^4 f_{i,diam}(v) = 1 \) by Fact 1(2). Let \( f_1, f_2 \) and \( f_4 \) be 3-faces. If one of the three 3-faces is a \((3, 4, 5)\)-face, then \( n_3(v) = 1 \) by Lemma 5 (Figure 1). It follows that \( c'(v) \geq 2 \times 5 - 6 - 2 - 1 \times 2 = 0 \) by R1. Otherwise, we firstly assume that \( f_{2,diam}(v) = 1 \). If the two bad 3-faces of \( v \) are adjacent \((3, 5, 5^+)-faces \( f_1 \) and \( f_2 \) with a common \((5, 5^+)-edge \), then each of the two 3-neighbors \( v_1 \) and \( v_2 \) receives at least \( \frac{1}{2} + \frac{1}{4} \) from its incident 7+-faces by R4, R5 and Fact 1. Hence, \( c'(v) \geq 2 \times 5 - 6 - 3 \times \frac{1}{2} + \frac{1}{2} + 1 \times \frac{1}{2} + 2 \times \frac{1}{3} \times \frac{1}{1} + \frac{1}{1} = \frac{23}{6} \) by R1 and R4-R7. If the two bad 3-faces of \( v \) are adjacent \((3, 5, 5^+)-faces \( f_1 \) and \( f_2 \) with a common \((3, 5)-edge \), then \( v_2 \) receives at least \( \frac{1}{2} \) from its incident 7+-faces if one of \( f_3 \) and \( f_5 \) is a 7-face and \( \frac{1}{4} \) otherwise by R4, R5 and Fact 1(4). Hence, \( c'(v) \geq 2 \times 5 - 6 - 3 \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{1} + \frac{1}{4} \times \frac{1}{1} + \frac{1}{4} \times \frac{1}{1} + \frac{1}{4} \times \frac{1}{1} + \frac{1}{4} \times \frac{1}{1} + \frac{1}{4} \times \frac{1}{1} = \frac{23}{6} \) if one of \( v_1 \) and \( v_2 \) is a 3-vertex and \( c'(v) \geq 2 \times 5 - 6 - 2 \times \frac{1}{2} - 1 = 0 \) otherwise by R1 and R4-R7. Secondly, assume that \( f_{2,diam}(v) = 0 \), then \( c'(v) \geq 2 \times 5 - 6 - \left( \frac{2}{3} + \frac{1}{2} \right) = 0 \) by Lemma 4 and R1-R2. Let \( n_2(v) = 1 \). Then \( f_3(v) + f_4(v) \leq 3 \) by Fact 1(2), (3). By Lemma 3(C4.2) and R1-R3, \( c'(v) \geq 2 \times 5 - 6 - 2 \times 1 - 1 = 0 \).

Suppose that \( d(v) = 6 \). Then \( c(v) = 2 \times 6 - 6 = 6 \) and \( n_2(v) \leq 4 \) by Lemma 3(C5.1). Let \( f_{3,4}(v) = (a, b) \) denote that \( f_3(v) = a \) and \( f_4(v) = b \) and let \( f_{3,4}(v) = (a', b') \) denote that \( f_3(v) \leq a \) and \( f_4(v) = b \). Similarly, we can define \( f_{3,4}(v) = (a, b^-) \). We will discuss this problem in terms of \( n_2(v) \).

If \( n_2(v) = 0 \), then \( f_3(v) \leq 4 \) by Fact 1(1), and it follows that \( c'(v) \geq 6 - 3 = 3 \) and \( f_{3,4}(v) = (0, 3^-) \) by R2: \( c'(v) \geq 6 - 3 - 2 = 1 \) for \( f_{3,4}(v) = (1, 2^-) \) by R1-R2. Assume that \( f_{3,4}(v) = (2, 1^-) \). \( \sum_{i=1}^4 f_{i,diam}(v) = 1 \), then \( c'(v) \geq 6 - 4 - 1 = 1 \) by Lemma 4 and R1-R2. Otherwise, \( c(v) = 2 \). If \( v \) is incident with at most one \((3, 3, 6)\)-face, then \( c'(v) \geq 6 - 3 - 2 - 1 = 0 \) by R1-R2. Otherwise, \( v \) is incident with two \((3, 3, 6)\)-faces. Let \( f_1 \) and \( f_2 \) be two 3-faces. Since every vertex is incident with at most one \( j \)-cycle, \( j \in \{6, 7\}, v_1, v_2, v_3 \)
and $v_4$ receive at least $2 \times \frac{1}{3} + 2 \times \frac{1}{6} = \frac{14}{18}$ from their incident 7-face. Then $c'(v) \geq 6 - 3 - 3 - 1 + \frac{1}{3} + 2 \times \frac{1}{6} = \frac{14}{18}$ by R1-R2 and R4-R6. For the case $f_{3,4}(v) = (3,0)$, we should consider the following subcases. The first subcase is that $\sum_{i=1}^{4} f_{dia}^i(v) = 1$, then $c'(v) \geq 6 - 3 - 3 + 1 = 2$ by Fact 1(3). If $v$ is not incident with a (3, 3, 6)-face, then $c'(v) \geq 6 - 2 \times 3 = 0$ by R1. Otherwise, let $f_1$ be a (3, 3, 6)-face. Then by Lemma 6 (Figure 2(1)), $v$ is incident with at most one (3, 4, 6)-face. Hence, by Fact 1(4), R1 and R4-R5, we have $c'(v) \geq 6 - 3 - 2 - \frac{2}{3} + \frac{1}{6} + 2 \times \frac{1}{6} = \frac{17}{18}$. For the case $f_{3,4}(v) = (4,0)$, $\sum_{i=1}^{4} f_{dia}^i(v) = 2$ by Fact 1(2). If $f_{dia}^1(v) = 0$, then $c'(v) \geq 6 - 2 \times 3 = 0$ by Lemma 4 and R1. Otherwise, by Lemma 7, $f_{dia}^1(v) = 1$. Assume that $f_1$ and $f_2$ construct a 1-diamond of $v$, then $f_3$ and $f_4$ are two bad 3-faces of $v$ by Fact 1(2). Furthermore, by Lemma 4, $d(v_3) = d(v_4) = 3$ and $d(v_2) = 4$. If $f_{dia}^2(v) = 0$, then by Lemma 8 (Figure 4(1)), $f_{dia}^3(v) = 0$ and $f_{dia}^4(v) = 1$. Hence, we have $c'(v) \geq 6 - 4 - 2 = 0$ by Lemma 4 and R1. If $f_{dia}^2(v) = 1$, then by Lemma 8 (Figure 4(1)), we have $d(v_3) = 3$ and $\min\{d(v_1), d(v_6)\} \geq 5$. Hence, by Fact 1, R1 and R4-R6, $c'(v) \geq 6 - 3 - 2 - \frac{2}{3} + \frac{1}{6} + 2 \times \frac{1}{6} = \frac{17}{18}$.

If $n_2(v) = 1$, then $f_{dia}^3(v) \leq 3$ by Fact 1(1). Hence, we have $c'(v) \geq 6 - 1 - 3 \times 1 = 2$ for $f_{3,4}(v) = (0,3')$ by R2-R3; $c'(v) \geq 6 - 3 - 1 - 2 = 0$ for $f_{3,4}(v) = (1,2')$ by R1-R3. If $f_{3,4}(v) = (2,1')$, then by Fact 1(2), $\sum_{i=1}^{4} f_{dia}^i(v) \leq 1$. If $\sum_{i=1}^{4} f_{dia}^i(v) = 1$, then by Lemma 4 and R1-R3, $c'(v) \geq 6 - 4 - 1 - 1 = 0$. Otherwise, by Lemma 6, $v$ is incident with at most one (3, 3, 6)-face. Hence, by Lemma 6 (Figure 2(1)), $v$ is not incident with a (3, 3, 6)-face, then by Lemma 7, $f_{dia}^1(v) = 0$. If the independent 3-face of $v$ is not a (3, 3, 6)-face, then $c'(v) \geq 6 - 3 - 2 - 1 = 0$ by R1-R3. Otherwise, assume that $f_1 = [v e_1 e_2]$ is an independent (3, 3, 6)-face, $f_3$ and $f_4$ are bad 3-faces of $v$ and $v_6$ is a 2-vertex. If $\sum_{i=1}^{4} f_{dia}^i(v) = 1$, then by Lemma 8 (Figure 4(2)), $f_{dia}^3(v) = 0$ and $f_{dia}^4(v) = 1$. Hence, by Lemma 4 and R1, $c'(v) \geq 6 - 3 - 2 = 1$. If $f_{dia}^2(v) = 1$, then by Lemma 8 (Figure 4(2)), $d(v_3) = 3$ and $\min\{d(v_1), d(v_6)\} \geq 5$. Hence, by Fact 1, R1 and R3-R6, $c'(v) \geq 6 - 4 - 3 + \frac{1}{6} + \frac{2}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{17}{18}$.

If $n_3(v) = 2$, then $f_{dia}^3(v) \leq 2$ by Fact 1(1). If $f_{3,4}(v) = (0,3')$, then $c'(v) \geq 6 - 3 - 2 - 2 = 1$ by R2-R3. For the case $f_{3,4}(v) = (1,2')$, if $v$ is not incident with a (3, 3, 6)-face, then $c'(v) \geq 6 - 2 - 2 = 0$ by R1-R3. Otherwise, $c'(v) \geq 6 - 3 - 2 - 2 + \min\{\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{17}{18}\} = 0$ by Fact 1(5) and R1-R6. For the case $f_{3,4}(v) = (2,1')$, if $\sum_{i=1}^{4} f_{dia}^i(v) = 1$, then by Lemma 7, $f_{dia}^1(v) = 0$. Hence, we have $c'(v) \geq 6 - 3 - 2 = 0$ by R1-R3. Otherwise, $v$ is not incident with a (3, 3, 6)-face and it follows that $n_3(v) \leq 2$. If $n_3(v) \leq 1$, then $c'(v) \geq 6 - 2 - 1 - 2 - 1 = 0$ by
R1-R3. If \( n_3(v) = 2 \), by Lemma 6, \( v \) is incident with at most one \((3, 4, 6)\)-face. It follows that \( c'(v) \geq 6 - \max\{2 \times \frac{3}{4} + 1, 2 + \frac{1}{2} + 2 + 1 - \frac{1}{3} - \frac{1}{3} \times 2\} = 0 \) by Fact 1 and R1-R5.

If \( n_3(v) = 3 \), then \( f_3(v) \leq 2 \) by Lemma 3(C7.1) and Fact 1(2). Then by R2 and R3, \( c'(v) \geq 6 - 3 - 3 = 0 \) for \( f_3(v) = (0, 3) \). If \( f_3(v) = 1 \), then \( f_3(v) \leq 2 \) by Fact 1(2), (3). By Lemma 7 and Lemma 3(C5.6), \( v \) is no incident with a \((3, 4^+, 6)\)-face. Hence, by Fact 1 and R1-R6, \( c'(v) \geq 6 - \frac{3}{4} - 3 - 2 + \frac{1}{3} \times 2 = 0 \) if \( v \) is incident with a \((3, 5^+, 6)\)-face and \( c'(v) \geq 6 - 1 - 3 - 2 = 0 \) otherwise.

If \( f_3(v) = 2 \), then \( f_4(v) \leq 1 \) and \( \sum_{i=1}^{4} f_{dia}(v) = 1 \) by Fact 1(2), (3) and Lemma 3(C7.1). Furthermore, \( f_{dia}^t(v) = 0 \) by Lemma 7. If \( f_{dia}^t(v) + f_{dia}^i(v) = 1 \), then by Lemma 8 (Figure 4(3)), \( f_{dia}^i(v) = 0 \) and \( f_{dia}^t(v) = 1 \). Hence, by R1-R5, \( c'(v) \geq 6 - 2 - 3 - 1 = 0 \). If \( f_{dia}^t(v) = 1 \), then by Lemma 8 (Figure 4(3)), the two bad 3-faces of the 2-diamond of \( v \) are two adjacent \((3, 5^+, 5^+)\)-faces with a common \((3, 6)\)-edge. Hence, by Fact 1 and R1-R6, \( c'(v) \geq 6 - 3 - 3 - 1 + \frac{1}{3} + \frac{1}{4} \times 3 + \frac{1}{4} \times \frac{1}{2} \times 3 = \frac{25}{6} \).

If \( n_2(v) = 4 \), then \( f_3(v) = 0 \) and \( f_4(v) \leq 3 \) by Lemma 3(C5.5) and Fact 1(2). Hence, we have \( c'(v) \geq 6 - 4 - 3 + \frac{1}{3} \times 2 + \frac{1}{4} \times \frac{1}{2} \times 4 = 0 \) by R2-R6.

Suppose that \( d(v) = 7 \). Then \( n_2(v) \leq 5 \) by Lemma 3(C6.1). At the same time, \( c(v) = 2 \times 7 - 6 = 8 \).

If \( n_2(v) = 0 \), then \( f_3(v) \leq 4 \) by Fact 1(1). If \( f_3(v) \leq 2 \), then by Fact 1, Lemma 4 and R1-R2, we have \( c'(v) \geq 8 - \max\{3, 3 + 2, 4 + 2, 3 + 3 + 1\} = 1 \). If \( f_3(v) = 3 \), then \( \sum_{i=1}^{4} f_{dia}(v) \leq 1 \). Hence, if \( \sum_{i=1}^{4} f_{dia}(v) = 1 \), then \( f_4(v) \leq 1 \) by Fact 1(2) and it follows that \( c'(v) \geq 8 - 4 - 3 - 1 = 0 \) by R1-R2; if \( \sum_{i=1}^{4} f_{dia}(v) = 0 \), then \( f_4(v) = 0 \) by Fact 1(2) and it follows that \( c'(v) \geq 8 - 2 \times 3 - 2 = 0 \) by Lemma 7 and R1. If \( f_3(v) = 4 \), then \( f_4(v) = 0 \) and \( 1 \leq \sum_{i=1}^{4} f_{dia}(v) \leq 2 \) by Fact 1(2), (3). If \( \sum_{i=1}^{4} f_{dia}(v) = 2 \), then by Lemma 7, Lemma 4 and R1, \( c'(v) \geq 8 - 4 - 3 = 1 \). If \( \sum_{i=1}^{4} f_{dia}(v) = 1 \), we will discuss this problem into three subcases. The first subcase is that \( f_{dia}^t(v) = 1 \), then by Lemma 7, Lemma 4 and R1, \( c'(v) \geq 8 - 4 - 2 = 2 \). The second subcase is that \( f_{dia}^t(v) = 1 \), then by Lemma 4, R1 and R4-R6, we have \( c'(v) \geq 8 - 3 - 3 - 2 = 0 \) if \( v \) is incident with at most one independent \((3, 3, 7)\)-face and \( c'(v) \geq 8 - 3 - 3 - 2 = 0 \) if \( v \) is incident with two independent \((3, 3, 7)\)-faces. The third subcase is that \( \sum_{i=1}^{4} f_{dia}(v) = 1 \), then by Lemma 4, R1 and R4-R6, \( c'(v) \geq 8 - \max\{2 + 3 + 3, \frac{5}{3} + 3 + 3 - 2 \times \frac{1}{2} \times 2\} = 0 \).

If \( n_2(v) = 1 \), then \( f_3(v) \leq 4 \) by Fact 1(1). If \( f_3(v) \leq 2 \), then by Fact 1(2), Lemma 4 and R1-R3, \( c'(v) \geq 8 - \max\{1 + 3, 3 + 1 + 2, 4 + 1 + 2, 3 + 3 + 1 + 1\} = 0 \). If \( f_3(v) = 3 \), then by Fact 1(2), (3) and Lemma 3(C7.1), \( \sum_{i=1}^{4} f_{dia}(v) \leq 1 \) and \( f_4(v) \leq 1 \). By Lemma 7 and R1-R3, \( c'(v) \geq 8 - \max\{3 + 3 + 1 + 1, 4 + 2 + 1 + 1\} = 0 \) if \( \sum_{i=1}^{4} f_{dia}(v) = 1 \) and \( c'(v) \geq 8 - 3 - 2 \times 2 - 1 = 0 \) if \( \sum_{i=1}^{4} f_{dia}(v) = 0 \). If \( f_3(v) = 4 \), then by Fact 1(2) and Lemma 3(C7.1), \( \sum_{i=1}^{4} f_{dia}(v) = 2 \) and \( f_4(v) = 0 \). Hence, by Lemma 7 and R1-R3, \( c'(v) \geq 8 - 4 - 3 - 1 = 0 \).
If $n_2(v) = 2$, then $f_d(v) \leq 3$ by Fact 1(1). If $f_d(v) \leq 2$, then $c'(v) \geq 8 - \max\{2 + 3, 3 + 2 + 2, 4 + 2 + 2, 3 + 2 + 1\} = 0$ by Fact 1(2), (3), Lemma 4, Lemma 7 and R1-R3. If $f_d(v) = 3$, then by Fact 1(2), (3) and Lemma 3(C7.1), $\sum_{i=1}^{4} f_{dia}(v) = 1$, $c(v) = 1$ and $f_d(v) \leq 1$. By Lemma 7, $f_{dia}(v) = 0$. Let $f_1 = [v_1v_2]$ be an independent 3-face of $v$. If $f_1$ is not a $(3,3,6)$-face, then $c'(v) \geq 8 - 3 - 2 - 2 - 1 = 0$ by Lemma 4 and R1-R3. Otherwise, $c'(v) \geq 8 - 3 - 2 - 1 + \frac{4}{2} + \frac{4}{2} + \frac{4}{2} + \frac{4}{2} + \frac{4}{2} = \frac{28}{2}$ by Lemma 4 and R1-R6.

If $n_2(v) = 3$, then $f_d(v) \leq 2$ by Fact 1(1). If $f_d(v) \leq 1$, then $c'(v) \geq 8 - \max\{3 + 3, 3 + 3, 3 + 2\} = 0$ by Fact 1 and R1-R3. If $f_d(v) = 2$, then by Fact 1, Lemma 7 and R1-R3, $c'(v) \geq 8 - 3 - 3 - 2 = 0$ if $\sum_{i=1}^{4} f_{dia}(v) = 1$ and $c'(v) \geq 8 - 2 - 2 - 3 - 1 = 0$ otherwise.

If $n_2(v) = 4$, then $f_d(v) \leq 2$ by Fact 1(1). If $f_d(v) = 0$, then $f_d(v) \leq 3$ by Fact 1(2) and it follows that $c'(v) \geq 8 - (3 + 4) = 1$ by R2-R3. If $f_d(v) = 1$, then $f_d(v) \leq 2$ by Fact 1(2), (3) and it follows that $c'(v) \geq 8 - 2 - 4 - 2 = 0$ by Lemma 7 and R1-R3. If $f_d(v) = 2$, then by Fact 1(2), (3) and Lemma 3(C7.1), $\sum_{i=1}^{4} f_{dia}(v) = 1$ and $f_d(v) \leq 2$. By Lemma 7, $f_{dia}(v) = 0$. Hence, we have $c'(v) \geq 8 - 3 - 4 - 2 + \frac{4}{2} + \frac{4}{2} = 0$ by Lemma 4 and R1-R6.

If $n_2(v) = 5$, then $f_d(v) = 0$ by Lemma 3(C6.3). Hence, $c'(v) \geq 8 - 5 - 3 = 0$ by Fact 1(2) and R2-R3.

Suppose that $d(v) = 8$. Then $n_2(v) \leq 6$ by Lemma 3(C8.1). At the same time, $c(v) = 2 \times 8 - 6 = 10$.

If $n_2(v) = 0$, then $f_d(v) \leq 5$ by Fact 1(1). If $f_d(v) \leq 3$, then $c'(v) \geq 10 - \max\{4 + 3, 4 + 2 + 2, 3 + 3 + 1\} = 0$ by Lemma 4, Fact 1(2), (3), R1-R3. If $f_d(v) = 4$, then $f_d(v) \leq 1$ and $\sum_{i=1}^{4} f_{dia}(v) \leq 2$ by Fact 1(2), (3). If $\sum_{i=1}^{4} f_{dia}(v) \geq 1$, then $c'(v) \geq 10 - \max\{4 + 4 + 1, 4 + 3 \times 3\} = 0$ by Lemma 4, R1-R3. Otherwise, $f_d(v) = 0$ by Fact 1(3) and it follows that $c'(v) \geq 10 - 3 \times 2 - 2 \times 2 = 0$ by Lemma 7, R1 and R3. If $f_d(v) = 5$, then $\sum_{i=1}^{4} f_{dia}(v) = 2$ and $f_d(v) = 0$ by Fact 1(2), (3). If follows that $c'(v) \geq 10 - 4 - 3 - 3 = 0$ by Lemma 4, Lemma 7 and R1.

If $n_2(v) = 1$, then $f_d(v) \leq 4$ by Fact 1(1). If $f_d(v) \leq 2$, then $c'(v) \geq 10 - \max\{1 + 4, 3 + 1 + 4, 4 + 1 + 3 \times 2 + 1 + 2\} = 1$ by Fact 1(2), (3), Lemma 4 and R1-R3. If $f_d(v) = 3$, then $f_d(v) \leq 1$ and $\sum_{i=1}^{4} f_{dia}(v) \leq 1$ by Fact 1(2), (3) and Lemma 3(C7.1). If $\sum_{i=1}^{4} f_{dia}(v) = 1$, then $c'(v) \geq 10 - 4 - 3 - 1 - 1 = 1$ by Lemma 4 and R1-R3. Otherwise, $c'(v) \geq 10 - 3 \times 2 - 2 - 1 = 0$ by Lemma 7 and R1-R3. If $f_d(v) = 4$, then $f_d(v) \leq 1$ and $\sum_{i=1}^{4} f_{dia}(v) \leq 2$ by Fact 1(2), (3) and Lemma 3(C7.1). If $\sum_{i=1}^{4} f_{dia}(v) = 2$, then $c'(v) \geq 10 - (4 \times 2 + 1 + 1) = 0$ by R1-R3. If $\sum_{i=1}^{4} f_{dia}(v) = 1$, then $c'(v) \geq 10 - \max\{4 + 2 + 1, 3 + 3 + 3 + 1\} = 0$ by Lemma 7 and R1-R3.

If $n_2(v) = 2$, then $f_d(v) \leq 4$ by Fact 1(1). If $f_d(v) \leq 2$, then $c'(v) \geq 10 - \max\{2 + 4, 3 + 2 + 3, 4 + 2 + 2, 3 \times 2 + 2 + 2\} = 0$ by Fact 1 and R1-R3. If $f_d(v) = 3$, then $f_d(v) \leq 1$ by Fact 1(2), (3) and it follows that $c'(v) \geq$
if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ and $c'(v) \geq 10 - 3 - 2 - 2 - 1 = 0$ by Lemma 4, R1-R3. If $f_{3}(v) = 4$, $f_{4}(v) \leq 1$ and $\sum_{i=1}^{4} f_{dia}^{i}(v) = 2$ by Fact (2) and Lemma 3(C7.1). It follows that $c'(v) \geq 10 - 4 - 3 - 2 - 1 = 0$ by Lemma 4, R1-R3.

If $n_{2}(v) = 3$, then $f_{3}(v) \leq 3$ by Fact 1(1). If $f_{3}(v) \leq 1$, then $c'(v) \geq 10 - \max\{4 + 2 + 2 + 1, 3 + 3 + 2 + 1\} = 1$ if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ and $c'(v) \geq 10 - 3 - 2 - 2 - 1 = 0$ if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 0$ by Lemma 4, Lemma 7 and R1-R3. If $f_{3}(v) = 4$, $f_{4}(v) \leq 1$ and $\sum_{i=1}^{4} f_{dia}^{i}(v) = 2$ by Fact (2) and Lemma 3(C7.1). It follows that $c'(v) \geq 10 - 4 - 3 - 2 - 1 = 0$ by Lemma 4, R1-R3.

If $n_{2}(v) = 4$, then $f_{3}(v) \leq 2$ by Fact 1(1). If $f_{3}(v) \leq 1$, then $c'(v) \geq 10 - \max\{4 + 4, 3 + 4 + 3\} = 1$ by Fact 1(2), (3) and R1-R3. If $f_{3}(v) = 2$, then $f_{4}(v) \leq 2$ by Fact 1(2), (3). It follows that $c'(v) \geq 10 - 4 - 3 - 2 - 1 = 0$ by Lemma 4 and R1-R3 if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ and $c'(v) \geq 10 - 3 - 2 - 3 - 2 = 0$ by Lemma 7 and R1-R3 if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 0$. If $f_{3}(v) = 3$, then $f_{4}(v) \leq 1$ and $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ by Fact 1(2), (3) and Lemma 3(C7.1). Hence, $c'(v) \geq 10 - (3 + 3) - 3 - 1 = 0$ by Lemma 4, Lemma 7 and R1-R3.

If $n_{2}(v) = 5$, then $f_{3}(v) \leq 2$ by Fact 1(1). If $f_{3}(v) = 0$, then $f_{4}(v) \leq 4$ by Fact 1(2) and $c'(v) \geq 10 - 5 - 4 - 1 = 0$ by R2-R3. If $f_{3}(v) = 1$, then $f_{4}(v) \leq 3$ by Fact 1(2), (3) and $c'(v) \geq 10 - 2 - 5 - 3 = 0$ by Lemma 7 and R1-R3. If $f_{3}(v) = 2$, then $f_{4}(v) \leq 2$ and $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ by Fact 1(2), (3) and Lemma 3(C7.1). Hence, $c'(v) \geq 10 - 3 - 5 - 2 = 0$ by Lemma 7 and R1-R3.

If $n_{2}(v) = 6$, then $f_{3}(v) = 0$ by Lemma 3(C8.2) and $f_{4}(v) \leq 4$ by Fact 1(2). Hence, $c'(v) \geq 10 - 6 - 4 = 0$ by R2-R3.

Suppose that $d(v) = 9$. Then $c'(v) = 2 \times 9 - 6 = 12$. By Lemma 9, $n_{2}(v) \leq 7$. We have some subcases below. If $n_{2}(v) = 0$, then $f_{3}(v) \leq 6$ by Fact 1(1). By Fact 1(2), (3), Lemma 4 and R2-R3, $c'(v) \geq 12 - \max\{4 + 4, 3 + 3 + 2 + 4, 4 + 3 + 3 + 3 + 1 + 4 \times 2 + 1, 4 \times 3 + 3 + 1, 3 + 2, 3 \times 3 + 1, 1, 4 \times 2 + 1, 1, 4 \times 3 + 2 + 1, 1, 3 \times 4\} = 0$. If $n_{2}(v) = 1$, then $f_{3}(v) \leq 5$ by Fact 1(1). If follows that $c'(v) \geq 12 - \max\{1 + 4, 3 + 1 + 3, 4 + 1 + 3, 3 \times 2 + 1, 4 + 3 + 3, 1 + 4 \times 2 + 1, 4 + 3 + 2 + 1, 4 \times 2 + 1, 1, 4 \times 2 + 3 + 1\} = 0$ by Fact 1(2), (3), Lemma 4 and R1-R3 if $c'(v) \neq 4$. Otherwise, $c'(v) \geq 12 - 3 \times 3 - 2 - 1 - 0 = 0$ by Lemma 7 and R1-R3. If $n_{2}(v) = 2$, then $f_{3}(v) \leq 4$ by Fact 1(1). If $f_{3}(v) \leq 3$, then $c'(v) \geq 12 - \max\{2 + 4, 3 + 2 + 3, 4 + 2 + 3, 3 \times 2 + 2 + 2, 4 + 3 + 2 + 2, 3 \times 3 + 2 + 1\} = 0$ by Fact 1(2), (3), Lemma 4 and R1-R3. If $f_{3}(v) = 4$, then $c'(v) \geq 12 - 4 \times 2 - 2 - 1 = 0$ by Lemma 4, Lemma 7 and R1-R3 otherwise. If $n_{2}(v) = 3$, then $f_{3}(v) \leq 4$ by Fact 1(1). If $f_{3}(v) \leq 2$, then $c'(v) \geq 12 - \max\{3 + 4, 3 + 3 + 3, 4 + 3 + 3, 3 \times 2 + 3 + 2\} = 1$ by Fact 1(2), (3), Lemma 4 and R1-R3. If $f_{3}(v) = 3$, then by Fact 1(2), (3) and R1-R3, we have $c'(v) \geq 12 - 4 - 3 - 3 - 2 = 0$ by Lemma 4 if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ and $c'(v) \geq 12 - 3 \times 2 - 2 - 3 - 1 = 0$ by Lemma 7 if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 0$. 

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If \( f_3(v) = 4 \), then \( f_4(v) \leq 1 \) and \( \sum_{i=1}^{4} f_{dia}^i (v) = 2 \) by Fact 1(2), (3) and Lemma 3(C7.1). It follows that \( c'(v) \geq 12 - 4 \times 2 - 3 = 1 = 0 \) by Lemma 4 and R1-R3. If \( n_2(v) = 4 \), then \( f_2(v) \leq 3 \) by Fact 1(1). If \( f_3(v) \leq 2 \), then \( c'(v) \geq 12 - \max\{4 + 4, 3 + 4, 4 + 3, 3 \times 2 + 4 + 2\} = 0 \) by Fact 1(2), (3), Lemma 4 and R1-R3. If \( f_3(v) = 3 \), then \( f_4(v) \leq 2 \) and \( \sum_{i=1}^{4} f_{dia}^i (v) = 1 \) by Fact 1(2), (3) and Lemma 3(C7.1) and it follows that \( c'(v) \geq 12 - 3 - 3 = 0 \) by Lemma 4, Lemma 7 and R1-R3. If \( n_2(v) = 5 \), then \( f_3(v) \leq 2 \) by Fact 1(1). If \( f_3(v) \leq 1 \), then \( c'(v) \geq 12 - \max\{5 + 4, 3 + 5 + 3\} = 1 \) by Fact 1(2), (3), Lemma 4 and R1-R3. If \( f_3(v) = 2 \), then \( f_4(v) \leq 3 \) by Fact 1(2), (3) and Lemma 3(C7.1). Hence, we have \( c'(v) \geq 12 - 3 - 2 = 0 \) by Lemma 7 and R1-R3 otherwise. If \( n_2(v) = 6 \), then \( f_3(0) \leq 2 \) and \( \sum_{i=1}^{4} f_{dia}^i (v) \leq 1 \) by Fact 1(2), (3) and Lemma 3(C7.1). If \( f_3(0) \leq 1 \), then \( c'(v) \geq 12 - \max\{6 + 4, 3 + 6 + 3\} = 0 \) by Fact 1(2), (3), R1 and R3. If \( f_3(v) = 2 \), then \( c'(v) \geq 12 - 6 = 0 \) by Fact 1(2), (3), Lemma 7 and R1-R3. If \( n_2(v) = 7 \), then \( f_3(v) = 0 \) by Lemma 9(C9.2) and \( f_3(v) \leq 4 \) by Fact 1(2). It follows that \( c'(v) \geq 12 - 4 = 8 \) by R2-R3.

Suppose that \( d(v) \geq 10 \). By the assumption of Theorem 2, we have \( \frac{4}{5} f_6(v) + 2c(v) + 2f_4(v) \leq d(v) \) and \( \frac{4}{5} f_6(v) + 2c(v) + n_2(v) \leq d(v) \). Note that \( f_3(v) = f_6(v) + c(v) \). Hence, by Fact 1, Lemma 4 and R1-R5, we can obtain that

\[
c'(v) \geq 2d(v) - 6 - \left( \frac{4}{5} f_6(v) + 3c(v) + f_4(v) + n_2(v) \right) \\
\quad + \frac{1}{5} (d(v) - f_6(v) - c(v) - f_4(v) - 1) \\
\quad = 2d(v) - 6 - \left( \frac{4}{5} f_6(v) + \frac{1}{5} (d(v) + 4) f_4(v) + n_2(v) - \frac{1}{5} d(v) + \frac{1}{5} \right) \\
\quad \geq 2d(v) - 6 - \left( \frac{4}{5} f_6(v) + \frac{1}{5} (d(v) + 4) f_4(v) + (d(v) - \frac{1}{5} f_6(v) - 2c(v)) \right) \\
\quad - \frac{1}{5} d(v) - \frac{1}{5} \\
\quad = 2d(v) - 6 - \left( \frac{6}{5} d(v) + \frac{4}{5} f_6(v) + \frac{5}{5} c(v) + \frac{7}{5} f_4(v) + \frac{4}{5} \right) \\
\quad \geq 2d(v) - 6 - \left( \frac{6}{5} d(v) + \frac{4}{5} f_6(v) + \frac{5}{5} \cdot \frac{1}{5} (d(v) - \frac{1}{5} f_6(v)) + \frac{4}{5} \right) \\
\quad \geq \frac{5}{5} d(v) - \frac{25}{5} \\
\quad \geq 0.
\]

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