b-GENERALIZED DERIVATIONS ON MULTILINEAR POLYNOMIALS IN PRIME RINGS

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Abstract. Let $R$ be a noncommutative prime ring of characteristic different from 2, $Q$ be its maximal right ring of quotients and $C$ be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$, $b \in Q$, $F$ a $b$-generalized derivation of $R$ and $d$ is a nonzero derivation of $R$ such that
\[ d([F(f(r)), f(r)]) = 0 \]
for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following holds:

1. There exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;
2. There exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in $R$.

1. Introduction

Throughout this paper $R$ always denotes an associative prime ring with center $Z(R)$. A ring $R$ is said to be a prime ring if for any $a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$. $Q$ denotes the maximal right ring of quotients of $R$. Then $C = Z(Q)$ is called the extended centroid of $R$. It is well known that when $R$ is a prime ring, then $Q$ is also a prime ring and $C$ is a field. We refer the reader to the book [1] for details. The commutator of $x$ and $y$ is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$ for $x, y \in R$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Evidently, for some $a, b \in R$, the map $F(x) = ax + xb$ for all $x \in R$ is an example of generalized derivation which is called as inner generalized derivation of $R$.

For a subset $S$ of $R$, a mapping $f : S \to R$ is called commuting (centralizing) on $S$ if $[f(x), x] = 0$ (resp. $[f(x), x] \in Z(R)$) for all $x \in S$. Posner [19] initiated...
the study of commuting and centralizing maps. Posner [19] proved that a prime ring must be commutative, if it possesses a nonzero centralizing derivation. Since then many authors investigated commuting and centralizing maps in different directions.

In [13], Lee and Lee proved that if \( R \) is a prime ring, \( I \) a nonzero ideal of \( R \) and \( d \) is a nonzero derivation of \( R \) such that \([d(f(r)), f(r)] \in Z(R)\) for all \( r = (r_1, \ldots, r_n) \in I^n\), then \( f(x_1, \ldots, x_n) \) is central-valued on \( R \), except when \( \text{char}(R) = 2 \) and \( R \) satisfies \( s_4(x_1, x_2, x_3, x_4) \).

Recently, De Filippis and Di Vincenzo (see [5]) studied the situation when \( \delta([d(f(r)), f(r)]) = 0 \) for all \( r = (r_1, \ldots, r_n) \in R^n \), where \( d \) and \( \delta \) are two derivations of \( R \). The statement of De Filippis and Di Vincenzo’s theorem is the following:

**Theorem A** ([5, Theorem 1]). Let \( K \) be a noncommutative ring with unity, \( R \) a prime \( K \)-algebra of characteristic different from 2, \( d \) and \( \delta \) two nonzero derivations of \( R \) and \( f(r_1, \ldots, r_n) \) a multilinear polynomial over \( K \). If

\[
\delta([d(f(r)), f(r)]) = 0
\]

for all \( r = (r_1, \ldots, r_n) \in R^n \), then \( f(r_1, \ldots, r_n) \) is central-valued on \( R \).

Then, De Filippis and Di Vincenzo [6] studied above result replacing derivation \( d \) with a generalized derivation \( F \) of \( R \). More precisely, authors proved the following:

**Theorem B.** Let \( R \) be a prime algebra over a commutative ring \( K \) with unity, and let \( f(x_1, \ldots, x_n) \) be a multilinear polynomial over \( K \), not central valued on \( R \). Suppose that \( d \) is a nonzero derivation of \( R \) and \( F \) is a nonzero generalized derivation of \( R \) such that

\[
d([F(f(r)), f(r)]) = 0
\]

for all \( r = (r_1, \ldots, r_n) \in R^n \). If the characteristic of \( R \) is different from 2, then one of the following holds:

1. There exists \( \lambda \in C \), the extended centroid of \( R \) such that \( F(x) = \lambda x \) for all \( x \in R \);
2. There exist \( a \in U \), the Utumi quotient ring of \( R \), and \( \lambda \in C \) such that \( F(x) = ax + xa + \lambda x \) for all \( x \in R \), and \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \).

Our motivation in the present paper is to consider \( F \) as a \( b \)-generalized derivation of \( R \). Let \( b \in Q \). An additive map \( G : R \to Q \) is called a \( b \)-generalized derivation of \( R \) if \( g(xy) = g(x)y + bx d(y) \) holds for all \( x,y \in R \), where \( d : R \to Q \) is an additive map. It is proved in [11] that if \( R \) is a prime ring and \( b \neq 0 \), then the associated map \( d \) must be a derivation of \( R \). Evidently, a generalized derivation is a 1-generalized derivation. For some \( a,b,c \in Q \), the map \( F(x) = ax + bxc \in Q \) is an example of \( b \)-generalized
derivation of $R$, which we call as inner $b$-generalized derivation of $R$. The $b$-
 generalized derivations appeared canonically in [3] and were introduced and
studied recently in [11,15,17].

More precisely, we prove the following theorem.

**Theorem 1.1.** Let $R$ be a noncommutative prime ring of characteristic dif-
ferent from 2, $Q$ be its maximal right ring of quotients and $C$ be its extended
centroid. Suppose that $f(x_1, \ldots , x_n)$ be a noncentral multilinear polynomial
over $C$, $b \in Q$, $F$ a $b$-generalized derivation of $R$ and $d$ is a nonzero derivation
of $R$ such that

$$d([F(f(r)), f(r)]) = 0$$

for all $r = (r_1, \ldots , r_n) \in R^n$. Then one of the following holds:

1. there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;
2. there exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all
   $x \in R$ with $f(x_1, \ldots , x_n)^2$ is central valued in $R$.

As an application of above theorem, we have the following corollary which
is a generalization of particular result of [4].

**Corollary 1.2.** Let $R$ be a noncommutative prime ring of characteristic dif-
ferent from 2, $Q$ be its maximal right ring of quotients and $C$ be its extended
centroid. Suppose that $f(x_1, \ldots , x_n)$ be a noncentral multilinear polynomial
over $C$, $b \in Q$, $F$ a $b$-generalized derivation of $R$ and $d$ is a nonzero derivation
of $R$ such that

$$[F(f(r)), f(r)] \in C$$

for all $r = (r_1, \ldots , r_n) \in R^n$. Then one of the following holds:

1. there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;
2. there exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all
   $x \in R$ with $f(x_1, \ldots , x_n)^2$ is central valued in $R$.

Let $\sigma$ be an automorphism of $R$. $\sigma$ is said to be inner automorphism of
$R$, if there exists an invertible element $p \in Q$ such that $\sigma(x) = pxp^{-1}$ for all
$x \in R$. If $\sigma$ is not inner, we say $\sigma$ as an outer automorphism of $R$. An additive
map $d : R \rightarrow R$ is called a $\sigma$-derivation, if $d(xy) = d(x)y + \sigma(x)d(y)$ holds
for all $x, y \in R$. For some $a \in Q$, $d(x) = ax - \sigma(x)a$ is an example of $\sigma$-
derivation, which is called as inner $\sigma$-derivation. An additive map $G : R \rightarrow R$
is called a generalized $\sigma$-derivation, if there exists a $\sigma$-derivation $d$ such that
$G(xy) = G(x)y + \sigma(x)d(y)$ holds for all $x, y \in R$. Note that generalized $1_R$-
derivation is called as generalized derivation, where $1_R$ denotes the identity automorphism of $R$. Generally, generalized $\sigma$-derivation is called as generalized skew derivation. If for some invertible $b \in Q$, $\sigma(x) = bx^{-1}$ for all $x \in R$, and
$d$ is inner $\sigma$-derivation of $R$, then $G(xy) = G(x)y + \sigma(x)d(y) = G(x)y + bx^{-1}(ay - by^{-1}a) = G(x)y + bx(b^{-1}ay - yb^{-1}a) = G(x)y + bx[a^{-1}a, y]$ for all $x, y \in R$, is nothing but a $b$-generalized derivation of $R$ with associated
derivation $d(x) = [b^{-1}a, x]$ for all $x \in R$. It is very easy to prove that any
generalized $\sigma$-derivation of $R$ with associated $\sigma$-derivation $d$, where $\sigma(x) = bxb^{-1}$ for all $x \in R$ and $b \in Q$ is an inner automorphism, is a $b$-generalized derivation of $R$ with the associated map $b^{-1}d$.

Thus as an application of Theorem 1.1, we have the following corollary.

**Corollary 1.3.** Let $R$ be a noncommutative prime ring of characteristic different from $2$, $Q$ be its maximal right ring of quotients, $C$ be its extended centroid and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$. Suppose that $F$ is a generalized $\sigma$-derivation of $R$ with $\sigma$ an inner automorphism of $R$ and $d$ is a nonzero derivation of $R$ such that
\[
d([F(f(r)), f(r)]) = 0
\]
for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following holds:

1. there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;
2. there exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in $R$.

Similarly, following the corollary also holds.

**Corollary 1.4.** Let $R$ be a noncommutative prime ring of characteristic different from $2$, $Q$ be its maximal right ring of quotients, $C$ be its extended centroid and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$. Suppose that $F$ is a generalized $\sigma$-derivation of $R$ with $\sigma$ an inner automorphism of $R$ such that
\[
[F(f(r)), f(r)] \in C
\]
for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following holds:

1. there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;
2. there exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in $R$.

2. The case of inner $b$-generalized derivation

First we consider the case when $F$ is the inner $b$-generalized derivation and $d$ is inner derivation of $R$. Let $F(x) = ax + brq$ for all $x \in R$ and $d(x) = [c, x]$ for all $x \in R$, for some $a, b, c, q \in Q$. Then by our hypothesis, we have
\[
[c, [ar + brq, r]] = 0
\]
for all $r \in f(R)$. This can be re-written as
\[
car^2 + cbqr - car - crbrq - ar^2c - brqrc + rarc + rbrqc = 0
\]
for all $r \in f(R)$.

We investigate this generalized polynomial identity in prime ring. In all that follows, let $R$ be a prime ring with extended centroid $C$, $\text{char}(R) \neq 2$ and $c \notin C$. Moreover, we assume that $f(x_1, \ldots, x_n)$ is a multilinear polynomial over $C$ which is not central valued on $R$. 
Lemma 2.1. If $b \in C$, then either $a, bq \in C$ or $a - bq \in C$ with $f(x_1, \ldots, x_n)^2$ is central valued in $R$.

Proof. If $b \in C$, then our hypothesis becomes
\[ [c, [ar + rbq, r]] = 0 \]
for all $r \in f(R)$. In this case by [6], one of the following holds: (i) $a, bq \in C$; (ii) $a - bq \in C$ with $f(x_1, \ldots, x_n)^2$ is central valued. \qed

Lemma 2.2. If $q \in C$, then $a + bq \in C$.

Proof. If $q \in C$, then our hypothesis becomes
\[ [c, [(a + bq)r, r]] = 0, \]
that is,
\[ [c, [a + bq, r]r] = 0 \]
for all $r \in f(R)$. In this case by [8, Corollary 2.9], $a + bq \in C$. \qed

Lemma 2.3 ([6, Lemma 1]). Let $C$ be an infinite field and $m \geq 2$. If $A_1, \ldots, A_k$ are not scalar matrices in $M_m(C)$, then there exists some invertible matrix $P \in M_m(C)$ such that any matrices $PA_1P^{-1}, \ldots, PA_kP^{-1}$ have all non-zero entries.

Proposition 2.4. Let $R = M_m(C)$, $m \geq 2$, be the ring of all $m \times m$ matrices over the infinite field $C$, $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over $C$ and $a, b, c, q \in R$. If
\[ car^2 + cbr - c - crbrq - ar^2c - brqrc + rarc + rbrqc = 0 \]
for all $r \in f(R)$, then either $b$ or $c$ or $q$ are central.

Proof. By our assumption $R$ satisfies the generalized polynomial identity
\[
\begin{aligned}
&caf(r_1, \ldots, r_n)^2 + cbf(r_1, \ldots, r_n)af(r_1, \ldots, r_n) \\
&- cf(r_1, \ldots, r_n)af(r_1, \ldots, r_n) - cf(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)q \\
&- af(r_1, \ldots, r_n)^2c - bf(r_1, \ldots, r_n)af(r_1, \ldots, r_n)c \\
&+ f(r_1, \ldots, r_n)af(r_1, \ldots, r_n)c + f(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)qc = 0.
\end{aligned}
\]
We assume first that $b \notin Z(R)$, $c \notin Z(R)$ and $q \notin Z(R)$. Now we shall show that this case leads to a contradiction.

Since $b \notin Z(R)$, $c \notin Z(R)$ and $q \notin Z(R)$, by Lemma 2.3 there exists a $C$-automorphism $\phi$ of $M_m(C)$ such that $\phi(b)$, $\phi(c)$ and $\phi(q)$ have all non-zero entries. Clearly $R$ must satisfies the condition
\[
\begin{aligned}
&\phi(ca)f(r_1, \ldots, r_n)^2 + \phi(cb)f(r_1, \ldots, r_n)\phi(q)f(r_1, \ldots, r_n) \\
&- \phi(c)f(r_1, \ldots, r_n)\phi(a)f(r_1, \ldots, r_n) \\
&- \phi(c)f(r_1, \ldots, r_n)\phi(b)f(r_1, \ldots, r_n)\phi(q) \\
&- \phi(a)f(r_1, \ldots, r_n)^2\phi(c) - \phi(b)f(r_1, \ldots, r_n)\phi(q)f(r_1, \ldots, r_n)\phi(c)
\end{aligned}
\]
+ f(r_1, \ldots, r_n)\phi(a)f(r_1, \ldots, r_n)\phi(c)
(2) \quad + f(r_1, \ldots, r_n)\phi(b)f(r_1, \ldots, r_n)\phi(q) = 0.

Here \(e_{kj}\) denotes the usual matrix unit with 1 in \((k, l)\)-entry and zero elsewhere. Since \(f(x_1, \ldots, x_n)\) is not central, by [14] (see also [16]), there exist \(u_1, \ldots, u_n \in M_m(C)\) and \(0 \neq \gamma \in C\) such that \(f(u_1, \ldots, u_n) = \gamma e_{kl}\), with \(k \neq l\). Moreover, since the set \(\{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in M_m(C)\}\) is invariant under the action of all \(C\)-automorphisms of \(M_m(C)\), then for any \(i \neq j\) there exist \(r_1, \ldots, r_n \in M_m(C)\) such that \(f(r_1, \ldots, r_n) = \gamma e_{ij}\), where \(0 \neq \gamma \in C\). Hence by (2) we have

\[
\phi(b)e_{ij}\phi(q)e_{ij} - \phi(c)e_{ij}\phi(a)e_{ij} - \phi(c)e_{ij}\phi(b)e_{ij}\phi(q) = 0
\]

and then left and right multiplying by \(e_{ij}\), it follows \(2e_{ij}\phi(c)e_{ij}\phi(b)e_{ij}\phi(q)e_{ij} = 0\), which is a contradiction, since \(\phi(b), \phi(c)\) and \(\phi(q)\) have all non-zero entries. Thus we conclude that either \(b\) or \(c\) or \(q\) are central. \(\square\)

**Proposition 2.5.** Let \(R = M_n(C), m \geq 2\) be the ring of all matrices over the field \(C\) with char\((R) \neq 2\) and \(f(x_1, \ldots, x_n)\) a non-central multilinear polynomial over \(C\) and \(a, b, c, q \in R\). If

\[
car^2 + cbqr - crar - cbqr - ar^2c - brcq + rarc + rbrqc = 0
\]

for all \(r \in f(R)\), then either \(b\) or \(c\) or \(q\) are central.

**Proof.** If one assumes that \(C\) is infinite, then the conclusions follow by Proposition 2.4.

Now let \(C\) be finite and \(K\) be an infinite field which is an extension of the field \(C\). Let \(\overline{R} = M_n(K) \cong R \otimes_C K\). Notice that the multilinear polynomial \(f(x_1, \ldots, x_n)\) is central-valued on \(R\) if and only if it is central-valued on \(\overline{R}\). Consider the generalized polynomial

\[
P(r_1, \ldots, r_n) = caf(r_1, \ldots, r_n)^2 + cbf(r_1, \ldots, r_n)qf(r_1, \ldots, r_n)
- cf(r_1, \ldots, r_n)af(r_1, \ldots, r_n) - cf(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)q
- af(r_1, \ldots, r_n)c - bf(r_1, \ldots, r_n)c
+ f(r_1, \ldots, r_n)af(r_1, \ldots, r_n)c + f(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)qc
\]

\[= 0\]

which is a generalized polynomial identity for \(R\).

Moreover, it is a multi-homogeneous of multi-degree \((2, 2, 2)\) in the indeterminates \(x_1, \ldots, x_n\).

Hence the complete linearization of \(P(x_1, \ldots, x_n)\) is a multilinear generalized polynomial \(\Theta(x_1, \ldots, x_n, y_1, \ldots, y_n)\) in \(2n\) indeterminates, moreover

\[
\Theta(x_1, \ldots, x_n, x_1, \ldots, x_n) = 2^n P(x_1, \ldots, x_n).
\]

Clearly the multilinear polynomial \(\Theta(x_1, \ldots, x_n, y_1, \ldots, y_n)\) is a generalized polynomial identity for \(R\) and \(\overline{R}\) too. Since char\((C) \neq 2\) we obtain \(P(r_1, \ldots, r_n) = 0\) for all \(r_1, \ldots, r_n \in \overline{R}\) and then conclusion follows from Proposition 2.4. \(\square\)
Corollary 2.6. Let $R = M_m(C)$, $m \geq 2$ be the ring of all matrices over the field $C$ with $\text{char}(R) \neq 2$ and $a, b, c, q \in R$. If
\[ c a r^2 + c b r q r - c a r - c b r q r + a r^2 c - b q r c + r a c + r b r q c = 0 \]
for all $r \in R$, then either $b$ or $c$ or $q$ are central.

Above corollary can be rewritten as:

Corollary 2.7. Let $R = M_m(C)$, $m \geq 2$ be the ring of all matrices over the field $C$ with $\text{char}(R) \neq 2$ and $a_1, a_2, a_3, a_4, a_6, a_7 \in R$. If
\[ a_1 r^2 + a_2 r a r - a_3 r a r - a_4 a r a_3 - a_4 r^2 a_5 - a_4 a_3 r a_5 + a_4 r a_5 + r a_6 r a_7 = 0 \]
for all $r \in R$, then either $a_3$ or $a_5$ or $a_6$ are central.

Lemma 2.8. Let $R$ be a primitive ring, which is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$, such that $\dim_C V = \infty$. Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in R$. If
\[ a_1 r^2 + a_2 r a r - a_3 r a r - a_4 a r a_3 = a_4 r^2 a_5 - a_4 a_3 r a_5 + a_4 r a_5 + r a_6 r a_7 = 0 \]
for all $x \in R$, then either $a_3$ or $a_5$ or $a_6$ are central.

Proof. We assume that $a_3, a_5$ and $a_6$ are noncentral central. Since $V$ is infinite dimensional over $C$, for any $e = e^2 \in \text{Soc}(R)$, we have $eRe \cong M_k(C)$ with $k = \dim_C V e$. Since $a_3, a_5 \notin C$ and $a_6 \notin C$, they do not centralize the nonzero ideal $\text{Soc}(R)$ of $R$, so $a_3 h_0 \neq h_0 a_3, a_5 h_1 \neq h_1 a_5$ and $a_6 h_2 \neq h_2 a_6$ for some $h_0, h_1, h_2 \in \text{Soc}(R)$. By Litoff’s theorem [12, p. 280] there exists an idempotent $e \in \text{Soc}(R)$ such that $h_0, h_1, h_2, h_0 a_3, a_3 h_0, h_1 a_5, a_5 h_1, h_2 a_6, a_6 h_2$ are all in $eRe$. We have $eRe \cong M_k(C)$ where $k = \dim_C V e$. Since $R$ satisfies GPI $e(a_1 ere)^2 + a_2 ere a_3 ere - a_3 ere a_4 ere - a_4 ere a_3 ere - a_4 (ere)^2 a_5 - a_4 ere a_3 ere + ere a_4 ere a_5 + ere a_4 ere a_7 e = 0$, the subring $eRe$ satisfies the GPI
\[ ea_1 e r^2 + ea_2 ere a_3 ere - ea_3 ere a_4 ere - ea_4 ere a_3 ere - ea_4 er^2 e a_5 \]
\[ - ea_4 ere a_3 ere + ere a_4 ere a_5 + ere a_4 ere a_7 e = 0. \]
Then by above finite dimensional case, we conclude that either $ea_3 e \in Z(eRe)$ or $ea_5 e \in Z(eRe)$ or $ea_6 e \in Z(eRe)$. Then
\[ a_3 h_0 = ea_3 h_0 = ea_3 e h_0 = h_0 e a_3 e = h_0 a_3, \]
\[ a_5 h_1 = ea_5 h_1 = ea_5 e h_1 = h_1 e a_5 e = h_1 a_5, \]
and
\[ a_6 h_2 = ea_6 h_2 = ea_6 e h_2 = h_2 e a_6 e = h_2 a_6. \]
All the cases lead to the contradiction. \qed

Lemma 2.9. Let $R$ be a noncommutative prime ring of characteristic different from 2, $Q$ be its maximal right ring of quotients, $C$ be its extended centroid and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$. Suppose for some
Proof. If $b \in C$ or $q \in C$, then result follows by Lemma 2.1 and Lemma 2.2 respectively. Thus we assume that $b \notin C$ and $q \notin C$.

By hypothesis, we have

$$\Psi(x_1, \ldots, x_n) = [c, [af(x_1, \ldots, x_n) + bf(x_1, \ldots, x_n)q, f(x_1, \ldots, x_n)]] = 0$$

for all $x_1, \ldots, x_n \in R$. Since $R$ and $Q$ satisfy same generalized polynomial identities (see [2]), $Q$ satisfies $\Psi(x_1, \ldots, x_n) = 0$. Since $c \notin C$, $b \notin C$ and $q \notin C$, $\Psi(x_1, \ldots, x_n)$ is a non-trivial GPI for $Q$. By the well known Martindale’s theorem [18], $Q$ is then a primitive ring with nonzero socle and with $C$ as its associated division ring. Then, by Jacobson’s theorem [9, p. 75], $Q$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$. Assume first that $V$ is finite dimensional over $C$, that is, $\dim_C V = m$. By density of $R$, we have $R \cong M_m(C)$. Since $f(r_1, \ldots, r_n)$ is not central valued on $R$, $R$ must be noncommutative and so $m \geq 2$. In this case, by Proposition 2.5, we get that $b$ or $q$ or $c$ are in $C$, a contradiction.

If $V$ is infinite dimensional over $C$, then by Lemma 2 in [20], the set $f(Q)$ is dense on $R$. Then by hypothesis, $Q$ satisfies

$$[c, [ar + brq, r]] = 0,$$

which gives

$$car^2 + cbrqr - carr - crbrq - ar^2c - brqrc + rarc + rbrqc = 0.$$

Then by Lemma 2.8, we conclude that either $b \in C$ or $q \in C$ or $c \in C$, which leads to a contradiction.

\hfill \Box

3. Result on $b$-generalized derivations

Lemma 3.1. Let $R$ be a noncommutative prime ring of characteristic different from 2, $Q$ be its maximal right ring of quotients and $C$ be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$, $b \in Q$, $F$ a $b$-generalized derivation of $R$ and $c \in R - C$ such that

$$[c, [F(f(r)), f(r)]] = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following holds:

(i) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;
(ii) there exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in $R$. 

Proof. By [11, Theorem 2.3], there exist a derivation \( d : R \to Q \) and \( a \in Q \) such that \( F(x) = ax + bd(x) \) for all \( x \in R \). By assumption, 
\[
[c, [af(r) + bd(f(r)), f(r)]] = 0
\]
for all \( r = (r_1, \ldots, r_n) \in R^n \).

If \( d \) is an inner derivation, that is \( d(x) = [p, x] \) for all \( x \in R \) and for some \( p \in Q \), then \( F(x) = (a + bp)x - bxp \) for all \( x \in R \) and hence by Lemma 2.9, we have:

(i) \( a + bp, b, bp \in C \). In this case \( F(x) = ax \) for all \( x \in R \), where \( a \in C \).

(ii) \( b \in C \), \( a + 2bp \in C \) and \( f(x_1, \ldots, x_n)^2 \) is central valued in \( R \). Let \( a + 2bp = \lambda \in C \). Then \( F(x) = \lambda x - bpx - xbp \) for all \( x \in R \).

(iii) \( p \in C \) and \( a \in C \). In this case also \( F(x) = ax \) for all \( x \in R \), where \( a \in C \).

Next assume that \( d \) is an outer derivation of \( R \). It is well known that any derivation of \( R \) can be uniquely extended to a derivation of \( Q \) (see [14, Lemma 2]). By hypothesis, we have

\[
[c, [af(r_1, \ldots, r_n) + bd(r_1, \ldots, r_n)] + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n), f(r_1, \ldots, r_n)] = 0
\]
for all \( r_1, \ldots, r_n \in Q \) by [1, Theorem 6.4.4]. By Kharchenko’s Theorem [10], \( Q \) satisfies

\[
[c, [bf(r_1, \ldots, r_n), f(r_1, \ldots, r_n)]] = 0.
\]

In particular, \( Q \) satisfies the blended component

\[
[c, [b \sum_i f(r_1, \ldots, s_i, \ldots, r_n), f(r_1, \ldots, r_n)] = 0. \tag{7}
\]

Assuming \( s_1 = r_1 \) and \( s_2 = \cdots = s_n = 0 \), \( Q \) satisfies

\[
[c, [bf(r_1, \ldots, r_n), f(r_1, \ldots, r_n)]] = 0 \tag{8}
\]

that is

\[
[c, [b, f(r_1, \ldots, r_n)]f(r_1, \ldots, r_n)] = 0. \tag{9}
\]

By [8, Corollary 2.9], since \( f(r_1, \ldots, r_n) \) is noncentral valued in \( R \) and \( c \notin C \), we have \( b \in C \). Then (7) yields

\[
[c, \sum_i f(r_1, \ldots, s_i, \ldots, r_n), f(r_1, \ldots, r_n)] = 0. \tag{10}
\]
Replacing $s_i$ with $[q,r_i]$ for some $q \notin C$, we get from above relation that $Q$ satisfies
\[(11) \quad [c,[[q,f(r_1,\ldots,r_n)],f(r_1,\ldots,r_n)]] = 0.\]

By [5, Theorem 1], either $c \in C$ or $q \in C$, a contradiction. \[\square\]

**Theorem 3.2.** Let $R$ be a noncommutative prime ring of characteristic different from 2, $Q$ be its maximal right ring of quotients and $C$ be its extended centroid. Suppose that $f(x_1,\ldots,x_n)$ be a noncentral multilinear polynomial over $C$, $b \in Q$, $F$ a $b$-generalized derivation of $R$ and $d$ is a nonzero derivation of $R$ such that
\[d([F(f(r)),f(r)]) = 0\]
for all $r = (r_1,\ldots,r_n) \in R^n$. Then one of the following holds:

(i) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;

(ii) there exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all $x \in R$ with $f(x_1,\ldots,x_n)^2$ is central valued in $R$.

**Proof.** By [11, Theorem 2.3], there exist a derivation $\delta : R \rightarrow Q$ and $a \in Q$ such that $F(x) = ax + b\delta(x)$ for all $x \in R$. If $d$ is inner derivation of $R$, then result follows by Lemma 3.1. Thus we assume that $d$ is outer derivation of $R$. By hypothesis $R$ satisfies
\[(12) \quad d([af(r_1,\ldots,r_n) + b\delta(f(r_1,\ldots,r_n)),f(r_1,\ldots,r_n)]) = 0.\]

Since any derivation of $R$ can be uniquely extended to a derivation of $Q$ (see [14, Lemma 2]), by [14] this differential identity is also satisfied by $Q$.

**Case-I:** Assume that $d$ and $\delta$ are $C$-dependent modulo inner derivations of $Q$, say $ad + \delta \beta = ad_q$, where $\alpha, \beta \in C$, $q \in Q$ and $ad_q(x) = [q,x]$ for all $x \in Q$.

**Subcase-i:** Let $\alpha \neq 0$.

Then $d(x) = \lambda \delta(x) + [c,x]$ for all $x \in Q$, where $\lambda = -\beta\alpha^{-1}$ and $c = \alpha^{-1}q$. Then $d$ can not be inner derivation of $Q$. From (12), we obtain
\[(13) \quad \lambda \delta([af(r) + b\delta(f(r)),f(r)]) + [c, [af(r) + b\delta(f(r)),f(r)]] = 0\]

that is,
\[
\lambda[af(r) + b\delta(f(r)),\delta(f(r))] \\
+ \lambda[\delta(a)f(r) + a\delta(f(r)) + \delta(b)\delta(f(r)) + b\delta^2(f(r)),f(r)] \\
+ [c,[af(r) + b\delta(f(r)),f(r)]] = 0
\]

for all $r = (r_1,\ldots,r_n) \in Q^n$. Let $f^\delta(r_1,\ldots,r_n)$ and $f^\delta(r_1,\ldots,r_n)$ be the polynomials obtained from $f(r_1,\ldots,r_n)$ replacing each coefficients $\alpha_\sigma$ with $\delta(\alpha_\sigma)$ and $\delta^2(\alpha_\sigma)$ respectively. Then we have
\[
\delta(f(r_1,\ldots,r_n)) = f^\delta(r_1,\ldots,r_n) + \sum_i f(r_1,\ldots,\delta(r_i),\ldots,r_n)
\]
and

\[ \delta^2(f(r_1, \ldots, r_n)) = f\delta^2(r_1, \ldots, r_n) + 2 \sum_i f(r_1, \ldots, \delta^2(r_i), \ldots, r_n) \]

\[ + \sum_{i} f(r_1, \ldots, \delta(r_i), \ldots, r_n) \]

\[ + \sum_{i \neq j} f(r_1, \ldots, \delta(r_i), \ldots, \delta(r_j), \ldots, r_n). \]

By applying Kharchenko’s Theorem [10] to (14), we can replace

\[ \delta(f(r_1, \ldots, r_n)) \]

with

\[ f\delta(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n) \]

and

\[ \delta^2(f(r_1, \ldots, r_n)) \]

with

\[ f\delta^2(r_1, \ldots, r_n) + 2 \sum_i f(r_1, \ldots, y_i, \ldots, r_n) \]

\[ + \sum_i f(r_1, \ldots, t_i, \ldots, r_n) + \sum_{i \neq j} f(r_1, \ldots, y_i, \ldots, y_j, \ldots, r_n) \]

in (14) and then \( Q \) satisfies blended component

\[ \lambda[b \sum_i f(r_1, \ldots, t_i, \ldots, r_n), f(r_1, \ldots, r_n)] = 0. \]

In particular, for \( t_2 = \cdots = t_n = 0 \) and \( t_1 = r_1 \), \( Q \) satisfies

\[ \lambda[bf(r_1, \ldots, r_n), f(r_1, \ldots, r_n)] = 0 \]

which is

\[ [\lambda b, f(r_1, \ldots, r_n)] f(r_1, \ldots, r_n) = 0. \]

By [7], it yields \( \lambda b \in C \).

Replacing \( t_i \) with \( [q, r_i] \) for some \( q \notin C \) in (15) and then using \( \lambda b \in C \), we have that \( Q \) satisfies

\[ [\lambda b, f(r_1, \ldots, r_n)] f(r_1, \ldots, r_n) = 0. \]

By [13, Theorem], this implies \( \lambda b \in C \). Since \( q \notin C \), we conclude that \( \lambda b = 0 \). This implies \( \lambda = 0 \) or \( b = 0 \). Both case leads to a contradiction.

Subcase-ii: Let \( \alpha = 0 \).

Then \( \delta(x) = [c, x] \) for all \( x \in Q \), where \( c = \beta^{-1}q \). From (12), we obtain

\[ d([a f(r) + b[c, f(r)], f(r)]) = 0 \]

that is

\[ [a f(r) + b[c, f(r)], d(f(r))] + [d(a) f(r) + ad(f(r)), f(r)] \]

\[ + [d(b)[c, f(r)] + b[d(c), f(r)] + b[c, d(f(r))], f(r)] = 0 \]

for all \( r = (r_1, \ldots, r_n) \in Q^n \).

Since

\[ d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n) \]
by Kharchenko’s Theorem [10], we can replace $d(f(r_1,\ldots,r_n))$ by $f^d(r_1,\ldots,r_n) + \sum_i f(r_i,\ldots,y_i,\ldots,r_n)$ in (20) and then $Q$ satisfies blended component

$$[af(r_1,\ldots,r_n) + b[c,f(r_1,\ldots,r_n)], \sum_i f(r_i,\ldots,y_i,\ldots,r_n)]$$

(21)

$$+ [a \sum_i f(r_i,\ldots,y_i,\ldots,r_n), f(r_1,\ldots,r_n)]$$

$$+ [b[c, \sum_i f(r_1,\ldots,y_i,\ldots,r_n)], f(r_1,\ldots,r_n)] = 0.$$  

In particular, for $y_1 = r_1$ and $y_2 = \ldots = y_n = 0$, we have

(22)  

$$2[af(r) + b[c, f(r)], f(r)] = 0$$  

for all $r = (r_1,\ldots,r_n) \in Q^n$. Since $\text{char}(R) \neq 2$, this can be written as

(23)  

$$[(a + bc)f(r) - bf(r)c, f(r)] = 0$$  

for all $r = (r_1,\ldots,r_n) \in Q^n$.

By Lemma 2.9, one of the following holds: (i) $a + bc, b, bc \in C$, that is $a,b, bc \in C$. In this case $F(x) = ax + b\delta(x) = ax + b[c,x] = ax$ for all $x \in R$, which is our conclusion (1). (ii) $b, a + 2bc \in C$ and $f(r_1,\ldots,r_n)^2$ is central valued. In this case $F(x) = ax + b\delta(x) = ax + b[c,x] = ax + [bc,x] = (a + bc)x - x(bc) = (a + 2bc)x - bcx - xbc$ for all $x \in R$. This gives conclusion (2). (iii) $c \in C$. In this case $F(x) = ax + b\delta(x) = ax + b[c,x] = ax$ for all $x \in R$ which is conclusion (1).

**Case-II:** Assume next that $d$ and $\delta$ are $C$-independent modulo inner derivations of $Q$.

From (12) we have

(24)  

$$[af(r) + b\delta(f(r)), d(f(r))]$$

$$+ ([d(a)f(r) + ad(f(r)) + d(b)\delta(f(r)) + b(d\delta)(f(r)), f(r)]) = 0$$

for all $r = (r_1,\ldots,r_n) \in Q^n$. By applying Kharchenko’s theorem [10] to (24), we can replace $d(f(r_1,\ldots,r_n))$ with $f^d(r_1,\ldots,r_n) + \sum_i f(r_i,\ldots,y_i,\ldots,r_n)$, $\delta(f(r_1,\ldots,r_n))$ with $f^\delta(r_1,\ldots,r_n) + \sum_i f(r_i,\ldots,s_i,\ldots,r_n)$ and $d\delta(f(r_1,\ldots,r_n))$ with

$$f^d(r_1,\ldots,r_n) + \sum_i f^\delta(r_1,\ldots,s_i,\ldots,r_n) + \sum_i f^d(r_1,\ldots,y_i,\ldots,r_n)$$

$$+ \sum_i f(r_1,\ldots,t_i,\ldots,r_n) + \sum_i f(r_1,\ldots,y_i,\ldots,s_j,\ldots,r_n)$$

in (24) and then $Q$ satisfies blended component

(25)  

$$[b \sum_i f(r_1,\ldots,t_i,\ldots,r_n), f(r_1,\ldots,r_n)] = 0.$$
Replacing $t_1 = r_1$ and $t_2 = \cdots = t_n = 0$ in (25), we have

$$[b, f(r_1, \ldots, r_n)]f(r_1, \ldots, r_n) = 0$$

for all $r_1, \ldots, r_n \in Q$. By [7], this yields $b \in C$. Since $b \neq 0$, again (25) yields

$$(26) \quad \sum f(r_1, \ldots, t_i, \ldots, r_n), f(r_1, \ldots, r_n) = 0$$

for all $r_1, \ldots, r_n \in Q$.

Let $q \notin C$ be an element of $Q$. Then replacing $t_i$ with $[q, r_i]$, we have that

$$\sum_{i=0}^{n} f(r_1, \ldots, [q, r_i], \ldots, r_n), f(r_1, \ldots, r_n) = 0$$

which gives,

$$[q, f(r_1, \ldots, r_n)]_2 = 0$$

for all $r_1, \ldots, r_n \in R$ implying $f(r_1, \ldots, r_n)$ is central-valued on $R$ [13, Theorem], a contradiction. □

References


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