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COMPARISON OF THE LEE AND HOMOGENOUS WEIGHTS OVER A FAMILY OF CHAIN RINGS

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Abstract. We compare the Lee and homogenous weights over the chain ring $S_{q,m} = \mathbb{F}_q[u]/(u^m)$ by computing the minimum distance of random codes for small values of n, q, m.

1. Introduction

A great deal of interest has been given to codes over finite rings in the last two decades. Particularly, the family of Frobenius rings, which proved to be the largest family of rings to study for coding theory, according to [14], have generated a lot of research. Moreover, MacWilliams identities hold for codes over these rings. Many different Frobenius rings were studied within that context for different reasons and motivations, leading to many different results. Among the often studied rings we can name $\mathbb{Z}_4$, $\mathbb{Z}_p^k$, Galois rings, finite chain rings, $\mathbb{F}_2 + v\mathbb{F}_2, \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2, R_k, R_{k,m}$ etc.

Different weight functions have also been of quite some interest in coding theory. Homogeneous weights were first introduced in [3] and later were considered within the confines of Frobenius rings in [8], [9], and [10]. In [16], the homogeneous weight for the ring family of R_k is finalized together with a Gray map, using first order Reed-Muller codes and using this map many optimal binary linear codes are found.

In [4], [6], [11], [12], [13] and [15], the Lee weights are defined for the rings together with suitable distance preserving gray maps that map the codes to the corresponding binary codes.

In this paper, we will introduce the basic materials about the codes over $S_{q,m}$ and the Lee weight over this ring with the gray map and proved some results. Also we study the homogeneous weight and related gray map on $S_{q,m}$. Finally, there is a comparison between these weights with statistical data.

In Section 2 contains the basic of the ring $S_{q,m}$ with linear codes over this ring. In Section 3 we introduce the Lee weight and the gray map over the ring

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$S_{q,m}$ with some examples. In Section 4 explains the homogeneous weight and the related gray map over the ring $S_{q,m}$. Finally, the last section contains a comparison between Lee weight and homogenous weight over $S_{q,m}$ with the statistical data.

2. The ring $S_{q,m}$

In this work, we will consider codes over a family of rings that can be described as $S_{q,m} := \mathbb{F}_q[u]/(u^m)$, $S_{q,m}$ also can be described as the $\mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{m-1}\mathbb{F}_q$ with $u^m = 0$, which will generalize some of the works that have been described above.

Note that we have $S_{q,1} = \mathbb{F}_q, S_{2,2} = \mathbb{F}_2 + u\mathbb{F}_2$ of size 4 where $\mathbb{F}_2 = \{0, 1\}$ and $S_{2,2} = \{0, 1, u, 1 + u\}, S_{3,2} = \mathbb{F}_3 + u\mathbb{F}_3$ of size 9 where $S_{3,2} = \{0, 1, 2, u, 1 + u, 2 + u, 1 + 2u, 2 + 2u, 2u\}$, and $S_{2,4} = S_4 = \mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2 + u^3\mathbb{F}_2$ of [11]. Here, q is a prime power and $S_{q,m}$ is a finite chain ring of q^m elements, i.e.,

$$0 = \langle u^m \rangle \subseteq \langle u^{m-1} \rangle \subseteq \cdots \subseteq \langle u \rangle \subseteq \langle 1 \rangle = S_{q,m},$$

with $uS_{q,m}$ as its unique maximal ideal.

An element $a_0 + a_1u + a_2u^2 + \cdots + a_{m-1}u^{m-1}$ in $S_{q,m}$ is a unit if and only if the coefficient a_0 not equal zero.

We denote the sets of units of $S_{q,m}$ by $\mathcal{U}(S_{q,m})$ and non-units by $\mathcal{D}(S_{q,m})$. It is clear that $|\mathcal{U}(S_{q,m})| = (q-1)q^{m-1}$ and $|\mathcal{D}(S_{q,m})| = q^{m-1}$.

Lemma 2.1. *For any $a \in S_{q,m}$, we have*:

$$au^{m-1} = \begin{cases} 0 & \textit{if } a \textit{ is a non-unit,} \\ iu^{m-1}, i \in \mathbb{F}_q^* & \textit{if } a \textit{ is a unit.} \end{cases}$$

2.1. Linear codes over the ring $S_{q,m}$

Definition 2.2. A linear code C of length n over the ring $S_{q,m}$ is an $S_{q,m}$-submodule of $(S_{q,m})^n$.

A non zero linear code C of length n over $S_{q,m}$ has a generator matrix which after a suitable permutation of the coordinates can be written in the form:

$$G = \begin{bmatrix} I_{k_1} & A_1 & A_2 & \cdots & A_m \\ 0 & uI_{k_2} & uB_1 & \cdots & uB_{m-1} \\ 0 & 0 & u^2I_{k_3} & \cdots & u^2C_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u^{m-1}I_{k_m} & u^{m-1}D \end{bmatrix},$$

where A_i, B_j, C_k, ... and D are all matrices over $S_{q,m}$.

A linear code that has a such generating matrix is said to be of type

$$(q^m)^{k_1}(q^{m-1})^{k_2} \cdots (q^2)^{k_{m-1}} q^{k_m},$$

and consequently has size $q^{mk_1+(m-1)k_2+\cdots+2k_{m-1}+k_m}$.

3. The Lee weight and the Gray map over the ring $S_{q,m}$

To define Lee weight and Gray maps for codes over $S_{q,m}$ with ω_L denoting the Lee weight for $S_{q,m}$ and ω_H denoting the Hamming weight, define:

For any $\bar{a} \in \mathbb{F}_q^m$, write $a = a_0 + a_1u + a_2u^2 + \cdots + a_{m-1}u^{m-1}$.

$$\omega_L(\bar{a}) = \omega_H(a_{m-1}, a_{m-1} + a_{m-2}, \ldots, a_{m-1} + a_{m-2} + \cdots + a_1 + a_0).$$

The definition of the weight immediately leads to a gray map from $S_{q,m}$ to $\mathbb{F}_q^m$ which can naturally be extended to $(S_{q,m})^n$.

$$\phi_L : ((\mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{m-1}\mathbb{F}_q), \text{Lee weight}) \to (\mathbb{F}_q^{mn}, \text{Hamming weight})$$

is defined as:

$$\begin{aligned} &\phi_L(a_0 + a_1u + a_2u^2 + \cdots + a_{m-1}u^{m-1}) \\ &= (a_{m-1} + a_{m-2} + \cdots + a_1 + a_0, \ldots, a_{m-1} + a_{m-2}, a_{m-1}), \end{aligned}$$

where $a_0, a_1, \ldots, a_{m-1} \in \mathbb{F}_q^m$.

It's clear that ϕ_L is a linear distance preserving isometry leading to the following theorem:

Theorem 3.1. *If C is linear code over $S_{q,m}$ of length n, size q^k and minimum Lee weight d, then $\phi_L(C)$ is a linear code over $\mathbb{F}_q$ with parameters $[mn, k, d]$. Moreover, C and $\phi_L(C)$ have the same weight enumerator.*

Recall that, the Lee weight of $S_{2,2} = \mathbb{F}_2 + u\mathbb{F}_2$ is defined by:

$$\phi_L(a + ub) = (b, a + b),$$

where a and $b \in \mathbb{F}_2^n$.

Similarly the Lee weight of $S_{3,2}$ is defined by:

$$\phi_L(a + ub) = (b, a + b),$$

where a and $b \in \mathbb{F}_3^n$.

For $\mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2 + u^3\mathbb{F}_2$, we can define the Lee weight by:

$$\phi_L(a + ub + u^2c + u^3d) = (d, d + c, d + b, d + c + b + a),$$

where a, b, c and $d \in \mathbb{F}_2^n$.

-If C is linear code of length 2 over $S_{3,2}$, then $\phi_L(C)$ is of parameters $[4, 0, 4]$, $[[4, 2, 1], 4, 1, 2]$, $[4, 2, 2]$, $[4, 1, 4]$, $[4, 4, 1]$, $[4, 3, 1]$ and $[4, 3, 2]$ with generator matrices respectively

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & u \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & u \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ u & u \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ u & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & u \\ 1 & 1 \end{bmatrix}.$$

- If C is linear code of length 3 over $S_{3,2}$, then $\phi_L(C)$ is of parameters $[6,0,6]$, $[6,2,1]$, $[6,1,2]$, $[6,2,2]$, $[6,1,4]$, $[6,2,3]$, $[6,2,4]$, $[6,4,2]$, $[6,1,6]$, $[6,6,1]$ with generator matrices respectively

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{u} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & u & u \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & u \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & u+2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 2u+2 \\ u & 2u+1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2u & u & u \end{bmatrix}.$$

- If C is linear code of length 4 over $S_{3,2}$, then $\phi_L(C)$ is of parameter $[8,1,8]$ with generator matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ u & u & u & 2u \end{bmatrix}.$$

m	q	n	$\phi_L(C)$
2	5	4	$[8,4,2]$ Linear code over $GF(5)$
2	9	3	$[6,4,2]$ Linear code over $GF(9)$
2	9	4	$[8,4,2]$ Linear code over $GF(9)$
2	11	3	$[6,4,2]$ Linear code over $GF(11)$
2	11	4	$[8,4,2]$ Linear code over $GF(11)$
3	7	4	$[12,9,2]$ Linear code over $GF(7)$
3	11	4	$[12,9,2]$ Linear code over $GF(11)$

4. The homogeneous weight and the related Gray map over the ring $S_{q,m}$

4.1. The homogeneous weight on Frobenius rings

Let R be a finite ring, by $J(R)$ we mean the Jacobson radical of R, the intersection of all maximal left ideals of R. The left socle of R is the sum of all simple left ideals of R and will be denoted by $soc({}_RR)$. Define $\hat{R} := \mathrm{Hom}(R, \mathbb{C}^\times)$ the character group of the additive group of R.

Theorem 4.1. *For a finite ring R the following are equivalent*:
(a) *$R/J(R)$ is isomorphic to $soc({}_RR)$ as left R-modules.*
(b) *$R/J(R)$ is isomorphic to $soc(R_R)$ as right R-modules.*
(c) *$soc({}_RR)$ is left principal.*
(d) *$soc(R_R)$ is right principal.*
(e) *$\hat{R}$ and R are isomorphic as left R-modules.*
(f) *$\hat{R}$ and R are isomorphic as right R-modules.*

Definition 4.2. A finite ring is called a *Frobenius ring* if it satisfies any (and hence all) of the equivalent statements in the previous theorem.

Note that for a finite Frobenius ring there exist characters χ and ψ such that

$$\hat{R} = \{{}^r\chi \mid r \in R\} = \{\psi^r \mid r \in R\}.$$

We call such χ a left-generating character and ψ a right generating character. It is worth mentioning that every left generating character is right generating at the same time. Furthermore, the kernel of a generating character does not contain any non-trivial left or right ideal of R.

We will be interested in the homogeneous weight and the related Gray map. Homogeneous weights were first introduced by Constantinescu and Heise [3].

Definition 4.3. A real valued function ω on the finite ring R is called a (left) homogeneous weight, if $\omega(0) = 0$ and the following is true.

(H1) For all $x, y \in R$, $Rx = Ry$ implies $\omega(x) = \omega(y)$.

(H2) There exists a real number γ such that

$$\sum_{y \in Rx} \omega(y) = \gamma\,|Rx| \text{ for all } x \in R\backslash\{0\}.$$

The number γ is the average value of ω on R, and from condition (H2) we can deduce that the average value of w is constant on every non-zero principal ideal of R.

Homogeneous weights for Frobenius rings can be described by using the Möbius function. For a finite poset P, consider the function $\mu : P \times P \to \mathbb{C}$ implicitly defined by $\mu(x, x) = 1$ and $\sum\limits_{y \le t \le x} \mu(t, x) = 0$ if $y < x$ and $\mu(y, x) = 0$ if y is not comparable to x. It is called the Möbius function on P and induces for arbitrary pairs of real-valued functions f, g on P the following equivalences, referred to as Möbius inversion

$$g(x) = \sum_{y \le x} f(y) \text{ for all } x \in P \Leftrightarrow f(x) = \sum_{y \le x} g(y)\mu(y, x).$$

Let R be a finite ring and μ be the Möbius function on the set $\{Rx \mid x \in R\}$ of its principal left ideals (partially ordered by inclusion). Further let $R^\times$ denote the set of units in R.

Theorem 4.4. *A real valued function w on the finite ring R is a homogeneous weight if and only if there exists a real number γ such that $w(x) = \gamma\left[1 - \frac{\mu(0,Rx)}{|R^\times x|}\right]$ for all $x \in R$.*

Honold [10] described the homogeneous weights on Frobenius rings in terms of generating characters.

Proposition 4.5. *Let R be a finite ring with generating character χ. Then every homogeneous weight on R is of the form*

$$w : R \to \mathbb{R},\ x \mapsto \gamma \left[1 - \frac{1}{|R^\times|} \sum_{u \in R^\times} \chi(xu) \right]. \tag{4.1}$$

By Property (H2), the average weight of a left principal ideal of R is γ. The following proposition shows that for any coset of either a left or a right ideal, the average weight is γ. This property is equivalent to R being Frobenius.

Proposition 4.6. *Let I be either a left or a right ideal of a finite Frobenius ring R, and let $y \in R$. Then $\sum_{r \in I+y} \omega(r) = \gamma\,|I|$.*

4.2. The homogeneous weight for $S_{q,m}$

We will find the homogeneous weight for the ring $S_{q,m}$. Let $T_r : \mathbb{F}_q \to \mathbb{F}_p$ be the trace of $\mathbb{F}_q$ to $\mathbb{F}_p$, then the generating character for the ring $S_{q,m}$ is defined as:

$$\chi(a_0 + a_1 u + \cdots + a_{m-1} u^{m-1}) = e^{\frac{2\pi i T_r(a_{m-1})}{p}},$$

where $a_i \in \mathbb{F}_q$, $i = 0, 1, \ldots, m-1$ and $q = p^r$.

Theorem 4.7. *The character χ defined above is a generating character for the ring $S_{q,m}$.*

Proof. Let a and $b \in S_{q,m}$ where $a = a_0 + a_1 u + \cdots + a_{m-1} u^{m-1}$ and $b = b_0 + b_1 u + \cdots + b_{m-1} u^{m-1}$. We have:

$$\begin{aligned}
\chi(a+b) &= e^{\frac{2\pi i (T_r(a_{m-1}) + T_r(b_{m-1}))}{p}} \\
&= e^{\frac{2\pi i T_r(a_{m-1}) + 2\pi i T_r(b_{m-1})}{p}} \\
&= e^{\frac{2\pi i T_r(a_{m-1})}{p}} \cdot e^{\frac{2\pi i T_r(b_{m-1})}{p}} \\
&= \chi(a) \cdot \chi(b).
\end{aligned}$$

This shows that χ is a character. To prove that χ is a generating character for the ring $S_{q,m}$, we need to show that χ is not trivial, when restricted to any non-zero ideal. Since $S_{q,m}$ has a unique minimal ideal which is contained in all the non-zero ideals, it is enough to show that χ restricted to the minimal ideal $\mathfrak{m} = \langle u^{m-1} \rangle$ is non-trivial. This is easy to verify, because $\chi(u^{m-1}) = e^{\frac{2\pi i}{p}} \neq 1$. □

Remark 4.8. For $S_{p,m}$ where p is prime, the character χ has the following form

$$\chi(a_0 + a_1 u + \cdots + a_{m-1} u^{m-1}) = e^{\frac{2\pi i a_{m-1}}{p}}.$$

Example 4.9. (1) For $S_{2,2} = \{0, 1, u, 1+u\}$, by previous remark we have:

$$\chi(a + bu) = e^{\frac{2\pi i b}{2}} = e^{\pi i b}.$$

Then $\chi(0) = 1$, $\chi(1) = 1$, $\chi(u) = \chi(1+u) = e^{\pi i} = -1$.

(2) For $S_{2,m} = \{a_0 + a_1u + \cdots + a_{m-1}u^{m-1} \,|\, a_i \in \mathbb{F}_2\}$:

$$\chi(a_0 + a_1u + \cdots + a_{m-1}u^{m-1}) = e^{\frac{2\pi i a_{m-1}}{2}} = e^{\pi i a_{m-1}} = \begin{cases} -1 & \text{if } a_{m-1} = 1, \\ 1 & \text{if } a_{m-1} = 0. \end{cases}$$

(3) For $S_{3,2} = \{a + ub \,|\, a \text{ and } b \in \mathbb{F}_3\} = \{0, 1, 2, u, 1 + u, 2u, 2 + u, 2 + 2u, 1 + 2u\}$, we have:

$$\chi(a + bu) = e^{\frac{2\pi i b}{3}}$$

then, $\chi(0) = \chi(1) = \chi(2) = 1$, $\chi(u) = \chi(1 + u) = \chi(2 + u) = e^{\frac{2\pi i}{3}}$, $\chi(2u) = \chi(1 + 2u) = \chi(2 + 2u) = e^{\frac{4\pi i}{3}}$.

(4) For $S_{3,m} = \{a_0 + a_1u + \cdots + a_{m-1}u^{m-1} \,|\, a_i \in \mathbb{F}_3\}$, we have:

$$\chi(a_0 + a_1u + \cdots + a_{m-1}u^{m-1}) = e^{\frac{2\pi i a_{m-1}}{3}} = \begin{cases} 1 & \text{if } a_{m-1} = 0, \\ \frac{-1}{2} + i\frac{\sqrt{3}}{2} & \text{if } a_{m-1} = 1, \\ \frac{-1}{2} - i\frac{\sqrt{3}}{2} & \text{if } a_{m-1} = 2. \end{cases}$$

(5) For $S_{4,2} = \{a + ub \,|\, a, b \in \mathbb{F}_4\} = \{0, 1, \omega, \bar{\omega}, u, 1 + u, 1 + \omega u, 1 + \bar{\omega}u, \omega + u, \omega + \omega u, \omega + \bar{\omega}u, \omega u, \bar{\omega}u, \bar{\omega} + \bar{\omega}u, \bar{\omega} + \omega u, \bar{\omega} + u\}$ then we have:

$$\chi(0) = \chi(1) = \chi(\omega) = \chi(\bar{\omega}) = e^{\pi i Tr(0)} = e^{\pi i(0)} = 1,$$

$$\chi(1 + u) = \chi(u) = \chi(\omega + u) = \chi(\bar{\omega} + u) = e^{\pi i Tr(1)} = e^{\pi i(0)} = 1,$$

$$\chi(1 + \omega u) = \chi(\bar{\omega} + \omega u) = \chi(\omega + \omega u) = \chi(\omega u) = e^{\pi i Tr(\omega)} = e^{\pi i(1)} = -1,$$

$$\chi(1 + \bar{\omega}u) = \chi(\bar{\omega}u) = \chi(\bar{\omega} + \bar{\omega}u) = \chi(\omega + \bar{\omega}u) = e^{\pi i Tr(\bar{\omega})} = e^{\pi i(1)} = -1.$$

The following lemma will be a key in proving the main theorem about the homogeneous weight on $S_{q,m}$:

Lemma 4.10. *Let x be any element in $S_{q,m}$ such that $x \neq 0$ and $x \neq iu^{m-1}$. Then*

$$\sum_{\alpha \in \mathcal{U}(S_{q,m})} \chi(\alpha x) = 0.$$

Proof. Since χ is a generating character, it is non-trivial when restricted any ideal, and thus we have,

$$\sum_{\alpha \in S_{q,m}} \chi(\alpha x) = 0. \tag{4.2}$$

But since $\alpha \in \mathcal{D}(S_{q,m})$ if and only if $i + \alpha \in \mathcal{U}(S_{q,m})$, and for $i \in \mathbb{F}_q^*$,

$$\mathcal{U}(S_{q,m}) = \cup_{\alpha \in \mathbb{F}_q^*}(\alpha + \mathcal{D}(S_{q,m}))$$

the above sum becomes

$$\begin{aligned} 0 &= \sum_{\beta \in \mathcal{D}(S_{q,m})} \chi(\beta x) + \sum_{\alpha \in \mathbb{F}_q^*} \sum_{\beta \in \mathcal{D}(S_{q,m})} \chi((\beta + \alpha)x) \\ &= \sum_{\beta \in \mathcal{D}(S_{q,m})} \chi(\beta x) + \sum_{\alpha \in \mathbb{F}_q^*} \chi(\alpha x) \sum_{\beta \in \mathcal{D}(S_{q,m})} \chi(\beta x) \end{aligned}$$

$$= \sum_{\beta\in\mathcal{D}(S_{q,m})} \chi(\beta x)[1+\sum_{\alpha\in\mathbb{F}_q^*}\chi(\alpha x)]$$
$$= \sum_{\beta\in\mathcal{D}(S_{q,m})} \chi(\beta x)\sum_{\alpha\in\mathbb{F}_q}\chi(\alpha x).$$

Let us label the sum:

$$F(x) = \sum_{\beta\in\mathcal{U}(S_{q,m})} \chi(\beta x).$$

As β runs through all the units of $S_{q,m}$, we can easily observe that $F(x) = F(\alpha x)$ for all $\alpha \in \mathcal{U}(S_{q,m})$. Recall that if β is a unit, then:

$$\{\alpha\beta : \alpha \in \mathcal{U}(S_{q,m})\} = \mathcal{U}(S_{q,m}).$$

So, the proof will be complete if we prove that $\sum_{\alpha\in\mathbb{F}_q}\chi(\beta x) \neq 0$ for at least one value of $\beta \in \mathcal{U}(S_{q,m})$. Now, we need to find a unit $\beta \in \mathcal{U}(S_{q,m})$ such that βx does not contain u^{m-1} in it. Let x be an element in $S_{q,m}$, we have two cases:

- If x is a unit, then $\{x\beta : \beta \in \mathcal{U}(S_{q,m})\} = \mathcal{U}(S_{q,m})$. Then there is a unit β such that $\beta x = 1$ and so $\sum_{\alpha\in\mathbb{F}_q}\chi(\beta x) \neq 0$.

- If x is a non-unit, then the coefficient a_0 equal zero. Put $x = a_i u^i + a_{i+1}u^{i+1} + \cdots + a_{m-1}u^{m-1}$ such that $a_i \neq 0$ and $1 \leq i < m-1$ because $x \neq 0$ and $x \neq iu^{m-1}$. Thus, $x = u^i(a_i + a_{i+1}u^i + \cdots + a_{m-1}u^{m-1-i}) \Rightarrow \exists$ a unit $\beta \in \mathcal{U}(S_{q,m})$ such that $\beta\cdot(a_i + a_{i+1}u^i + \cdots + a_{m-1}u^{m-1-i}) = 1$ and so $\beta x = u^i \Rightarrow \sum_{\alpha\in\mathbb{F}_q}\chi(\beta x) \neq 0$. □

We are now ready to describe the homogeneous weight for $S_{q,m}$:

Theorem 4.11. *The homogeneous weight on $S_{q,m}$ is found be*:

$$w_{\text{hom}}(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{q}{q-1}\gamma & \text{if } x = iu^{m-1}, \\ \gamma & \text{otherwise.} \end{cases}$$

Proof. Suppose $x = iu^{m-1}$, $i \neq 0$. Thus

$$\sum_{\alpha\in\mathcal{U}(S_{q,m})} \chi(\alpha i u^{m-1}) = q^{m-1}\sum_{\alpha\in\mathbb{F}_{q^*}}\chi(iu^{m-1}) = q^{m-1}\sum_{\alpha\in\mathbb{F}_{q^*}} e^{\frac{2\pi i tr(\alpha)}{p}}$$
$$= q^{m-1}(-1) = -q^{m-1}.$$

Hence, by Proposition 3.3, we have

$$w_{\text{hom}}(x) = \gamma\left[1 - \frac{1}{|\mathcal{U}(S_{q,m})|}\sum_{\alpha\in\mathcal{U}(S_{q,m})}\chi(\alpha x)\right]$$
$$= \gamma[1 - \frac{1}{(q-1)(q^{m-1})} - q^{m-1}]$$
$$= \gamma[1 + \frac{1}{q-1}]$$

$$= \gamma \frac{q}{q-1}.$$

If $x \neq 0$ and $x \neq iu^{m-1}$, then by Lemma 4.10, we have

$$\sum_{\alpha \in \mathcal{U}(S_{q,m})} \chi(\alpha x) = 0.$$

Thus we obtain

$$w_{\text{hom}}(x) = \gamma \left[1 - \frac{1}{|\mathcal{U}(S_{q,m})|} 0 \right] = \gamma. \qquad \square$$

The parameter γ is the average weight value. The case $\gamma = (q-1)q^{m-2}$ is used to given an isometry using a special type of Gray map [1]. Let ω_{hom} be a length q vector list of all the elements of $\mathbb{F}_q$, and let 1 be the one-vector. Define $b = 1 \otimes 1 \otimes \cdots \otimes 1$, where the product is over $m-1$ vectors. Let c_i be similar to b where the $i+1$ factor is replaced by ω_{hom} for $i = 0, \ldots, m-1$. These m vectors span a m-dimensional subspace of $\mathbb{F}_q^{q^{m-1}}$. For an element $a = a_0 + a_1 u + \cdots + a^{m-1}u^{m-1} \in S_{q,m}$, let $a^{(i)} = \mu(a_i)$ where μ is the natural projection from the field $\mathbb{F}_q$ to the base field F_p. The Gray map is defined then as:

$$\phi_{hom}(a_0 + a_1 u + \cdots + a_{m-1}u^{m-1}) = (a^{(0)}c_0 + a^{(1)}c_1 + \cdots + a^{(m-1)}c_{m-1}).$$

The image code is the generalized Reed-Muller code $GRM(1, m-1)$ over $\mathbb{F}_q$. Now the homogenous weight of $S_{q,m}$ can be written as [1]:

$$\omega_{\text{hom}}(x) = \begin{cases} 0 & \text{if } x = 0, \\ q^{m-1} & \text{if } x = iu^{m-1}, \\ (q-1)q^{m-2} & \text{otherwise.} \end{cases}$$

Example 4.12. The homogeneous weight on $S_{3,2}$ is given by the following:

$$\omega_{\text{hom}}(x) = \begin{cases} 0 & \text{if } x = 0, \\ 3 & \text{if } x = u \text{ or } x = 2u, \\ 2 & \text{otherwise.} \end{cases}$$

The Gray map from $S_{3,2}$ to $\mathbb{F}_3^3$ is $\phi_{hom}(1) = (0,1,2)$ and $\phi_{hom}(u) = (1,1,1)$. Then extend the map to all of $S_{3,2}$ by letting the map be $\mathbb{F}_3$-linear. Thus we would have

$$\phi_{hom}(a + bu) = a\phi_{hom}(1) + b\phi_{hom}(u) = (b, a+b, 2a+b).$$

- If C is linear code of length 2 over $S_{3,2}$, then $\phi_{hom}(C)$ is of parameters $[6,2,4], [6,1,6], [6,4,2]$ with generator matrices respectively

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2u+2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2u & u \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}.$$

- If C is linear code of length 3 over $S_{3,2}$, then $\phi_{hom}(C)$ is of parameters $[9,1,9]$, $[9,3,6]$, $[9,2,6]$ with generator matrices respectively

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u & u & u \end{bmatrix}, \begin{bmatrix} 0 & u & 2u \\ 2u+1 & u+2 & u+2 \\ u & 2u & 2u \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2u & 1 \end{bmatrix}.$$

- If C is linear code of length 4 over $S_{3,2}$, then $\phi_{hom}(C)$ is of parameters $[12,4,6],[12,2,9]$ with generator matrices respectively

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & u+2 & 2u+2 \\ 2u & u & 1 & u+2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & u \end{bmatrix}.$$

m	q	n	$\phi_{hom}(C)$
2	5	3	$[6,4,2]$ Linear code over $GF(5)$
2	7	4	$[8,4,2]$ Linear code over $GF(7)$
2	9	3	$[6,4,2]$ Linear code over $GF(9)$
2	9	4	$[8,4,2]$ Linear code over $GF(9)$
2	11	3	$[6,4,2]$ Linear code over $GF(11)$
2	11	4	$[8,4,2]$ Linear code over $GF(11)$
3	11	4	$[12,9,2]$ Linear code over $GF(11)$

5. A comparison between the two weights with statistical data

In this section, we could choose a few rings of manageable size from $S_{q,m}$ and construct in a systematic way many codes with respect to both weights. Then comparing the Gray images and see which ones are optimal or near-optimal with reference to [7], we could make a conclusion on which weight to be more advantageous. We could consider this problem from several angles. We could explain the strong and weak sides of each weight in comparison with the other. In doing so, we will construct codes using magma [2].

The gray map of the Lee weight ϕ_L is bijective since for each element $b \in \mathbb{F}_q^m$ there is an element in $S_{q,m}$ such that $\phi_L(b) = a$, but the gray map of the homogeneous weight ϕ_{hom} is not bijective.

5.1. Some theoretical comparison

Proposition 5.1. *Let μ be a canonical projection from the ring $S_{q,m}$ to $\mathbb{F}_q$*

$$\mu : S_{q,m} \to \mathbb{F}_q.$$

Then $\mu(a_0 + a_1u + \cdots + a_{m-1}u^{m-1}) = a_0$ module u.

A code $C \subset R^n$ over R is called a free code if C is a free R-module, that is C is isomorphic to R-module R^k for some k [5].

Definition 5.2. A free code over $S_{q,m}$ is a code which is free module.

Recall that if C is a free code of dimension k, then $\mu(C)$ is a q-ray code of dimension k.

Proposition 5.3. *If C is a free code with $d_H(\mu(C)) = d$, then $d_L(C) \leq dm$.*

Proof. Suppose that $\bar{x}$ is a codeword in $\mu(C)$ with weight d, $\exists d$ non zero in $\bar{x}$ and $\exists \bar{y} \in C$ such that $\mu(\bar{y}) = \bar{x}$ and $\bar{y}$ has d units coordinates, then $u^{m-1}\bar{y} \in C$ has d entries of the form iu^{m-1}, and the rest will be zero. Hence

$$\omega_L(u^{m-1}y) = m\omega_H(\mu(\bar{y})) = md.$$

Thus $d_L(C) \leq dm$. □

Proposition 5.4. *If C is a free code with $d_H(\mu(C)) = d$, then $d_{hom}(C) \leq dq^{m-1}$.*

Proof. Suppose that $\bar{x}$ is a codeword in $\mu(C)$ with weight d, $\exists$ d non zero in $\bar{x}$ and $\exists$ $\bar{y} \in C$ such that $\mu(\bar{y}) = \bar{x}$ and $\bar{y}$ has d units coordinates, then $u^{m-1}\bar{y} \in C$ has d entries of the form iu^{m-1}, and the rest will be zero. Hence

$$\omega_{hom}(u^{m-1}y) = q^{m-1}\omega_H(\mu(\bar{y})) = q^{m-1}d.$$

Thus $d_{hom}(C) \leq q^{m-1}d$. □

Definition 5.5. A code C over $S_{q,m}$ is said to be completely non-free code if $\mu(C) = 0$. In other word, there are no units in C.

Remark 5.6. A non-free code C is said to be that not free, but $\mu(C) \neq 0$, i.e., it has some units entries.

Bound on minimum distance.

Corollary 5.7. *Let C be a linear code over $S_{q,m}$ of length n and let $\phi_L(C)$ be a gray map $d_L(C)$ be a minimum Lee distance and $d_H(C)$ be a minimum Hamming distance, then we have*:

$$d_H(C) \leq d_L(C) \leq md_H(C).$$

Corollary 5.8. *Let C be a linear code over $S_{q,m}$ of length n and let $\phi_{Hom}(C)$ be a Gray map $d_{hom}(C)$ be a minimum Homogeneous distance and $d_H(C)$ be a minimum Hamming distance, then we have*:

$$(q-1)q^{m-2}d_H(C) \leq d_{Hom}(C) \leq q^{m-1}d_H(C).$$

5.2. Some statistical comparison

Take a code C over $S_{q,m}$ and take I_1, I_2 be information rate of $\phi_L(C)$, $\phi_{hom}(C)$ respectively, and d_1, d_2 be a minimum distance of $\phi_L(C), \phi_{hom}(C)$ respectively, then if we compare $\frac{d_2}{d_1}$ with $\frac{I_1}{I_2}$, we have:

(1) If $\frac{d_2}{d_1} > \frac{I_1}{I_2}$, we can say $\phi_{hom}(C)$ is better.
(2) If $\frac{d_2}{d_1} < \frac{I_1}{I_2}$, we can say $\phi_L(C)$ is better.

Example 5.9. - Let $\phi_L(C)$ be linear code over $\mathbb{F}_3$ of length 6 with minimum distance 2, $\phi_{hom}(C)$ be a linear code over of length 9 with minimum distance 3, and generator matrices respectively

$$\begin{bmatrix} 0 & 0 & u \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & u \\ 0 & 2 & u+2 \\ 2 & u & 2 \end{bmatrix}.$$

Then we have:

d_1	I_1	d_2	I_2	d_2I_2	d_1I_1
2	$\frac{5}{6}$	3	$\frac{5}{9}$	$\frac{15}{9}$	$\frac{10}{6}$

- Let $\phi_L(C)$ be linear code over $\mathbb{F}_3$ of length 3 with minimum distance 2, $\phi_{hom}(C)$ be a linear code over of length 9 with minimum distance 4, and generator matrices respectively

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

Then we have:

d_1	I_1	d_2	I_2	d_2I_2	d_1I_1
2	$\frac{1}{3}$	4	$\frac{2}{9}$	$\frac{8}{9}$	$\frac{2}{3}$

- Let $\phi_L(C)$ be linear code over $\mathbb{F}_3$ of length 2 with minimum distance 2, $\phi_{hom}(C)$ be a linear code over of length 3 with minimum distance 3, and generator matrices respectively

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ 0 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ 0 & 1 & 2 \end{bmatrix}.$$

Then we have:

d_1	I_1	d_2	I_2	d_2I_2	d_1I_1
2	$\frac{1}{2}$	3	$\frac{1}{3}$	$\frac{3}{3}=1$	$\frac{2}{2}=1$

By using magma program to count the number of homogenous weight that is better (i.e., the number of d_2I_2) and similarly for Lee weight. For example $S_{3,2}$ of length 2 we count 5328 time for d_2I_2 and nothing for d_1I_1 which mean the homogenous weight is better.

MDS code. Because the homogeneous weight is divisible and dimension n is bounded then its less likely to get MDS codes. With Lee weight there are MDS codes.

A non-trivial MDS code obtained from Lee weight which can not be obtained from homogeneous weight. If G is a generating matrix of a linear code C defined

as:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & \vdots & \vdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & u \end{bmatrix}.$$

Then a linear code of $\phi_L(C)$ is $[mn, mn-1, 2]$ which is MDS code.

Upper bound for the information rate of the homogeneous image.

If

$$\omega_{\text{hom}}(x) = \begin{cases} 0 & \text{if } x = 0, \\ q^{m-1} & \text{if } x = iu^{m-1}, \\ (q-1)q^{m-2} & \text{otherwise,} \end{cases}$$

and if C is linear code of length n and $\phi_{hom}(C)$ be a gray map of size $q^{m-1}n$, then the size of a code C length n over $S_{q,m}$ can be at most q^{mn}.

Proposition 5.10. *If a C is a linear code over $S_{q,m}$ of length n, then*

$$\dim(\phi_{hom}(C)) \leq mn.$$

Corollary 5.11. *If C is a linear code over $S_{q,m}$ of length n, and if $IR(\phi_{hom}(C))$ denotes the information rate of $\phi_{Hom}(C)$, then we have*:

$$IR(\phi_{Hom}(C)) \leq \frac{mn}{q^{m-1}n} = \frac{m}{q^{m-1}}.$$

Example 5.12.
(1) If C is a linear code over $\mathbb{F}_{25} + u\mathbb{F}_{25}$, $q = 25$, $m = 2$, then we have: $IR(\phi_{hom}(C)) \leq \frac{2}{25} < 1$ which is very small.
(2) If C is a linear code of length n over $\mathbb{F}_{25} + u\mathbb{F}_{25}$, $q = 25$, $m = 2$, then we have: $\dim(\phi_{hom}(C)) \leq 2n$ while the length of $\phi_{hom}(C) = 25n$. In special case, if $n = 2$, then we can get: $[50, 1 \leq \dim \leq 4, d \geq 20]$.

A special case. When $q = 2^m$ (characteristic is 2), then $\phi_{hom}(C)$ is a binary code. when $q = 2$, we will have binary code as the images $\phi_{hom}(C)$, where $S_{2,m} = \mathbb{F}_2 + u\mathbb{F}_2 + \cdots + u^{m-1}\mathbb{F}_2$ and

$$\omega_{hom}(x) = \begin{cases} 0 & \text{if } x = 0, \\ 2^{m-1} & \text{if } x \in \langle u^{m-1} \rangle - \{0\}, \\ 2^{m-2} & \text{otherwise.} \end{cases}$$

If C is any linear code over $S_{2,m}$, then $\phi_{hom}(C)$ is a binary code of length $2^{m-1}n$ and is divisible by 2^{m-2}.

Proposition 5.13. *If C is a binary code, which is divisible by 4, then C is self-orthogonal.*

Corollary 5.14. *If C is any linear code over $S_{2,m}$, with $m \geq 4$, then $\phi_{hom}(C)$ is a binary self-orthogonal code.*

Proposition 5.15. *If a ternary linear code is divisible by* 3*, then it is self-orthogonal.*

When $q = 3$, $S_{3,m} = \mathbb{F}_3 + u\mathbb{F}_3 + \cdots + u^{m-1}\mathbb{F}_3$ and

$$\omega_{hom}(x) = \begin{cases} 0 & \text{if } x = 0, \\ 3^{m-1} & \text{if } x \in \langle u^{m-1} \rangle - \{0\}, \\ 2.3^{m-2} & \text{otherwise.} \end{cases}$$

Then we have $\phi_{hom}(C)$ is divisible by 3^{m-2}.

Corollary 5.16. *If C is any linear code over $S_{3,m}$ with $m = 3$, then $\phi_{hom}(C)$ is a ternary self-orthogonal code.*

6. Conclusion

We get all dimensions over ϕ_L, but not all the dimensions in ϕ_{hom}.

We get all codes with all types of information rate over ϕ_L, but with ϕ_{hom} the information rate is less than.

We get codes of length of weights kinds over ϕ_L, but in ϕ_{hom} we get a divisible codes.

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