REAL HYPERSURFACES WITH MIAO-TAM CRITICAL METRICS OF COMPLEX SPACE FORMS

XIAMIN CHEN

ABSTRACT. Let $M$ be a real hypersurface of a complex space form with constant curvature $c$. In this paper, we study the hypersurface $M$ admitting Miao-Tam critical metric, i.e., the induced metric $g$ on $M$ satisfies the equation: $-(\Delta g)\lambda + \nabla^2 g\lambda - \lambda \text{Ric} = g$, where $\lambda$ is a smooth function on $M$. At first, for the case where $M$ is Hopf, $c = 0$ and $c \neq 0$ are considered respectively. For the non-Hopf case, we prove that the ruled real hypersurfaces of non-flat complex space forms do not admit Miao-Tam critical metrics. Finally, it is proved that a compact hypersurface of a complex Euclidean space admitting Miao-Tam critical metric with $\lambda > 0$ or $\lambda < 0$ is a sphere and a compact hypersurface of a non-flat complex space form does not exist such a critical metric.

1. Introduction

Recall that on a compact Riemannian manifold $(M^n, g)$, $n > 2$ with a smooth boundary $\partial M$ the metric $g$ is referred as Miao-Tam critical metric if there exists a smooth function $\lambda : M^n \to \mathbb{R}$ such that

\begin{equation}
-(\Delta_g)\lambda + \nabla^2 g\lambda - \lambda \text{Ric} = g
\end{equation}

on $M$ and $\lambda = 0$ on $\partial M$, where $\Delta_g, \nabla^2 g\lambda$ are the Laplacian, Hessian operator with respect to the metric $g$ and $\text{Ric}$ is the $(0, 2)$ Ricci tensor of $g$. The function $\lambda$ is known as the potential function. The equation (1) is called as Miao-Tam equation. Applying this equation, Miao-Tam in [12] classified Einstein and conformally flat Riemannian manifolds. In particular, they proved that any Riemannian metric $g$ satisfying the equation (1) must have constant scalar curvature. Recently, Patra-Ghosh studied the Miao-Tam equation on certain class of odd dimensional Riemannian manifolds, namely contact metric
manifolds (see [15, 16]). It was proved that a complete $K$-contact metric satisfying the Miao-Tam equation is isometric to a unit sphere. Wang-Wang [18] also considered an almost Kenmotsu manifold with Miao-Tam critical metric.

An $n$-dimensional complex space form is an $n$-dimensional Kähler manifold with constant sectional curvature $c$. A complete and simple connected complex space form is complex analytically isometric to a complex projective space $\mathbb{C}P^n$ if $c > 0$, a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$, a complex Euclidean space $\mathbb{C}^n$ if $c = 0$. The complex projective and complex hyperbolic spaces are called non-flat complex space forms and denoted by $\tilde{M}^n(c)$. Let $M$ be a real hypersurface of a complex space form, then there exists an almost contact structure $(\phi, \eta, \xi, g)$ on $M$ induced from the complex space form. In particular, if $\xi$ is an eigenvector of shape operator $A$, then $M$ is called a Hopf hypersurface.

Since there are no Einstein real hypersurfaces in non-flat complex space forms ([3, 13]), Cho and Kimura [4, 5] considered a generalization of Einstein metric, called Ricci soliton, which satisfies

$$\frac{1}{2} \mathcal{L}_V g + \text{Ric} - \rho g = 0,$$

where $V$ and $\rho$ are the potential vector field and some constant on $M$, respectively. They proved that a compact contact-type hypersurface with a Ricci soliton in $\mathbb{C}^n$ is a sphere and a compact Hopf hypersurface in a non-flat complex space form does not admit a Ricci soliton.

From the Miao-Tam equation (1), we remark that the Miao-Tam critical metric can also be viewed as a generalization of the Einstein metric since the critical metric will become an Einstein metric if the potential function $\lambda$ is constant. Thus the above results intrigue us to study the real hypersurfaces admitting Miao-Tam critical metrics of complex space forms. In this article, we mainly study the Hopf hypersurfaces in complex space forms as well as a class of non-Hopf hypersurfaces of non-flat complex space forms. For a compact real hypersurface with Miao-Tam critical metric, we also get a result.

This paper is organized as follows: In Section 2 we recall some basic concepts and related results. In Section 3, we consider respectively the Hopf hypersurfaces with Miao-Tam critical metrics of non-flat complex space forms and complex Euclidean spaces, and one class of non-Hopf hypersurfaces of non-flat complex space forms is considered in Section 4. In the last section we will prove the result of compact real hypersurfaces with Miao-Tam critical metrics.

## 2. Some basic concepts and related results

Let $(\tilde{M}^n, \tilde{g})$ be a complex $n$-dimensional Kähler manifold and $M$ be an immersed, without boundary, real hypersurface of $\tilde{M}^n$ with the induced metric $g$. Denote by $J$ the complex structure on $\tilde{M}^n$. There exists a local defined unit normal vector field $N$ on $M$ and we write $\xi := -JN$ by the structure vector field of $M$. An induced one-form $\eta$ is defined by $\eta(\cdot) = \tilde{g}(J\cdot, N)$, which
is dual to \( \xi \). For any vector field \( X \) on \( M \) the tangent part of \( JX \) is denoted by \( \phi X = JX - \eta(X)N \). Moreover, the following identities hold:

\[
\begin{align*}
(2) & \quad \phi^2 = -Id + \eta \otimes \xi, \quad \eta \circ \phi = 0, \quad \phi \circ \xi = 0, \quad \eta(\xi) = 1, \\
(3) & \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \\
(4) & \quad g(X, \xi) = \eta(X),
\end{align*}
\]

where \( X, Y \in \mathfrak{X}(M) \). By (2)-(4), we know that \((\phi, \eta, \xi, g)\) is an almost contact metric structure on \( M \).

Denote by \( \nabla, A \) the induced Riemannian connection and the shape operator on \( M \), respectively. Then the Gauss and Weingarten formulas are given by

\[
\begin{align*}
(5) & \quad \tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX, \\
(6) & \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX.
\end{align*}
\]

In particular, \( M \) is said to be a \textit{Hopf hypersurface} if the structure vector field \( \xi \) is an eigenvector of \( A \).

From now on we always assume that the sectional curvature of \( \tilde{M}^n \) is constant \( c \). When \( c = 0 \), \( \tilde{M}^n \) is complex Euclidean space \( \mathbb{C}^n \). When \( c \neq 0 \), \( \tilde{M}^n \) is a non-flat complex space form, denoted by \( \tilde{M}^n(c) \), then from (5), we know that the curvature tensor \( R \) of \( M \) is given by

\[
R(X, Y)Z = \frac{c}{4} \left( g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y + 2g(X, \phi Y)\phi Z \right) + g(A Y, Z)AX - g(A X, Z)AY,
\]

and the shape operator \( A \) satisfies

\[
(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4} \left( \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \right)
\]

for any vector fields \( X, Y, Z \) on \( M \). From (7), we get the Ricci tensor \( Q \) of type \((1, 1)\):

\[
(9) \quad QX = \frac{c}{4} \left( (2n + 1)X - 3\eta(X)\xi \right) + h AX - A^2 X,
\]

where \( h \) denotes the mean curvature of \( M \) (i.e., \( h = \text{trace}(A) \)). We denote by \( S \) the scalar curvature of \( M \), i.e., \( S = \text{trace}(Q) \).

If \( M \) is a Hopf hypersurface of \( \tilde{M}^n(c) \), \( A\xi = \alpha \xi \), where \( \alpha = g(A \xi, \xi) \). Due to [14, Theorem 2.1], \( \alpha \) is constant. Remark that when \( c = 0 \), \( \alpha \) is also constant (see the proof of [5, Lemma 1]). Using the equation (8), we obtain

\[
(10) \quad (\nabla_X A)\xi = \alpha \phi AX - A \phi AX + \frac{c}{4} \phi X
\]
for any vector field $X$. Since $\nabla_\xi A$ is self-adjoint, by taking the anti-symmetry part of (10), we get the relation:

\begin{equation}
2A\phi AX - \frac{c}{2} \phi X = \alpha (\phi A + A\phi) X.
\end{equation}

As the tangent bundle $TM$ can be decomposed as $TM = \mathbb{R} \xi \oplus \mathcal{D}$, where $\mathcal{D} = \{ X \in TM : X \perp \xi \}$, the condition $\mathcal{A} \xi = \alpha \xi$ implies $A\mathcal{D} \subset \mathcal{D}$, thus we can pick up $X \in \mathcal{D}$ such that $AX = fX$ for some function $f$ on $M$. Then from (11) we obtain

\begin{equation}
(2f - \alpha) A\phi X = \left( f \alpha + \frac{c}{2} \right) \phi X.
\end{equation}

If $2f = \alpha$, then $c = -4f^2$, which shows that $M$ is locally congruent to a horosphere in $CH^n$ (see [2]).

Next we recall an important lemma for a Riemannian manifold satisfying Miao-Tam equation (1).

**Lemma 2.1** ([7]). Let a Riemannian manifold $(M^n, g)$ satisfy the Miao-Tam equation. Then the curvature tensor $R$ can be expressed as

\begin{equation}
R(X,Y) \nabla \lambda = X(\lambda)QY - Y(\lambda)QX + \lambda \left[ (\nabla_X Q)Y - (\nabla_Y Q)X \right] + X(\beta)Y - Y(\beta)X
\end{equation}

for any vector fields $X, Y$ on $M$ and $\beta = -\frac{3A+1}{n-1}$.

Applying this lemma we obtain:

**Lemma 2.2.** For a Hopf real hypersurface $M^{2n-1}$ with Miao-Tam critical metric of a complex space form, the following equation holds:

\begin{equation}
\lambda \left[ X(h) - \xi(h) \eta(X) \right] = \mu \left( \xi(\lambda) \eta(X) - X(\lambda) \right) + \alpha^2 \xi(\lambda) \eta(X) - \alpha AX(\lambda),
\end{equation}

where $\mu = \frac{\xi}{4}(2n-1) + \alpha h - \alpha^2 - \frac{S}{2n-2}$.

**Proof.** Replacing $Z$ in (7) by $\nabla \lambda$, we have

\begin{equation}
R(X,Y) \nabla \lambda = \frac{c}{4} \left( Y(\lambda)X - X(\lambda)Y + \phi Y(\lambda)\phi X - \phi X(\lambda)\phi Y \right) + 2g(X, \phi Y)\phi \nabla \lambda \right) + AY(\lambda)AX - AX(\lambda)AY.
\end{equation}

By combining with Lemma 2.1, we get

\begin{equation}
X(\lambda)QY - Y(\lambda)QX + \lambda \left[ (\nabla_X Q)Y - (\nabla_Y Q)X \right]
\end{equation}

\begin{equation}
= \left( \frac{c}{4} - \frac{S}{2n-2} \right) \left( Y(\lambda)X - X(\lambda)Y \right) + \frac{c}{4} \left( \phi Y(\lambda)\phi X - \phi X(\lambda)\phi Y \right)
\end{equation}

\begin{equation}
+ 2g(X, \phi Y)\phi \nabla \lambda \right) + AY(\lambda)AX - AX(\lambda)AY.
\end{equation}

Now making use of (9), for any vector fields $X, Y$ we first compute

\begin{equation}
(\nabla_Y Q)X = \frac{c}{4} \left( -3(\nabla_Y \eta)(X)\xi - 3\eta(X)\nabla_Y \xi \right) + Y(h)AX + h(\nabla_Y A)X.
\end{equation}
By (8), we thus obtain
\[(\nabla_X Q)Y - (\nabla_Y Q)X\]
\[= -\frac{3c}{4} \{g(\phi AX + A\phi X, Y)\xi + \eta(Y)\phi AX - \eta(X)\phi AY\} + \frac{hc}{4} (\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi)\]

Therefore, taking the product of (15) with \(\xi\) and using (16), we conclude
\[-\frac{3c}{4} g(\phi AX + A\phi X, Y)\xi + \eta(Y)\phi AX - \eta(X)\phi AY\]
\[-g(\nabla_X A)AY + (\nabla_Y A)AX, \xi\]
\[-\frac{hc}{4} (\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi)\]

Substituting this into (17) we arrive at
\[-\frac{c + 2\alpha^2}{4} (\phi AX + A\phi X) + \alpha X(h)\xi - \alpha Y(h)\eta(X)\]
\[+ 2hc - \frac{\alpha c^2}{4} \phi X\]
\[= \frac{\mu}{\lambda} \left(\eta(X)\nabla \lambda - \lambda(\lambda)\xi\right) + \frac{\alpha}{\lambda} \eta(X)AX - \frac{\alpha}{\lambda} AX(\lambda)\xi.\]

Finally, taking an inner product of (18) with \(\xi\) gives (13). \(\square\)

3. Hopf real hypersurfaces of complex space forms

First of all, we assume \(c \neq 0\), i.e., \(M^{2n-1}\) is a Hopf real hypersurface of non-flat complex space form \(\tilde{M}^n(\alpha)\). We first consider \(\alpha = 0\), i.e., \(A\xi = 0\), then the relation (13) yields
\[-\frac{S}{2n-2} + \frac{c}{4} (2n-1)\left(\xi(\lambda)\xi - \nabla \lambda\right) = 0.\]
If $-\frac{S}{2n-2} + \frac{\xi}{2}(2n-1) = 0$, i.e., $S = \frac{1}{2}c(n-1)(2n-1)$, then from (18) we find

$$c \frac{\phi A X + A \phi X}{4} = 0,$$

which yields $\phi A X + A \phi X = 0$ for all vector field $X$. This is contradictory with [14, Corollary 2.12]. Thus $S \neq \frac{\xi}{2}(n-1)(2n-1)$, and it follows from (19) that $\nabla \lambda = \xi(\lambda)$. Differentiating this along $X$ gives

$$\nabla_X \nabla \lambda = X(\xi(\lambda))\xi + \xi(\lambda)\phi A X.$$

On the other hand, from (1) we can obtain

$$\nabla_X \nabla \lambda = (1 + \Delta \lambda)X + \lambda QX.$$

Comparing (21) and (22), we have

$$X(\xi(\lambda))\xi + \xi(\lambda)\phi A X = (1 + \Delta \lambda)X + \lambda QX.$$

Moreover, by (9), putting $X = \xi$ gives

$$\xi(\xi(\lambda)) = 1 + \Delta \lambda + \frac{\lambda c}{2}(n-1).$$

Choose a local orthonormal frame $\{e_i\}$ such that $e_{2n-1} = \xi$ and $e_{n-1+i} = \phi e_i$ for $i = 1, \ldots, n-1$. Using the frame to contract over $X$ in (23), we also derive that

$$\xi(\xi(\lambda)) = (1 + \Delta \lambda)(2n-1) + \lambda S.$$

Comparing with (24), we find

$$(2n-2)(1 + \Delta \lambda) + \lambda S = \frac{\lambda c}{2}(n-1).$$

Furthermore, by taking the trace of Miao-Tam equation (1), we get

$$(2 - 2n)\Delta \lambda - \lambda S = 2n - 1,$$

which, together with (25), yields

$$\frac{\lambda c}{2}(n-1) + 1 = 0.$$

This shows that $\lambda$ is constant. Thus $M$ is Einstein, but as is well-known that there are no Einstein hypersurfaces in a non-flat complex space form as in introduction, hence we immediately obtain:

**Proposition 3.1.** A real hypersurface with $A \xi = 0$ of a non-flat complex space form does not admit Miao-Tam critical metric.

Next we consider the case where $\alpha \neq 0$. If for every $X \in \mathcal{D}$ such that $AX = \frac{\alpha}{2}X$, as before we know that $M$ is locally congruent a horosphere in $\mathbb{C}H^n$ and $c = -\alpha^2$. Moreover, the mean curvature $h = \alpha$ is constant. Then from (18) we can obtain $nc = -\frac{\alpha^2}{2}$. This implies $2n = 1$. It is impossible.
Now choose $X \in \mathcal{D}$ such that $AX = fX$ with $f \neq \frac{2}{\alpha}$, so from (18) we have
\[\frac{c + 2\alpha^2}{4}(f \phi X + \tilde{f} \phi X) + \alpha X(h)\xi + \frac{\alpha}{2}(f^2 \phi X + \tilde{f}^2 \phi X) - \frac{2hc - ac}{4}\phi X\]
\[= -\frac{\mu}{\lambda}X(\lambda)\xi - \frac{\alpha}{\lambda}AX(\lambda)\xi.\]

Here we have used $A\phi X = \tilde{f} \phi X$ with $\tilde{f} = \frac{f + \alpha}{2\alpha - \alpha}$ followed from (12). Since $\phi X \in \mathcal{D}$, we further derive
\[\frac{c + 2\alpha^2}{8}(f + \tilde{f}) + 2\alpha(f^2 + \tilde{f}^2) - (2hc - ac) = 0.\]
Moreover, inserting $\tilde{f} = \frac{f + \alpha}{2\alpha - \alpha}$ into the equation (28), we have
\[8\alpha f^4 - 4(c + 4\alpha^2)f^3 + (6ac + 8\alpha^3 - 8hc)f^2 \]
\[+ (8hca - 4\alpha^2c - c^2)f + \alpha c^2 + 2\alpha^3c - 2hca^2 = 0.\]

Now we denote the roots of the polynomial by $f_1, f_2, f_3, f_4$, then from the relation between the roots and coefficients we obtain
\[\begin{cases}
f_1 + f_2 + f_3 + f_4 = \frac{c + 4\alpha^2}{8}, \\
f_1f_2 + f_1f_3 + f_1f_4 + f_2f_3 + f_2f_4 + f_3f_4 = \frac{3\alpha c + 4\alpha^3 - 4hc}{4\alpha}, \\
f_1^2f_2f_3 + f_1f_2f_4 + f_2f_3f_4 + f_3f_4 = -\frac{8hca - 4\alpha^2c - c^2}{8\alpha}, \\
f_1f_2f_3f_4 = \frac{c^2 + 2\alpha^2c - 2hca}{8}. \end{cases}\]

As the proof of [5, Lemma 4.2], we can also get the following.

**Lemma 3.2.** The mean curvature $h$ is constant.

Hence from (13) we conclude
\[AX\nabla\lambda = \frac{\mu}{\alpha}\phi^2\nabla\lambda + \alpha\xi(\lambda)\xi.\]
By taking the inner product with the principal vector $X \in \mathcal{D}$, we obtain
\[(f + \frac{\mu}{\alpha})X(\lambda) = 0.\]
If $X(\lambda) = 0$ for all $X \in \mathcal{D}$, then $\nabla\lambda = \xi(\lambda)\xi$. As the proof of Proposition 3.1, we see that $M$ is Einstein, which is impossible.

If $X(\lambda) \neq 0$ for all $X \in \mathcal{D}$, then $f + \frac{\mu}{\alpha} = 0$, i.e., $M$ has only two distinct constant principal curvatures $\alpha, -\frac{\mu}{\alpha}$. Further, we see from (12) that
\[2f^2 - 2\alpha f - \frac{c}{2} = 0.\]
Since the hypersurface $M$ has two distinct constant principle curvatures: $\alpha$ of multiplicity 1 and $f$ of multiplicity $2n - 2$, it is easy to get that the mean curvature $h = \alpha + (2n - 2)f$ and the scalar curvature $S = c(n^2 - 1) + 2\alpha(2n - 2)f + (2n - 2)(2n - 3)f^2$. Thus
\[\mu = -\frac{3c}{4} + (2n - 4)\alpha f - (2n - 3)f^2.\]
Inserting this into the relation \( f + \frac{4c}{r} = 0 \), we obtain

\[
(2n - 3)(\alpha f - f^2) = \frac{3c}{4}.
\]

Combining (31) with (32), we find \( nc = 0 \), which is a contradiction.

If \( X(\lambda) \neq 0 \) for some principle vector \( X \in \mathcal{D} \), and without loss general, we suppose \( e_1(\lambda) \neq 0 \), then \( Ae_1 = -\frac{4c}{\alpha}e_1 \) and \( A\phi e_1 = \frac{\alpha u - \frac{4c}{\alpha}}{2u + \alpha^2} \phi e_1 \).

Notice that if the hypersurface \( M \) of \( \mathbb{C}H^n \) has constant principal curvatures, the classification is as follows:

**Theorem 3.3** ([2]). Let \( M \) be a Hopf real hypersurface in \( \mathbb{C}H^n (n \geq 2) \) with constant principal curvatures. Then \( M \) is locally congruent to the following:

1. \( A_2 \): Tubes around a totally geodesic \( \mathbb{C}H^{n-1} \subset \mathbb{C}H^n \).
2. \( B \): Tubes of radius \( r \) around a totally geodesic real hyperbolic space \( \mathbb{R}H^n \subset \mathbb{C}H^n \).
3. \( N \): Horospheres in \( \mathbb{C}H^n \).

Since the horospheres have two distinct principal curvatures, it is impossible.

By Theorems 3.9 and 3.12 in [14], the Type \( A_2, B \) hypersurfaces have three distinct principal curvatures: \( \lambda_1 = \frac{1}{2} \tanh(u), \lambda_2 = \frac{1}{2} \coth(u) \) and \( \alpha = \frac{3}{2} \tan(2u) \). Then \( h = \alpha + (n-1)(\lambda_1 + \lambda_2) = \alpha + \frac{(2n-1)}{\coth(2u)} \). On the other hand, from Corollary 2.3(ii) in [14], we also have \( \frac{1}{\tau^2} = \frac{\lambda_1 + \lambda_2}{2} \alpha + \frac{c}{\alpha} \), i.e., \( c = -\frac{4}{\tau^2} \). This implies from the last relation in (30) that

\[
\frac{1}{\tau^4} = \frac{c^2 + 2\alpha^2 c - 2\alpha c}{8} = \frac{4n - 2}{\tau^4}.
\]

Thus \( n = \frac{3}{4} \), that is impossible.

For the case of \( \mathbb{C}P^n \), the classification is as follow:

**Theorem 3.4** ([9, 12]). Let \( M \) be a Hopf hypersurface in \( \mathbb{C}P^n (n \geq 2) \) with constant principal curvatures. Then \( M \) is an open part of

1. \( A_2 \): a tube over a totally geodesic complex projective space \( \mathbb{C}P^k \) of radius \( \frac{\tau}{\sqrt{r}} \) for \( 0 \leq k \leq n - 1 \), where \( r = \frac{\tau}{\sqrt{c}} \), or
2. \( B \): a tube over a complex quadric \( Q_{n-1} \subset \mathbb{R}P^n \), or
3. \( C \): a tube around the Segre embedding of \( \mathbb{C}P^1 \times \mathbb{C}P^k \) into \( \mathbb{C}P^{2k+1} \) for some \( k \leq 2 \), or
4. \( D \): a tube around the Plücker embedding into \( \mathbb{C}P^9 \) of the complex Grassmann manifold \( G_2(\mathbb{C}^5) \) of complex 2-planes in \( \mathbb{C}^5 \), or
5. \( E \): a tube around the half spin embedding into \( \mathbb{C}P^{15} \) of the Hermitian symmetric space \( SO(10) = U(5) \).

The Type \( A_2 \) and \( B \) hypersurfaces have three distinct principal curvatures: \( \lambda_1 = -\frac{1}{2} \cot(u), \lambda_2 = \frac{1}{2} \tan(u), \alpha = \frac{3}{2} \tan(2u) \) (see [14, Theorems 3.14 and 3.15]). From the first relation of (30), we have

\[
\lambda_1 + \lambda_2 = \frac{c + 4\alpha^2}{4\alpha} \Rightarrow -\frac{16}{\tau^2} = c + 4\alpha^2.
\]
It gives a contradiction since $c > 0$.

For the Type $C, D$ and $E$ hypersurfaces, they have five distinct principal curvatures (see [14, Theorems 3.16, 3.17, and 3.18]). We compute

$$\frac{1}{r} \left( -\cot(u) + \tan(u) + \cot\left(\frac{\pi}{4} - u\right) + \cot\left(\frac{3\pi}{4} - u\right) \right) = \frac{2}{r} (1 + \cot^2(2u)).$$

Thus the first relation of (30) implies

$$-\frac{24}{r^2} \cot^2(2u) = c + \frac{8}{r^2}.$$

It is impossible since $c > 0$. So the hypersurfaces of Type $C, D, E$ do not admit Miao-Tam critical metrics.

Summarizing the above discussion, we thus assert the following:

Proposition 3.5. A real hypersurface with $A\xi = \alpha \xi, \alpha \neq 0$ in a non-flat complex space form does not admit Miao-Tam critical metric.

Together Proposition 3.1 with Proposition 3.5, we prove:

Theorem 3.6. There exist no Hopf real hypersurfaces with Miao-Tam critical metric in non-flat complex space forms.

In the following we always assume $c = 0$. That is to say that $M$ is a real hypersurface of complex Euclidean space $\mathbb{C}^n$. First of all, if $A\xi = 0$, we obtain from (19)

$$S(\xi(\lambda)\xi - \nabla\lambda) = 0.$$

If $S \neq 0$, we have $\nabla\lambda = \xi(\lambda)\xi$. As before we can also lead to (27), but it yields a contradiction since $c = 0$. Thus the scalar curvature $S = 0$, and the relation (26) implies $\Delta\lambda = -\frac{2n-1}{2n-2}$. Actually, $\lambda = -\frac{2n-1}{4n-4}|x|^2$ on $\mathbb{R}^{2n-1}$. Since $R(\xi, X, \xi, X) = 0$ for all $X$, the sectional curvature of $M$ is also zero. By Hartman and Nirenberg’s theorem in [8], $M$ is a hyperplane or a cylinder, hence we have the following:

Theorem 3.7. Let $M^{2n-1}$ be a complete real hypersurface with $A\xi = 0$ of complex Euclidean space $\mathbb{C}^n$. If $M$ admits Miao-Tam critical metric, it is a generalized cylinder $\mathbb{R}^{2n-1-p} \times \mathbb{S}^p$ or $\mathbb{R}^{2n-1}$.

When $\alpha \neq 0$. Let us choose $X \in \mathcal{D}$ such that $AX = \beta X$ for a smooth function $\beta$, then we know $\beta \neq \frac{\alpha}{2}$, otherwise, if $\beta = \frac{\alpha}{2}$, then $-4\beta^2 = c = 0$ from (12), i.e., $\beta = 0$. This is a contradiction with $\alpha \neq 0$. Further, from (12) we have

$$(33) \quad A\phi X = \frac{\beta \alpha}{2\beta - \alpha} \phi X.$$ 

Therefore we find that the equation (29) holds, and for $c = 0$ and $f = \beta$ it becomes

$$(\beta^2 - \alpha \beta)^2 = 0.$$
So $\beta^2 = \alpha \beta$, that means that $\beta$ is constant and further $h$ is also constant. If $\alpha = \beta$, from (33) we see that the shape operator can be expressed as $A = \alpha I$, where $I$ denotes the identity map. In this case, $M$ is locally congruent to a sphere.

If $\beta = 0$, $A = \alpha \eta \otimes \xi$, as the proof of [11, Theorem 1.1], we know that $M$ is $\mathbb{S}^1 \times \mathbb{R}^{2n-2}$. Therefore we assert the following:

**Theorem 3.8.** Let $M^{2n-1}$ be a complete real hypersurface with $A \xi = \alpha \xi$, $\alpha \neq 0$, of complex Euclidean space $\mathbb{C}^n$. If $M$ admits Miao-Tam critical metric, it is locally congruent to a sphere, or $\mathbb{S}^1 \times \mathbb{R}^{2n-2}$.

4. Ruled hypersurfaces of non-flat complex space forms

In this section we study a class of non-Hopf hypersurfaces with Miao-Tam critical metric of non-flat complex space forms. Let $\gamma : I \to \tilde{M}^n(c)$ be any regular curve. For $t \in I$, let $\tilde{M}_{(t)}^n(c)$ be a totally geodesic complex hypersurface through the point $\gamma(t)$ which is orthogonal to the holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Write $M = \{\tilde{M}_{(t)}^n(c) : t \in I\}$. Such a construction asserts that $M$ is a real hypersurface of $M^n(c)$, which is called a ruled hypersurface. It is well-known that the shape operator $A$ of $M$ is written as:

$$A \xi = \alpha \xi + \beta W (\beta \neq 0),$$
$$AW = \beta \xi,$$
$$AZ = 0 \text{ for any } Z \perp \xi, W,$$

where $W$ is a unit vector field orthogonal to $\xi$, and $\alpha, \beta$ are differentiable functions on $M$. From (9), we have

$$Q \xi = \left(\frac{1}{2}(n - 1)c - \beta^2\right) \xi,$$
$$Q W = \left(\frac{1}{4}(2n + 1)c - \beta^2\right) W,$$
$$Q Z = \left(\frac{1}{4}(2n + 1)c\right) Z \text{ for any } Z \perp \xi, W.$$

From these equations we know the scalar curvature $S = (n^2 - 1)c - 2\beta^2$. Since $S$ is constant, this shows that $\beta$ is also constant. Further, the following relation $\nabla \beta = (\beta^2 + c/4) \phi W$ is valid (see [10]), which yields

$$\beta^2 + c/4 = 0 \text{ and } S = -4(n^2 - 2)\beta^2.$$

Further, the following lemma holds:

**Lemma 4.1 ([10]).** For all $Z \in \{X \in TM : \eta(X) = g(X, W) = g(X, \phi W) = 0\}$, we have the following relations:

$$\nabla_W \phi W = -2\beta W, \quad \nabla_W W = (\beta + \beta^2) \phi W,$$
$$\nabla_Z \phi W = -\beta Z, \quad \nabla_Z W = \beta \phi Z.$$
\[ \nabla_{\varphi W} \phi W = 0. \]

Now putting \( Y = \xi \) and \( X = W \) in (15) yields

\[
W(\lambda)(\frac{1}{2}(n-1)c - \beta^2)\xi - \xi(\lambda)\left(\frac{1}{4}(2n+1)c - \beta^2\right)W + \lambda\{(\nabla W Q)\xi - (\nabla \xi W)\} \]
\[
= \left( \frac{c}{4} - \frac{S}{2n-2} \right) \left( \xi(\lambda)W - W(\lambda)\xi \right) + A\xi(\lambda)AW - AW(\lambda)A\xi.
\]

Because \( \beta \) is constant, from (35) and (34), by Lemma 4.1 we compute

\[
(\nabla W Q)\xi - (\nabla \xi W) = \nabla W (Q\xi) - Q\nabla W \xi - \nabla \xi (QW) + Q\nabla \xi W = -W(\beta^2)\xi + \xi(\beta^2)W = 0.
\]

Inserting this into (38), we conclude that

\[
\begin{align*}
W(\lambda)\left( \frac{1}{4}(2n-1)c - 2\beta^2 - \frac{S}{2n-2} \right) & = 0, \\
\xi(\lambda)\left( \frac{1}{2}(n+1)c - 2\beta^2 - \frac{S}{2n-2} \right) & = 0.
\end{align*}
\]

(39)

From (39), we get \( \xi(\lambda) = W(\lambda) = 0 \) since \( \frac{1}{2}(n+1)c - 2\beta^2 - \frac{S}{2n-2} \neq 0 \), which is followed from (37).

Putting \( Y = \xi \) and \( X = Z \) in (15), we have

\[
Z(\lambda)\left( \frac{1}{2}(n-1)c - \beta^2\right)\xi - \xi(\lambda)\left(\frac{1}{4}(2n+1)c\right)Z + \lambda\{(\nabla Z Q)\xi - (\nabla \xi Z)\} \\
= \left( \frac{c}{4} - \frac{S}{2n-2} \right) \left( \xi(\lambda)Z - Z(\lambda)\xi \right).
\]

(40)

By Lemma 4.1, we also obtain

\[
(\nabla Z Q)\xi - (\nabla \xi Z) = -Z(\beta^2)\xi + \xi(\beta^2)Z = 0.
\]

Since \( \xi(\lambda) = 0 \), the relation (40) becomes

\[
Z(\lambda)\left[ \frac{1}{4}(2n-1)c - \beta^2 - \frac{S}{2n-2} \right] = 0.
\]

Thus \( Z(\lambda) = 0 \) since \( \frac{1}{4}(2n-1)c - \beta^2 - \frac{S}{2n-2} \neq 0 \) as before.

By taking \( X = \varphi W \) and \( Y = \xi \) in (15), a similar computation gives

\[
-\lambda\beta\left( \frac{1}{2}(n+2)c + \beta^2 \right) = \left( -\frac{S}{2n-2} + \frac{1}{4}(2n-1)c - \beta^2 \right)\varphi W(\lambda).
\]

(41)

Inserting (37) into (41), we find

\[
\varphi W(\lambda) = \frac{\lambda\beta(2n+3)(n-1)}{2n-1}.
\]

Consequently, we obtain

\[
\nabla \lambda = \frac{\lambda\beta(2n+3)(n-1)}{2n-1} \varphi W.
\]

(42)
On the other hand, as we known $\nabla_X \nabla \lambda = \lambda Q X + (1 + \Delta \lambda) X$ by Miao-Tam equation (1). When $X = Z$ and $W$ respectively, by Lemma 4.1 it follows respectively from (35), (36) and (42) that
\[
-\frac{\lambda \beta^2 (2n + 3)(n - 1)}{2n - 1} = -\lambda(2n + 1)\beta^2 + (1 + \Delta \lambda),
\]
\[
-2\frac{\lambda \beta^2 (2n + 3)(n - 1)}{2n - 1} = -\lambda(2n + 2)\beta^2 + (1 + \Delta \lambda).
\]
It will give $\lambda \beta^2 = 0$, which is a contradiction with $\lambda, \beta \neq 0$. Hence the following theorem is proved.

**Theorem 4.2.** There exist no ruled hypersurfaces with Miao-Tam critical metrics of non-flat complex space forms.

5. Compact hypersurfaces of complex space forms

For the case where $M$ is compact, we immediately obtain the following result:

**Theorem 5.1.** Let $M^{2n-1}$ be a compact real hypersurface admitting Miao-Tam critical metric with $\lambda > 0$ or $\lambda < 0$ of complex Euclidean space $\mathbb{C}^n$, then $M$ is a sphere. In the compact real hypersurfaces of a non-flat complex space form $\tilde{M}^n(c)$ there does not exist such a critical metric.

**Proof.** Write $\check{\text{Ric}} = \text{Ric} - \frac{S}{2n-1} g$. It is proved the following relation(see the proof of [1, Lemma 5]):
\[
\text{div}(\check{\text{Ric}}(\nabla \lambda)) = \lambda |\check{\text{Ric}}|^2.
\]
Thus integrating it over $M$ gives $\check{\text{Ric}} = 0$ if $\lambda > 0$ or $\lambda < 0$, that means that $\text{Ric} = \frac{S}{2n-1} g$. Namely $M$ is Einstein. For the case of complex Euclidean space $\mathbb{C}^n$, it is proved that $M$ is a sphere, a hyperplane, or a hypercylinder over a complete plane curve (cf. [6]). But the latter two cases are not compact. For $c \neq 0$, it is impossible since there are no Einstein hypersurfaces in a non-flat complex space form. Therefore we complete the proof.

**Acknowledgement.** The author would like to thank the referees for the helpful suggestions.

**References**

REAL HYPERSURFACES WITH MIAO-TAM CRITICAL METRICS


XIAOMIN CHEN
COLLEGE OF SCIENCE
CHINA UNIVERSITY OF PETROLEUM-BEIJING
BEIJING 102249, P. R. CHINA
Email address: xmchen@cup.edu.cn