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Jo Tae-gu's Juseo Gwan-gyeon and Jihe Yuanben

趙泰耉의 籌書管見과 幾何原本

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Matteo Ricci and Xu Gwangqi translated the first six Books of Euclid's Elements and published it with the title Jihe Yuanben, or Giha Wonbon in Korean in 1607. It was brought into Joseon as a part of Tianxue Chuhan in the late 17th century. Recognizing that Jihe Yuanben deals with universal statements under deductive reasoning, Jo Tae-gu completed his Juseo Gwan-gyeon to associate the traditional mathematics and the deductive inferences in Jihe Yuanben. Since Jo served as a minister of Hojo and head of Gwansang-gam, Jo had a comprehensive understanding of Song–Yuan mathematics, and hence he could successfully achieve his objective, although it is the first treatise of Jihe Yuanben in Joseon. We also show that he extended the results of Jihe Yuanben with his algebraic and geometric reasoning.

Keywords: Euclid's Elements, Jihe Yuanben (幾何原本, 1607), Matteo Ricci (利瑪竇, 1552–1610), Xu Guangqi (徐光啓, 1562–1633), Jo Tae-gu (趙泰耉, 1660–1723), Juseo Gwan-gyeon (籌書管見, 1718)

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1 Introduction

As is well known, Matteo Ricci (利瑪竇, 1552–1610) of the Society of Jesus made a great impact on the development of the western astronomy and mathematics in China [15, 16]. Since his arrival in Zhaoqing (肇慶) of Guangdong (廣東) in 1582, Ricci published Zuchuan Tianzhu Shijie (祖傳天主十誡, 1584), Yudi Shanhai Quantu (輿地山海全圖, 1584) in Zhaoqing and then Jiaoyou Lun (交友論, 1595) in Nanchang (南昌), Siyuan Xinglun (四元行論, 1599–1600) and Shanhai Yudi Quantu (山海輿地全圖, 1600) in Nanjing (南京) [15]. In 1601, Ricci was permitted to stay in Beijing and then published more books as follows:

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Jingtian Gai (經天該, 1601), Xiqin Quyi Bazhang (西琴曲意八章, 1601), Kunyu Wanguo Quantu (坤輿萬國全圖, 1602), Liangyi Xuanlantu (兩儀玄覽圖, 1603), Tianzhu Shiyi (天主實義, 1603), Ershiwu Yan (二十五言, 1605), Tianzhu Jiaoyao (天主教要, 1605), Xizi Qiji (西字奇迹, 1605), Riqiu Dayu Diqiu Diqiu Dayu Yueqiu (日球大于地球地球大于月球, 1607–1608).

In the meantime, Ricci had very fruitful collaborations with Chinese scholars, notably Li Zhizao (李之藻, 1565–1630) and Xu Guangqi (徐光啓, 1562–1633). In 1607, Hungai Tongxian Tushuo (渾盖通憲圖說) and Jihe Yuanben (幾何原本) [2] were published. The former is a Chinese translation of Astrobium (1593) of Christophorus Clavius (克拉維斯, 1538–1612), which was assisted by collaborator Li Zhizao. Ricci was one of the best students of Clavius at the Roman College. Translating the first six Books of Euclid's Elements compiled by Clavius into Jihe Yuanben, Ricci and Xu Guangqi worked together and tried to introduce new terminologies accommodating Chinese reasoning [15].

Li Zhizao also played an important role for Ricci's translations, Yuanrong Jiaoyi (圜容較義, 1609), the translation of Clavius' Trattato della figura isoperimetre, and Tongwen Suanzhi (同文算指, 1613) based on Clavius' Epitome arithmeticae practicae (1583) and Suanfa Tongzong (算法統宗, 1592) of Cheng Dawei (程大位, 1533–1606).

After Jihe Yuanben, Ricci and Xu Guangqi completed Celiang Fayi (測量法義) in 1608 which was published in 1618. We should note that Xu Guangqi completed Celiang Yitong (測量異同, 1608) and Gougu Yi (勾股義, 1608?).

Liu Hui (劉徽) added Haidao Suanjing (海島筭經) after his completing the commentary on Jiuzhang Suanshu (九章筭術, 263), which deals with surveying (測量) with two poles extending the traditional one with a single pole. Yang Hui (楊輝) included proofs for them in Xugu Zhaiqi Suanfa (續古摘奇筭法, 1275) of his Yang Hui Suanfa (楊輝筭法, 1274–1275) and those proofs were quoted in Suanfa Tongzong. Xu Guangqi related these in Celiang Yitong. He applied the geometrical proofs in Jihe Yuanben to traditional problems of right triangles in Gougu Yi.

Ricci also wrote Jiren Shipian (疇人十篇, 1609), Bianxue Yidu (辨學遺牘, 1615) and Xiguo Jifa (西國記法, 1625).

For our further development, we briefly recall Tianxue Chuhan (天學初函, 1629) compiled by Li Zhizao. It consists of two parts, Libian (理編) and Qibian (器編). The former includes Jiren Shipian, Jiaoyou Lun, Ershiwu Yan, Tianzhu Shiyi, Bianxue Yidu among others. Qibian contains Taixi Shuifa (泰西水法, 1612, Ursis (熊三拔, 1575–1620)), Hungai Tongxian Tushuo, Jihe Yuanben, Biaodu Shuo (表度說, 1614, Ursis), Tianwen Luo (天問略, 1615, Junior (陽瑪諾 1574–1659)), Jianpingyi Shuo (簡平儀說, 1614, Ursis), Tongwen Suanzhi, Yuanrong Jiaoyi, Celiang Fayi, Gougu Yi ([10] for its history in Japan).

The most important development in the history of Chinese mathematics, namely the theory of equations with tianyuanshu (天元術) to siyuanshu (四元術) and zengcheng kaifangfa (增乘開方法), was achieved in Song–Yuan era. But it was forgotten in the Ming dynasty along with calculating rods. Further, Ming had failed to update Shoushili (授時曆, 1280) of the Yuan dynasty until the 17th century. It is plausible that Ricci might have studied Suanfa Tongzong without any mathematical works completed in Song–Yuan era. Therefore he might have chosen Elements first for the Chinese translation and then completed Celiang Fayi based on the western geometry with rigorous proofs. Translating Elements, Ricci and Xu Guangqi had to introduce completely new concepts and terminologies. In the traditional East Asian mathematics, mathematical structures were revealed by solving practical problems such as well assumed examples with *specific numerical* data. But Elements deal with *universal* statements so that they first *define* terminologies. These led Ricci and Xu to introduce the word jieshuo (界說), qiuzuo (求作) and gonglun (公論) for definitions, postulates and axioms respectively. Nowadays, it is well known that postulates involve infinite processes or indicate the relationship between undefined terms, which were defined rather intuitively in jieshuo. Moreover, one can find many a Clavius' idiosyncrasy. Thus they might have resulted severe obstacle for East Asian mathematicians to understand the theory in Jihe Yuanben.

Ricci and Li Zhizao also quoted Celiang Fayi in terms of the traditional terminologies in Tongwen Suanzhi. Indeed, Book Six of Tongwen Suanzhi, Celiang Sanlüfa (測量三率法) and Gougu Lue (勾股略), deals with topics based on Celiang Fayi and Gougu Yi, respectively. As Xu did in Celiang Yitong, Ricci and Li Zhizao also noticed traditional proofs for surveying in Suanfa Tongzong and therefore, they revised Celiang Fayi and Gougu Yi along with Chinese proofs. Celiang Sanlüfa changed the order of presentations in Celiang Fayi but Gougu Lue retained the order exactly the same with that in Gougu Yi. But Ricci and Li tried to avoid terminologies of Jihe Yuanben as much as possible in Celiang Sanlüfa and Gougu Lue. Xu Guangqi strictly followed Jihe Yuanben in most of proofs, called lun (論) in his Gougu Yi. But Ricci and Li either deleted them when they are trivial for Chinese scholars or revised them for the convenience of Chinese readers. They mentioned Jihe Yuanben as a reference in those alterations though.

We also recall that Shuli Jingyun (數理精蘊, 1723) [2] is a compendium for the western mathematics whose editors included the introductory part for geometry with the *same* name Jihe Yuanben but based on a school textbook, Elemens de Geometrie (1671 or 1673) [12] of Ignace Gaston Pardies (1636–1673), a French Jesuit. It is well known that Elemens de Geometrie was a text book for Kangxidi (康熙帝, 1654–1722, r. 1661–1722) taught by French Jesuits Joachim Bouvet (白晉, 1656–1730)

and Jean-Francois Gerbillon (張誠, 1654–1707), who translated Elemens de Geometrie into Manchu language (滿文) [2, 15]. One of the important purposes for Shuli Jingyun is to introduce the western mathematical basis for the study of the western astronomy and hence Jihe Yuanben of Shuli Jingyun had to include far more subjects than those in the first six Books of Elements. Thus they could not retain logically strict processes but explained topics rather intuitively and replaced many terminologies of plane figures in Jihe Yuanben of Ricci and Xu by those which can be easily figured out by traditional Chinese literatures. We don't know why the translators of Shuli Jingyun changed the term xiang shi (相似) for similar figures into tongshi (同式). Because of full contents for the astronomy and mathematics and intuitive approaches, Jihe Yuanben in Shuli Jingyun became more popular for East Asian mathematicians and astronomers after 18th century than the translation of Ricci and Xu.

The purpose of this paper is to investigate the history of Jihe Yuanben in Joseon. We discuss the initial stage of the western mathematics in the Joseon dynasty and then Juseo Gwan-gyeon (籌書管見, 1718) [9] of a noble class Jo Tae-gu (趙泰耉, 1660–1723). The book is the uniquely extant one which deals with Jihe Yuanben in the Joseon dynasty.

For the Chinese books included in [2], they will not be numbered as an individual reference.

2 Initial stage of the western mathematics in the Joseon Dynasty

In this section, we discuss the history of introduction of the western mathematics in Joseon.

As is well known, Joseon mathematics has been developed by Suanxue Qimeng (筭學啓蒙, 1299) of Zhu Shijie (朱世傑), Yang Hui Suanfa and Xiangming Suanfa (詳明筭法, 1373) of An Zhizhai (安止齊). Because of Japanese invasion (1592–1598), most of mathematical works before the 16th century was completely lost.

The 15th King Gwanghae (光海, 1575–1641, r. 1608–1623) of the Joseon dynasty was dethroned in a coup by his nephew, King Injo (仁祖, 1595–1649, r. 1623–1649). Meanwhile, Nurhaci (1559–1626) founded a new state, Later Jin (後金, 1616–1636) in Manchuria and then the new dynasty Daqing (大淸) or simply the Qing dynasty by his son Hong Taiji (皇太極, 1592–1643). Manchuria is located in between Joseon and the mainland of China so that Later Jin tried to block the close relationhip between Joseon and Ming. Gwanghae took the policy of friendly relations with Later Jin, but King Injo reversed the policy. This move has resulted two invasions by the Later Jin, called Jeongmyo Horan (丁卯胡亂, 1627) and Byeongja Horan (丙子胡亂, 1636). The second one by Hong Taiji was really devastating and forced the King

Injo surrender. We should point out that Tianzhu Shiyi was brought into Joseon in the second decade of the 17th century and that the influences by Jesuits in China were perceived by a few Joseon scholars of the time.

Joseon did not have any information on the other works by Ricci and his collaborators and Chongzen Lishu (崇禎曆書) initiated by Xu Guangqi until the last period of the Ming dynasty.

Adam Schall von Bell (湯若望, 1591–1666) amended Chongzen Lishu into Xiyang Xinfa Lishu (西洋新法曆書, 1645), later called Xinfa Suanshu (新法算書). The Qing dynasty adopted a new calendar system, Shixianli (時憲曆) in 1645 which was brought into Joseon by an envoy to Qing. In 1646, another envoy Lee Gyeongseok (李景奭) to Qing was able to buy Xiyang Xinfa Lishu from Schall von Bell privately [11, 17]. It contains Dace (大測, 1631), Geyuan Baxianbiao (割圓八線表, 1631) translated by J. Terrenz (1576–1630); Jihe Yaofa (幾何要法, 1631) by G. Aleni (1582–1649); Bili Juijie (比例規解, 1630), Chousuan (籌算, 1628), Celiang Quanyi (測量全義, 1631) by J. Rho (羅雅谷, 1590–1638).

Joseon mathematicians and astronomers could not understand Xiyang Xinfa Lishu so that they did have a great difficulty to figure out the Shixianli based on the western mathematics. However, Joseon also adopted Shixianli, called Siheonlyeog as an official calendar in 1653.

In the mid-17th century, Gyeong Seon-jing (慶善徵, 1616–1690) wrote Muksa Jibsanbeob (默思集算法) for training mathematical officials. Although he kept Suanxue Qimeng, he did not deal with tianyuanshu which has been the most important topic in Joseon mathematics. Since Kim Si-jin (金始振, 1618–1667) republished Suanxue Qimeng in 1660, the theory of equations based on tianyuanshu and zengcheng kaifangfa (增乘開方法) became once again the main subject for Joseon mathematicians. Park Yul (朴繘, 1621–1668) completed Sanhak Wonbon (筭學原本, 1701) [7, 13]. It is the first book with tianyuanshu in Joseon transmitted to the present.

We quote the following in Seungjeongwon Ilgi (承政院日記), i.e., The Daily Records of Royal Secretariat of Joseon Dynasty (1699, the 25th year of King Sukjong (肅宗, r. 1674–1720)) [18]. It is a memorial addressed to the King by Gwon Tag (權倬) which informs Taixi Shuifa (泰西水法). Gwon referred to Taixi Shuifa as Ricci's book elucidated by Ming scholar Ursis (熊三拔) in 1611. As mentioned above, Ursis is its author and Xu Guangqi is its transcriber. This indicates that Gwon Tag did not have an exact idea on the book and that Tianxue Chuhan might be brought into Joseon in the last decade of the 17th century.

> 記昔萬曆辛丑年, 西洋國毆邏巴人 利瑪竇, 航海朝天, 通博物理,
> 明儒 熊三拔演利氏之意, 作爲一書, 名曰 泰西水法, 專論水車制度,
> 江河淵井之水, 無不激而上之

Gusu Ryag (九數略) [1] is the first book in Joseon which deals with the western mathematics. It is written by Choe Seog-jeong (崔錫鼎, 1646–1715). Choe Seog-jeong was a well-known politician and eventually served as a prime minister, yeonguijeong (領議政). His main reference for Gusu Ryag is Tongwen Suanzhi and he tried to relate mathematics with sixiang (四象). He quoted Tianxue Chuhan and Chousuan as references for the book and described the former as a book by Ricci and Li Zhizao (西士 利瑪竇授 明李之藻演) and the latter as one by Rho.

He transcribed Book Six of Tongwen Suanzhi, Celiang Sanlüfa with Gougu Lue discussed in the previous section. Choe reordered Celiang Sanlüfa and deleted the sentences relating Celiang Sanlüfa with Jihe Yuanben and Celiang Fayi in Tongwen Suanzhi, namely "務取極方詳見幾何原本" "詳見徐太士測量法義". He added that his transcription is made from Tianxue Chuhan as saying "出天學初函". Before Choe Seog-jeong transcribed Book Six of Tongwen Suanzhi, he also discussed basic operations of fractions based on those in Tongwen Suanzhi. We don't know why he identified Tianxue Chuhan with Tongwen Suanzhi. Further, he didn't transcribe Gougu Lue but classified gougu chapter (勾股章) of Jiuzhang Suanshu unjustly as one related to surveying by similar triangles or Yang Hui's proof for Haidao Suanjing in his classification of Jiuzhang by sixiang (九章分配四象).

Choe also transcribed a part of Chousuan in Xiyang Xinfa Lishu. Books on the astronomy should be kept in the national observatory, Gwansang-gam (觀象監) so that it was accessible to the high official Choe. In the East Asian mathematics, chou (籌) means the calculating rods which are used to represent coefficients of polynomial representations in tianyuanshu along with the main tool for basic calculations. It was prevailed throughout the Joseon dynasty so that sanhak (suanxue in China, 算學) has been also called juhak (chouxue, 籌學). Thus Choe might have taken rather Chousuan than the other much more important books in Xiyang Xinfa Lishu. His transcription is almost illegible because he omitted too much. The original book has seven sections for the basic constructions of Napier's bones, but he included parts of section 4, 1 and 7 without any logical sequence and then the methods for multiplications, divisions and extractions of square roots. Choe missed the fact that Napier's bones is a calculating *device* for calculations up to extractions of cube roots. But he added its origin as follows though:

今按 籌筭一法卽西洋國筭法也 極西耶穌會士羅雅谷作一書論籌筭成於崇禎戊辰

He was probably too busy to study mathematics in detail due to his official duties but he left the first book with the western mathematics in the history of Joseon mathematics. Choe is also well known by his remarkable contributions to magic squares and Euler's square which are discussed in the final section, Harag Byeonsu (河洛变數) of the appendix in Gusu Ryag.

For gougushu, the theory of right triangles in Joseon, there are two mathematicians, Hong Jeong-ha (洪正夏, 1684–1727) and his contemporary Yu Su-seog (劉壽錫) who left really important contributions based on tianyuanshu. Comparing Xu's geometrical proving approach to gougushu with Joseon mathematicians' ones, we can easily observe that algebraic approach is much more efficient to get the structure of gougushu [8].

3 Jihe Yuanben in Juseo Gwan-gyeon

As Choe Seog-jeong's Gusu Ryag, Jo Tae-gu (趙泰耇, 1660–1723) also completed a book, Juseo Gwan-gyeon (籌書管見, 1718) [5, 7] dealing with the western mathematics. Jo Tae-gu belongs to a noble class. He passed the preliminary examination, called sogwa (小科), saengwonsi (生員試) in 1683 and then the national civil service examination, called gwageo (科擧) in 1686.

After various positions in the government, Jo Tae-gu was appointed the minister, panseo (判書) of Hojo (戶曹) in 1712 and also served as jejo (提調), the head of Gwansang-gam. He became yeonguijeong (領議政) in 1721. Hojo deals with taxation, finance and national census and is the only ministry in the Joseon dynasty with mathematics officials who are selected by the examination, called chuijae (取才) and belong to jungin (中人) class. Jo Tae-gu recommended Hong Jeong-ha, the lowest rank in Hojo to discuss mathematics with the Qing envoys, He Guozhu (何國柱) and A Qitu (阿齊圖) in 1713. Hong Jeong-ha completed Gu-il Jib (九一集, 1713-1724)[4] which is the most important book in the history of Joseon mathematics and drew a great impression to the then minister Jo. Hong included the detail about the discussions with the envoys together with his friend Yu Su-seog (劉壽錫), the author of Gugo Sulyo (句股術要) in the appendix of Gu-il Jib. Indeed, they competed with each other on their mathematics by asking 22 problems. He Guozhu was surprised by Hong's solving problems by tianyuanshu with calculating rods. Hong added the final remarks by Sili (司曆), He Guozhu and Hong himself:

> 司曆曰筭家諸術中 方程正負之法極爲最難 君能知之乎
> 余曰方程之術卽中等之法何難之有
> 余布筭之際 司曆曰中國無如此筭子可得而誇中國乎
> 余卽以與之卽擇其中四十箇而去
> 司曆曰君之姓名書去 吾當以示大國因書吾與劉生姓名以去

Indeed, Sili said that fangcheng zhengfu is the most difficult part in mathematics and asked whether you know it. We don't know that He Guozhu meant fangcheng zhengfu by solving polynomial equations or systems of linear equations. It means the latter by the usual sense but they didn't discuss systems of linear equations.

Hong retorted that fangcheng befalls around the halfway difficulty among mathematics and hence it is not that difficult. When I used calculating rods for solving problems, Sili said that we don't have the calculating rods in China so that he wanted to have some and show them in China. Hong gave him the rods out of which He took 40 and left. He Guozhu also wanted Hong and Yu to leave their names to report the discussion in China. It seems quite plausible by the above comments that He Guozhu might have kindled Chinese mathematicians' attentions on tianyuanshu and calculating rods.

He Guozhu also asked the problems on regular polygons which are inscribed and circumscribed in a circle. He Guozhu added a comment that they could also be solved by trigonometry and its table (八線表). He further stated that the detail could be obtained by the study of Jihe Yuanben and Celiang Quanyi and that he would send them to Hong and Yu but he did not keep the promise. As discussed in the above, both books were already available in Joseon Gwansang-gam but not to mathematical officials (算員) in Hojo.

Besides the above meeting, Jo Tae-gu also arranged meetings for He Guozhu and officials in Gwansang-gam. Indeed, He brought an astronomical instrument, sanghandaeui (象限大儀) and measured the latitude (北極高度) of Seoul [14]. He Guozhu taught a Gwansang-gam official Heo Won (許遠) astronomical instruments and mathematics relating to the astronomy and promised to send books and instruments to him. The minister Jo Tae-gu eventually sent Heo Won as a member of the envoy (節使) to Yeon-gyeong (燕京) in 1715. Heo brought back nine astronomy books including Ilsig Boyu (日食補遺), Gyosig Jeungbo (交食增補), Yeogcho Byeongji (曆草駢枝) and six surveying and calculating instruments through He Guozhu's help [16, 17].

In all, the above records indicate that Jo Tae-gu paid great attentions to improve mathematics and astronomy of Joseon and that he himself studied Joseon mathematics together with Chinese mathematics and astronomy influenced by the western ones. These efforts paved the way for him to complete the book Juseo Gwan-gyeon in 1718.

Juseo Gwan-gyeon consists of the four parts: basic terminologies and mathematical mnemonics, gugyeol or koujue in Chinese (口訣); multiplications and divisions by calculating rods; nine chapters (gujang, 九章); gujang mundab (九章問答).

The first part includes mostly well-known mathematical mnemonics. In the late 17th century, Joseon adopted a new system of farm land tax (田制法) and Juseo Gwan-gyeon is the first mathematics book quoting the system [6]. Further, he included the second approximation of π of Zu Chongzhi (祖冲之, 429-500) in the miscellaneous rules (雜法). Zhu Shijie (朱世傑) says that $\frac{22}{7}$ is Chongzhi's milyul or milü (冲之密率) in his Suanxue Qimeng. But it is known as yakyul or yuelü (約

率) and milü is $\frac{355}{113}$. Jo Tae-gu renamed the milü as milhuyul or mihoulü (密後率). He might have needed the better approximation of π in his study of astronomy. Finally Jo quoted approximate rates of areas of regular polygons with those of inscribed and circumscribed circles in Suanfa Tongzong. Cheng's results are based on the area of an equilateral triangle, where the ratio between its side and height is approximated by $7:6$, called zhengliu mianqi (正六面七), i.e., $\sqrt{3} \approx \frac{12}{7}$. Cheng quoted this from Jiuzhang Suanfa Bilei Daquan (九章筭法比類大全, 1450) of Wujing (吳敬). Interestingly, Jo Tae-gu replaced the ratio of areas of a circle and its inscribed regular hexagon by $8:7$ (圓容六角八分之七) instead of $7:6$. Using the approximation $\sqrt{3} \approx \frac{7}{4}$, Jo had the result. So far we can't find his approximation in any other mathematical literature and it is not a repeating decimal with the almost same error with $\frac{12}{7}$ (also see [5, 7]). He also added the ratio between volumes of a cube and its inscribed and circumscribed sphere. For the case of the inscribed cube in a sphere, he mentioned the rate is $3:1$. We should remind that the volume v of a sphere with the diameter d in the East Asian mathematics is $v = \frac{9d^3}{16}$ and that for a side a of the inscribed cube to the sphere, $d = \sqrt{3}a$, one has $a^3 = \frac{64}{189} \times v$, where Jo used approximation $\sqrt{3} \approx \frac{7}{4}$ and $\frac{64}{189} \approx \frac{1}{3}$.

The second part deals with traditional methods of multiplications and divisions by calculating rods, the theory of fractions and proportions. As the other traditional mathematics books, Jo also explained multiplications and divisions through word problems and the exact processes of calculating rods (依圖布籌) and then Jo dealt with direct proportionality, called iseung dongje, or yicheng tongchu (異乘同除) as their first applications. For the theory of fractions, called jibunlon, or zhifenlun (之分論), Jo mostly quoted the theory of fractions in Tongwen Suanzhi as Choe Seogjeong did in his Gusu Ryag and then continued to study again iseung dongje in the section Sayulbeob (四率法) along the context of Tongwen Suanzhi and Jihe Yuanben. Tongwen Suanzhi deals with proportions in Book One and Book Six of tongbian (通編). Book One concerns with the algebraic structures of proportions and Book Six with geometrical structures as discussed in the previous section. Since the theory of ratios and proportions in Book Five of Jihe Yuanben is introduced mainly in the context of geometrical structures, the readers in East Asia might have great difficulty to grasp their structures. Most of them are quite familiar with yicheng tongchu and tongcheng yichu (同乘異除). Tongwen Suanzhi related proportions with them in Book One. Thus Jo Tae-gu begins the section Sayulbeob with the following:

> 四率者異乘同除之法理者也 其法有比例有互視有合率
> 皆以二率三率相乘而以一率除之得四率也

It says: sayul, i.e., proportionality is nothing but yicheng tongchu. It has three kinds, namely direct (比例), inverse (互視) and compound (合率) ones but they all mean $a_4 = \dfrac{a_2 a_3}{a_1}$ for $a_1 : a_2 \simeq a_3 : a_4$.

The above sentence is the first instance for Jo's quotations from Jihe Yuanben. Hosi, or hushi is clearly the abbreviation of huxiang shizhi (互相視之) appearing in the second definition (第二界) of Book Six of Jihe Yuanben. After the above definition, he gave examples to explain the three cases. For the direct proportions, he added the following example after the usual yicheng tongchu example.

> 假如有大小句股相似之形 以大句大股(小句)求小股
> 則以大句爲一率大股爲二率小句爲三率求得四率小股
> 其術爲以大句比大股若小句與小股 亦爲以大句比小句若大股與小股也
> 如求小句則爲大股比大句若小股與小句也

Namely, suppose that there are two *similar* right triangles with gou, gu $a_i, b_i, i = 1, 2$ then find b_2 by a_1, a_2, b_1. Since $a_1 : b_1 \simeq a_2 : b_2$, one has b_2. One can also have it by $a_1 : a_2 \simeq b_1 : b_2$, and a_2 by $b_1 : a_1 \simeq b_2 : a_2$.

The above quote also indicates that Jo Tae-gu comprehended exactly the most important subject, similar triangles, i.e., *xiangshizhixing* (相似之形) in Jihe Yuanben and the basic structures of proportions.

He explained huxiang shizhi as follows:

> 互視者比例之變例也 比例之法今物多於原物則今價亦多於原價
> 今股小於原股則今句亦小於原句矣
> 若今布狹於原布則所還之長反長於原長 今布廣於原布則所還之長反小於原長
> 此不可以比例求也 故變其名曰互視 其術爲以今濶比原濶若原長與今長也

Jo Tae-gu used the well-known example for tongcheng yichu in East Asian mathematics literatures to explain the inverse proportion. For two rectangles with the *same area*, their lengths, say a_1, a_2 and widths b_1, b_2, one has $a_1 : a_2 \simeq b_2 : b_1$. We point out that the second definition for huxiang shizhi of Jihe Yuanben does not make any sense because it misses the condition of the *same area* (also see Proposition 15 and Proposition 16 in Book Six and [3]). He also explained compound proportion by an example.

The third part deals with mathematics along Jiuzhang Suanshu as in Suanfa Tongzong. It begins with basic natures of the nine chapters. They are basically quotes of jiuzhang mingyi (九章名義) in Suanfa Tongzong, which are precisely the quotes of jiuzhang mingyi in Suan Jing (算經) of Xia Chawei (謝察微). The second chapter sumi (粟米) is mistakenly replaced by subu (粟布) as in Xiangming Suanfa (詳明筭法, 1373). Choe Seog-jeong stated it sumi with subu (粟米 云粟布) but yingnu (盈朒) for yingbuzu (盈不足). It shows that he studied Yang Hui Suanfa and Suanfa Tongzong.

Jiuzhang Suanshu was not available to Joseon mathematicians until the mid-19th century so that they had to follow the format and contents in Suanfa Tongzong. Jo Tae-gu presented the third part along them as well. Since he is quite familiar with Suanxue Qimeng and Gu-il Jib, his presentation is quite different from those in Suanfa Tongzong. First, he used calculating rods as far as possible and revealed his theory of equations by tianyuanshu and zengcheng kaifangfa in the chapter shaoguang (少廣) based on the results of Zhu Shijie and Hong Jeong-ha. Indeed, Jo presented the process of solving equations by the exactly same way with those in Hong Jeong-ha's Gu-il Jib and also included the approximate solutions of equations. As in Suanxue Qimeng, the theory of finite series was associated with that of areas and volumes. In the end of gougu (勾股), Jo included the methods of surveying with poles in Yang Hui Suanfa and proofs. We just quote the problem with one pole as follows because Jo discussed the case with two poles in the next part:

> 解曰 勾股之法二段求一段 此只有句而無弦 無積不可以求股也
> 故退作小句股 其形與大句股相似 以異乘同除求之得大股
> 以直形補成倒順兩句股則弦之內外各有直積一段兩積相同
> 故以小股除之得大句 小句除之得大股

Noting that information of two sides of a right triangle gives rise to the remaining side, one can't apply gougushu to have the height, gu and hence one should have another right triangle which is made of the pole and the observer's one eye and is *similar* to the given problem. Further, he explains the traditional method of Yang Hui.

Finally, Jo discussed the area of an arbitrary triangle, but his example is a right triangle with three sides 30, 40 and 50. We recall that Zhu Shijie also discussed the same problem with three sides 75, 60 and 45 and the height 36 in Problem 13 of Tianmuxingduan (田畝形段) in Suanxue Qimeng. Both are clearly redundant. We don't know how Zhu did find the height but Jo just included the process of finding the height. Since Jo discussed the detailed proof for arbitrary triangles in the next part, we postpone its detail.

The fourth part is titled by Gujang Mundab, questions and answers on Jiuzhang (九章問答). Jo Tae-gu presented his personal views, gwan-gyeon (管見) on mathematical structures discussed in the third part and revealed his thorough understanding along the mathematical logic which should be obtained by his study of Jihe Yuanben. We remind that most of authors in East Asia followed the format of questions (問) and processes of solving them (術) since Jiuzhang Suanshu and that readers should find their intrinsic structures by themselves. The part consists of 60 items. As we mentioned already, the term gujang (九章) in Juseo Gwan-gyeon is irrelevant to Liu Hui's Jiuzhang Suanshu but indicates divisions of mathematics along jiuzhang mingyi (九章名義). Jo's main references are Suanxue Qimeng, Gu-il Jib, Suanfa Tongzong and Jihe Yuanben. Since our main concern of this paper is Jihe Yuanben, we will just discuss Jo's discourses relating to Jihe Yuanben in this section and will deal with those of the remaining part in a separate paper.

The author began with the units of length and area in the first question. He stated that they are related with seon, xian (線) and myeon, mian (面), respectively and originated from the western mathematics as "卽西士所謂線(面, resp.)也". In the East Asian mathematics, they have been using the same units for the lengths and areas. He indicated $1(chi)^2 = 100(cun)^2$ by the *the area* of 100 cun, 百寸之積. Further, he described $10(cun)^2 = 1chi \times 1cun$, $1(cun)^2 = 1chi \times 1fen$ and $1(fen)^2 = 1chi \times 1li$, where $1chi$ (尺) $= 10cun$(寸), $1cun = 10fen$(分) and $1fen = 10li$(釐) for units of lengths. Applying the change of old and new units of lengths with the ratio $m : n$, one should apply the ratio $m^2 : n^2$ for the corresponding change of areas. By these, Jo emphasized the difference of xian and mian (線面之異也). Dealing with 12 items for fangtian (方田), Jo discussed mostly better approximations of various areas.

Until junshu (均輸) chapter, we can't find anything related to Jihe Yuanben except Jo's deductive inferences. In the chapter, he explained again proportions, namely bilei, huxi and helü (合率).

As one may expect, the gougu chapter in Gujang Mundab is the part mostly influenced by Jihe Yuanben. Jo Tae-gu dealt with 8 items and except the item 1, the others are all related to Jihe Yuanben. In fact, the first one deals with algebraic consequences of the Pythagorean theorem and the identity $x^2 - y^2 = (x - y)(x + y)$.

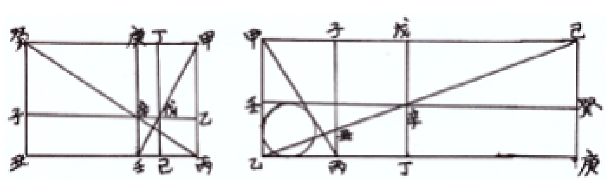

Figure 1. Geometric Constructions

In the item 2 and 3, Jo illustrated the *geomeric constructions* of inscribed square and circle in a right triangle. He has already confirmed $s = \dfrac{ab}{a+b}$ and $d = \dfrac{2ab}{a+b+c}$ for the side s (diameter d, resp.) of the square (circle, resp.) in a right triangle with a, b, c for its gou, gu and xian. In Figure 1, 丙丑$= a+b$ for the left one; 乙丙=丙丁$= a$, 乙庚$= a + b + c$ for the right one. Jo might be drawn by inferences by geometric constructions in Jihe Yuanben for universal statements which are the most distinct and unfamiliar features of Jihe Yuanben to East Asian mathematicians including Jo. He said that it is difficult to explain the construction by words so that he is using a figure, i.e., 此難用言喩可以圖明之. Both cases are problems of solving linear equations which are done by areas divided by the diagonal of a rectangle introduced by Yang Hui. After the discussion for the inscribed circle in a right triangle, he added the following statement regarding the same method to find the inscribed square and circle in an *arbitrary triangle*.

> 曰若以圭形求容圓容方則其法亦與句股不異乎
> 曰以圭形求容方則以濶爲句以中長爲股幷長濶以除倍積
> 其法與句股求容方同而所得容方亦無不同矣
> 若求容圓則以濶幷兩斜弦以除四積其法亦不異
> 但其濶幷兩弦之數不與句股弦相幷者等
> 則其仍得圓經不得與句股容圓同矣

For a triangle with three sides a, b, c and its height h for a base a, namely a side containing a side of the inscribed square, $s = \dfrac{ah}{a+h}$ and hence its construction can be done by taking h as gu for the case of a right triangle. Further for the inscribed circle, $d = \dfrac{2ah}{a+b+c}$. The last two lines are slightly ambiguous, for his gu (股) denotes the height as that in a right triangle. But their geometric constructions are exactly the same with the case of a right triangle because they used areas of triangles. Incidentally, the problem for the inscribed square in an arbitrary triangle and then that in a right triangle, were treated in Clavius's addenda in Book Six of Jihe Yuanben. Xu Guangqi discussed the proof for the inscribed square in a right triangle with specific dimensions, $27, 36$ for gou and gu and then $\dfrac{27 \times 36}{27+36} \approx 15.428$ in his Gouguyi. Xu did not use the dimensions for the proof but it indicates that he drew the diagrams using the dimensions with the approximation. The same problem was quoted in Tongwen Suanzhi but the approximation was replaced by $15\dfrac{3}{7}$. They did not deal with the inscribed square and circle in an arbitrary triangle and geometric construction of those in a right triangle either.

We should point out that the incenter of a triangle was dealt in Proposition 4 of Book Four and its radius can be constructed by the height from the incenter to a side. But it does not say anything about the algebraic relation between the radius and three sides. On the other hand, Jo's construction of the diameter does not give any information of its incenter. Since the ancient times, East Asian mathematicians have never paid any attention to the study of angles except the right angle. Although Jo studied astronomy and mentioned trigonometry in the end of Juseo Gwan-gyeon, he did not include any mathematical structures of angles in the book.

The item 4 deals with the inscribed square's side x in a right triangle with the given remaining sides $a-x, b-x$, yugou (餘句) and yugu (餘股). Jo discussed first the process by the areas and then by the proportion $b-x : x \simeq x : a-x$ from the similar triangles which implies $x = \sqrt{(a-x)(b-x)}$. Since $a-x, b-x$ can't be expressed by tianyuanshu, one can't have *universal* statements, $x = \dfrac{ab}{a+b}$ with tianyuanshu, while we have immediately $x = \dfrac{ab}{a+b}$ from the equation $x^2 = (a-x)(b-x)$.

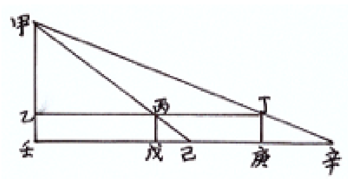

Figure 2. Haidao Suanjing and proportions

In the item 5, Jo discussed the structure of Liu Hui's haidao (海島) problem with two poles (重表) by similar triangles and a property of proportions. Indeed, he used the fact that two right triangles with a common hypotenuse and one parallel sides are similar and that proportions $x : h \simeq y : a_1, x : h \simeq y + b : a_2$ implies $x : h = b : a_2 - a_1$. Here, 甲乙$= x$, 丙戊$=$丁庚$= h$, 乙丙$= y$, 戊己$= a_1$, 庚辛$= a_2$ and 戊庚$= b$ in Figure 2. Also, he compared the above with the proof given by Yang Hui.

The next two items 6 and 7 deal with the problem of the area of an arbitrary triangle by its three sides. We first note that the basic field of numbers in the East Asian mathematics is the field $\mathbb{Q}$ of rational numbers since Jiuzhang Suanshu. As we have already discussed, the area of an *equilateral* triangle can't be obtained for its height involves the irrational number $\sqrt{3}$. Thus the area of an arbitrary triangle has never been discussed except trivial one in Suanxue Qimeng. Jo Tae-gu is the first mathematician in Joseon, if not unique and he explains the procedure of finding the height with deductive and didactic inferences as follows.

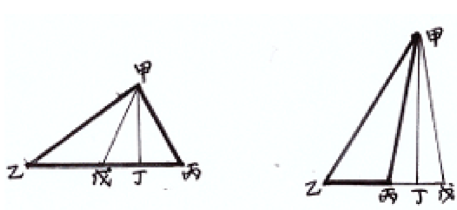

Figure 3. Area of an arbitrary triangle

For the modern readers' convenience, we relabel a triangle jiayibing (甲乙丙) in the left one of Figure 3 by ABC with their lengths $AB = a, AC = b, BC = c$ with $a > b$ and its height 甲丁$= AH = h$. He begins with the statement that one needs to find the height for the area, namely the perpendicular line (垂線) AH from A to the base BC, which results two *right* triangles ABH and ACH. Although we have their hypotenuses a, b, we can't apply Pythagorean theorem to them because we don't have their gou, say $BH = x_1, HC = x_2$. From the identities $x_1^2 + h^2 = a^2$, $x_2^2 + h^2 = b^2$, he has $x_1^2 - x_2^2 = a^2 - b^2$. Applying the well-known identity $x^2 - y^2 = (x-y)(x+y)$ to the above, he has $(x_1 - x_2)(x_1 + x_2) = (a-b)(a+b)$ and hence $x_1 - x_2 = \dfrac{(a-b)(a+b)}{c}$ for $c = x_1 + x_2$. Let 戊$= D$ denote the point on BH with $HD = x_2$, then $BD = x_1 - x_2$ and hence $2x_2 = c - BD$ implies $x_2 = \dfrac{c^2 - (a^2 - b^2)}{2c}$. Thus $h^2 = b^2 - x_2^2$ can be obtained by a, b, c. He inserted $x_1 + x_2 : a + b \simeq a - b : x_1 - x_2$ for the equation $(x_1 - x_2)(x_1 + x_2) = (a-b)(a+b)$, called hushizhifa (互視之法) but it is clearly redundant. He also added that the above procedure can be applied to any choice of a base. Further in the item 7, the author discussed how to find the height of a triangle ABC whose perpendicular line AH to the base BC is outside of the triangle in the right one of Figure 3. Using the exactly same notions in the item 6 and $c = x_1 - x_2$, he obtained the height h. Jo also explained the case of the item 7 with specific dimensions, $a = 63, b = 52, c = 25$ but did not mention that the triangle is an *obtuse* one. He may be perplexed by the fact that the height is out of the triangle. Thus in item 8, he said that the height is imaginary (虛設之數) but the case must be necessary. He take an example of a right triangle and then consider the triangle inside the right triangle made of the half of gou. It gives a perfect example for the item 7. All of the items in gougu part, Jo used figures to explain his statements with proofs.

In the next two items, Jo Tae-gu expressed his views on his own book Juseo Gwan-gyeon as his concluding remarks. First, he discussed the traditional mathematics referred to as Jiuzhang, and new mathematics. The question is that Jiuzhang contains everything of mathematics as "或曰 數之爲術九章盡之矣 是外固無餘法乎" His answer is as follows:

> 曰凡籌書所有小數雜法之贅賸 而不切者悉掃去而不論矣 獨朱氏有立天元之法
> 西人有愚平三角 弧三角之法 皆刱智而得其巧者也

My book (籌書) includes far less unnecessary and miscellaneous rules, unrelated topics and discussions on them. One may say that tianyuanshu of Zhu Shijie, and plane and spherical trigonometry of western mathematicians are all ingenious inventions with full of consequences and applications. Clearly, Jo did neither have any information of the history of tianyuanshu except that in Suanxue Qimeng nor that of trigonometry. He related unjustly tianyuanshu with the chapter shaoguang (少廣之演) and trigonometry to the chapter gougu (句股之奧) in Jiuzhang Suanshu. But he continued to state that one has to study first Jiuzhang Suanshu for a further understanding of the above (領悟者必須九章貫通無疑 然後可進於此矣).

In the final item, Jo emphasized the thorough understanding of the mathematical principles or structures (問如扣鍾 析理如破竹 所以解剝其肯綮 發揮其理致者：乃見潺流

而不知有江河也) and also the proofs by figures (有未盡數爲圖).

We discussed those items directly related by the influence of Jihe Yuanben but we can easily recognize that the remaining items are also chosen and discussed by the same principle. The discourses indicate the great change of the view on mathematics in Juseo Gwan-gyeon from that in traditional Joseon mathematical literatures.

4 Conclusions

The republication of Suanxue Qimeng in 1660 stimulated a successful revival of mathematics in Joseon. Indeed, the country had experienced a long and inactive period of study on mathematics after the great achievement of her own mathematics and astronomy under the directive of the King Sejong (世宗, 1397–1450, r. 1417–1450). It focused on the theory of equations based on tianyuanshu and then culminated in Gu-il Jib (九一集, 1713–1724) of Hong Jeong-ha (洪正夏, 1684–1727). In the last decade of the 17th century, western mathematical and astronomical literatures, notably Tianxue Chuhan (天學初函, 1729) were brought into Joseon and then their study were begun by high officials, Choe Seog-jeong (崔錫鼎, 1646–1715), the author of Gusu Ryag (九數略) and Jo Tae-gu (趙泰耉, 1660–1723), the author of Juseo Gwan-gyeon (籌書管見, 1718). Choe's main reference is Tongwen Suanzhi and Jo's is Jihe Yuanben.

Jo Tae-gu figured out through his study of Jihe Yuanben that the main structures used in the book are proportions and similar triangles. Since Jo Tae-gu has studied Suanxue Qimeng and Gu-il Jib, he understood sufficiently well the algebraic structures of East Asian mathematics. Thus, he had much easier and perfect approach to proportions than those in Jihe Yuanben. Further, he tried to reorganize the traditional mathematics in conjunction with Jihe Yuanben and hence his attentions to similar triangles were restricted to similar *right* triangles. In 1718, Jo Tae-gu was appointed the minister of hyeong jo (刑曹) and later that of gong jo (工曹) but he could not accept the appointment because of his illness and stayed away from the capital. Instead, he completed his unique mathematical work Juseo Gwan-gyeon. In spite of all these restrictions, Jo did demonstrate that the main ingredient of Jihe Yuanben is the deductive reasoning based on the sound logic for *universal* statements. Jo Tae-gu indeed dealt with universal statements as far as possible and geometric constructions in Juseo Gwan-gyeon. Indeed, the book is the first and unique mathematics book with such an inquiry in the history of Joseon mathematics. As a consequence of his study, he emphasized mathematical structures or its principles rather than procedures of obtaining final results by calculations.

In his final years, he served the King Gyeongjong (景宗, 1688–1724, r. 1720–1724) whose health was really poor and his heir apparent was his step brother, later the

King Yeongjo (英祖, 1694–1778, r. 1724–1776). His succession was not that smooth and a calamity of literati, called Sinim Sahwa (辛壬士禍, 1721–1722) was believed to be instigated by Jo Tae-gu. In fact, Jo Tae-gu's cousin Jo Tae-chae (趙泰采, 1660–1722) among others was executed by the death sentence at the calamity. As Yeongjo ascended the throne, Jo became a persona non grata although he was already dead, his official titles were all stripped away in 1746 and his family were all demoted to the lowest rank, slaves in 1755 (至丙寅始命追奪官爵, 乙亥擧孥籍之典 [17]).

Although he served as a prime minister, his contributions including mathematics became completely forgotten. It is really pity that his wonderful, if not one of the most important contributions in the history of Joseon mathematics, work could not contribute to the development of mathematics in Joseon.

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