DOMAINS WITH INVERTIBLE-RADICAL FACTORIZATION

MALIK TUSIF AHMED AND TIBERIU DUMITRESCU

Abstract. We study those integral domains in which every proper ideal can be written as an invertible ideal multiplied by a nonempty product of proper radical ideals.

In [15] Vaughan and Yeagy introduced and studied the notion of SP-domain, i.e., an integral domain whose ideals are products of radical (also called semi-prime) ideals. They proved that an SP-domain is always almost Dedekind (i.e., every localization at a maximal ideal is a rank one discrete valuation domain (DVR)). They also gave an example of an SP-domain which is not Dedekind. For examples of almost Dedekind domains which are not SP, see [16] and [6, Example 3.4.1]. The study of SP-domains was continued by Olberding (in [12]) who gave several characterizations for SP-domains inside the class of almost Dedekind domains and also gave a method to construct SP-domains starting from Boolean topological spaces.

In a sequence of papers ([10], [11], [13]) Olberding introduced and studied the concept of ZPUI (Zerlegung Prim und Umkehrbaridealen) domain, i.e., a domain for which every proper nonzero ideal can be factored as a product of an invertible ideal times a nonempty product of pairwise comaximal prime ideals (Olberding did his study for commutative rings, but we are interested here only in domain case). He showed that a domain $A$ is ZPUI if and only if every proper nonzero ideal can be factored as a product of a finitely generated ideal times a nonempty finite product of prime ideals if and only if $A$ is a strongly discrete h-local Pru¨fer domain [13, Theorem 1.1]. Let $A$ be a domain. We recall that $A$ is h-local if the factor ring $A/I$ is local (resp. semilocal) for each nonzero prime ideal (resp. nonzero ideal) $I$ of $A$. Also $A$ is a Pru¨fer domain if its nonzero finitely generated ideals are invertible. A Pru¨fer domain is strongly discrete if it has no idempotent prime ideal except zero.

In this paper we study a new class of domains. Call a domain $A$ an ISP-domain (invertible semiprime domain) if each proper ideal of $A$ is can be written as an invertible ideal multiplied by a nonempty product of proper radical ideals. So any SP-domain (resp. ZPUI-domain) is an ISP-domain.
In Section 1 we prove the following results. If $A$ is an ISP-domain, then any factor domain of $A$ and any (flat) overring of $A$ are also ISP-domains (Propositions 2 and 3, see also Proposition 9). Any one-dimensional ISP-domain is almost Dedekind and, consequently, any Noetherian ISP-domain is a Dedekind domain (Corollary 4). In Section 2 we prove that if $A$ is an ISP-domain, then $A$ is a strongly discrete Prüfer domain and every nonzero prime ideal of $A$ is contained in a unique maximal ideal (Theorem 5). Consequently, an ISP-domain such that every ideal has finitely many minimal prime ideals is a ZPUI-domain (Corollary 10). In Section 3 we consider the question whether every one-dimensional ISP-domain is an SP-domain. We provide a positive answer for domains in which every nonzero element is contained in at most finitely many noninvertible maximal ideals (Theorem 13). In particular, a one-dimensional ISP-domain having only finitely many noninvertible maximal ideals is an SP-domain (Corollary 14). In Section 4 we give an example of a two-dimensional ISP-domain $A$ which is not h-local. Hence $A$ is neither an SP-domain nor a ZPUI-domain.

Throughout this paper, our rings are commutative and unitary. For any undefined terminology, we refer the reader to [8] or [9].

1. Basic results

We recall the key definition of our paper.

**Definition 1.** We say that a domain $A$ is an *ISP-domain* (invertible semiprime domain) if every proper nonzero ideal $I$ of $A$ can be written as $JQ_1 \cdots Q_n$ where $n \geq 1$, $J$ is an invertible ideal and each $Q_i$ is a proper radical ideal.

Clearly a ZPUI-domain or an SP-domain is an ISP-domain. The well-known Bezout domain $A = \mathbb{Z} + X\mathbb{Q}[X]$ (see [4] for its basic properties) is not an ISP-domain. Indeed, consider the ideal $I = X\mathbb{Z}[1/2] + X^2\mathbb{Q}[X]$. The radical ideals containing $I$ are $X\mathbb{Q}[X]$ and $nA = n\mathbb{Z} + X\mathbb{Q}[X]$ with $n$ a positive square-free integer. So there is no element $f \in A$ such that $I \subseteq fA$ and $If^{-1}$ is a product of radical ideals. Note that every proper nonzero principal ideal $gA$ can be written in the form required by Definition 1. Indeed, if $g \not\in X\mathbb{Q}[X]$, then $g$ is a product of principal primes and if $g \in X\mathbb{Q}[X]$, then $g = 2(g/2)A$. Note also that $A$ is strongly discrete.

In this section we prove a few basic properties of ISP-domains.

**Proposition 2.** If $A$ is an ISP-domain and $P$ a prime ideal of $A$, then $A/P$ is an ISP-domain.

**Proof.** Let $I \supset P$ be a proper ideal of $A$. As $A$ is an ISP-domain, we can write $I = JH_1 \cdots H_n$ with $J$ an invertible ideal, $n \geq 1$ and each $H_i$ a proper radical ideal. Since all ideals $I, H_1, \ldots, H_n$ contain $P$, we get $I/P = (J/P)(H_1/P) \cdots (H_n/P)$ with $J/P$ invertible and each $H_i/P$ a proper radical ideal. □
Proposition 3. Let $A$ be an ISP-domain and $B$ a flat overring of $A$. Then $B$ is an ISP-domain.

Proof. Let $H$ be a proper nonzero ideal of $B$ and $I = H \cap A$. By [2, Theorem 2], $IB = H$. As $A$ is an ISP-domain, we can write $I = JQ_1 \cdots Q_n$ with $J$ an invertible ideal, $n \geq 1$ and all $Q_i$’s proper radical ideals. Then $H = IB = (JB)(Q_1B)\cdots(Q_nB)$, where $JB$ is invertible and each $Q_iB$ is a radical ideal. Indeed, since $A_M \cap A = B_M$ for every $M \in \text{Max}(B)$ (cf. [2, Theorem 2]), it is easy to check locally that a radical ideal of $A$ extends to a radical ideal of $B$. If every $Q_iB$ is equal to $B$, then $H = JB$ and $WB = B$ where $W = Q_1 \cdots Q_n$. Hence $J \subseteq JB \cap A = H \cap A = I = JW \subseteq J$, so $J = JW$, thus $W = A$ (because $J$ is invertible), a contradiction. □

We give a simple application of Proposition 3.

Corollary 4. Any one-dimensional ISP-domain is almost Dedekind. Consequently, a Noetherian ISP-domain is a Dedekind domain.

Proof. Let $A$ be a one-dimensional ISP-domain. By Proposition 3, we may assume that $A$ is local with maximal ideal $M$. Let $x \in M - \{0\}$. Since the radical ideals of $A$ are 0 and $M$, we get $xA = yM^k$ for some $y \in A$ and $k \geq 1$, so $M$ is invertible, hence $A$ is a DVR. For the “Consequently” part, assume, by the contrary, that $A$ is a Noetherian ISP-domain which is not Dedekind. By the first part, $\dim(A) \geq 2$, so, using Proposition 3, we may assume that $A$ is a two-dimensional local domain (with maximal ideal $M$). Let $x \in M - M^2$, $P$ a height one prime ideal containing $x$ and let $y \in M - P$. Since $P \not\subseteq M^2$, $M$ is minimal over $(P,y^2)$ and $A$ is an ISP-domain, we get $(P,y^2) = M$. Modding out by $P$, we get a contradiction. □

2. ISP domains are Prüfer strongly discrete

The following theorem is the main result of this paper.

Theorem 5. If $A$ is an ISP-domain, then
(a) $A$ is a strongly discrete Prüfer domain, and
(b) every nonzero prime ideal of $A$ is contained in a unique maximal ideal.

In particular, a local domain is an ISP-domain if and only if it is a strongly discrete valuation domain.

We need a string of three lemmas.

Lemma 6. If $A$ is an ISP-domain and $P \subset M$ are nonzero prime ideals of $A$, then $P \subseteq M^2A_M$.

Proof. By Proposition 3, we may assume that $A$ is local with maximal ideal $M$. Assume that $P \not\subseteq M^2$ and take $x \in M - P$. Since $A$ is an ISP-domain and $P \not\subseteq M^2$, we get that $(P,x^2)$ is a radical ideal, so $(P,x^2) = (P,x)$ which gives a contradiction after modding out by $P$. □
Lemma 7. Let $A$ be an ISP-domain, $P \subset M$ prime ideals and $x \in M - P$ such that $M$ is minimal over $(P, x)$. Then $MA_M$ is a principal ideal.

Proof. By Proposition 3, we may assume that $A$ is local with maximal ideal $M$. We show first that $M$ is not idempotent. On contrary assume that $M^2 = M$. Note that $\sqrt{(P, x)} = M$ is the only radical ideal containing $(P, x)$. As $A$ is an ISP-domain and $M = M^2$, we get $(P, x) = yM$ for some $y \in A$. As $P \subseteq yM$, we get $y \notin P$ (otherwise $P = yA \subseteq yM$), hence $P = Py$. From $x \in yM$, we get $x = yz$ for some $z \in M$. Now from $(Py, yz) = yM$, we get $(P, z) = M$, so $M/P$ is a principal idempotent nonzero maximal ideal of $A/P$, a contradiction. Thus $M$ is not idempotent and let us pick $w \in M - M^2$. By Lemma 6, $M$ is the only prime ideal containing $w$, so $wA = M$ because $A$ is an ISP-domain. □

Lemma 8. If $A$ is an ISP-domain and $I$ an invertible radical proper ideal of $A$, then $A/I$ is zero-dimensional.

Proof. On contrary assume that $\dim(A/I) \geq 1$. Then there exist two prime ideals $P \subset M$ and $x \in M - P$ such that $I \subseteq P$ and $M$ is minimal over $(P, x)$. By Lemma 7, $MA_M$ is principal. Localizing at $M$, we may assume that $A$ is local with maximal ideal $M$. Then $I = yA$ and $M = zA$ for some $y, z \in A$. As $I \subset M$, we get $y = az^2$ for some $a \in A$, so $az \in \sqrt{yA} = yA$, hence $y = az^2 \in yzA$, thus $1 \in zA = M$, a contradiction. □

Proof of Theorem 5. (a) By [13, Lemma 3.2], it suffices to show that $PA_P$ is a principal ideal for every nonzero prime ideal $P$ of $A$. Set $B = A_P$ and $M = PA_P$. By Proposition 3, $B$ is an ISP-domain. Given $x \in M - \{0\}$, we write $xB = yH_1 \cdots H_n$ with $y \in B$, $n \geq 1$ and $H_i$ a proper radical ideal for $i = 1$ to $n$. Then each $H_i$ is invertible hence principal, because $B$ is local. By Lemma 8, we have $Spec(B/H_i) = \{M/H_1\}$, hence $H_1 = \sqrt{H_1} = M$.

(b) By Proposition 3, we may assume that $A$ is semilocal. Indeed, if $M_1$ and $M_2$ are two distinct maximal ideals containing a nonzero prime ideal, then (b) fails for $A_2$, where $S = A - (M_1 \cup M_2)$. Now let $I$ be a nonzero radical ideal. Since $A$ is a semilocal Prüfer domain, it follows that $I$ has finitely many minimal primes, say $P_1, \ldots, P_n$. Then $I = P_1 \cap \cdots \cap P_n = P_1 \cdot P_n$ because $P_1, \ldots, P_n$ are incomparable prime ideals in a Prüfer domain, hence pairwise comaximal. Since $A$ is an ISP-domain and every nonzero radical ideal is a product of primes, $A$ is a ZPUI-domain. By [13, Theorem 1.1], $A$ is h-local, so (b) holds. The “in particular” assertion follows from [13, Theorem 1.1]. □

We give two corollaries of Theorem 5.

Corollary 9. Any overring of an ISP-domain is also an ISP-domain.

Proof. Let $A$ be an ISP-domain and $B$ an overring of $A$. By Theorem 5, $A$ is a Prüfer domain, so $B$ is $A$-flat, cf. [14, page 798]. Apply Proposition 3. □

Corollary 10. For a domain $A$, the following are equivalent.

(a) $A$ is a ZPUI-domain.
(b) $A$ is an $h$-local strongly discrete Prüfer domain.
(c) $A$ is an $h$-local ISP-domain.
(d) $A$ is a generalized Dedekind ISP-domain.
(e) $A$ is an ISP-domain such that $\text{Min}(I)$ is finite for each ideal $I$.

Proof. (a) $\iff$ (b) is a part of [13, Theorem 1.1]. Implications [(a) and (b)] $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) are well-known. For (e) $\Rightarrow$ (a), repeat the second half of the proof of Theorem 5 part (b). $\square$

3. Almost Dedekind ISP-domains

In this section, we consider the question whether any one-dimensional ISP-domain is an SP-domain. First, we recall some terminology from [12]. Let $A$ be an almost Dedekind domain. The maximal ideals of $A$ containing a radical invertible ideal are called non-critical, while the others are called critical. Given $I$ an ideal of $A$ and $n \geq 1$, we set $V_n(I) = \{M \in \text{Max}(A) \mid I \subseteq M^n\}$. Note that $V_{n+1}(I) \subseteq V_n(I)$ and $V_1(I)$ is the usual Zariski closed set $V(I)$. Next, we recall [12, Theorem 2.1] and add a new assertion (g).

Theorem 11 ([12, Theorem 2.1]). For an almost Dedekind domain $A$, the following assertions are equivalent.

(a) $A$ is an SP-domain.
(b) $A$ has no critical maximal ideals.
(c) The radical of an invertible ideal is invertible.
(d) Every principal ideal is a product of radical ideals.
(e) For every nonzero proper (principal) ideal $I$ and $n \geq 1$, the set $V_n(I)$ is (Zariski) closed in $\text{Spec}(A)$ and $V_m(I)$ is empty for some large $m$.
(f) Every nonzero proper ideal $I$ can be factorized (uniquely) as $I = J_1 J_2 \cdots J_n$ with radical ideals $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n$.
(g) For every nonzero proper ideal $I$, we have $I = \sqrt{IH}$ for some ideal $H$.

Proof. Since only (g) is new, it suffices to prove the equivalence of (f) and (g).

(g) $\Rightarrow$ (f) We have $I = \sqrt{IH_1}$ and $H_1 = \sqrt{H_1 H_2}$ for some ideals $H_1$ and $H_2$. Set $J_1 = \sqrt{I}$ and $J_2 = \sqrt{H_1}$, so $I = J_1 J_2 H_2$. From $I \subseteq H_1$, we get $J_1 \subseteq J_2$. Repeating, we get $I = J_1 J_2 \cdots J_n$ with radical ideals $J_1 \subseteq \cdots \subseteq J_n$. If some $H_n$ is $A$, we are done. If not, let $M$ be a maximal ideal containing all $J_i$'s. Then $I = J_1 J_2 \cdots J_n H_n \subseteq H_n$ for each $n \geq 1$, which is a contradiction because $A_M$ is a DVR. Conversely, from $I = J_1 \cdots J_n$ with $J_1 \subseteq \cdots \subseteq J_n$ radical ideals, we get $\sqrt{I} = J_1$, so we are done. $\square$

In the next lemma, we recall two known facts.

Lemma 12. If $A$ is an almost Dedekind domain which is not Dedekind, then

(a) Every noninvertible nonzero ideal of $A$ is contained in some noninvertible maximal ideal.
(b) Every infinite closed subset of $\text{Max}(A)$ contains some noninvertible maximal ideal.
Proof. (a) is a well-known application of Zorn’s Lemma (every non finitely generated ideal is contained in a non finitely generated prime ideal).

(b) Let \( I \) be a nonzero ideal such that \( V(I) \) is infinite. By (a), we may assume that \( I \) is invertible, so the assertion follows from \([6, \text{Proposition 3.2.2}]\). We give an alternative proof. For each \( P \in V(I) \), we have \( IA_P = (PA_P)^{n_P} \) for some (unique) positive integer \( n_P \). Consider the ideal \( H = \sum_{P \in V(I)} IP^{-n_P} \). It suffices to show that \( H \) is not finitely generated, because \( I \subseteq H \) implies \( V(H) \subseteq V(I) \), so part (a) applies. Suppose that \( H \) is finitely generated. Then there exist distinct ideals \( P_1, \ldots, P_{k+1} \in V(I) \) such that \( IP_{k+1}^{-n_{k+1}} \subseteq \sum_{i=1}^{k} IP_i^{-n_i} \) where \( n_j = n_{P_j} \). Since the ideals \( P_j \) are mutually comaximal, we have \( IP_{k+1}^{-n_{k+1}} \subseteq I(\cap_{i=1}^{k} P_i^{n_i})^{-1} \), cf. \([10, \text{Lemma 5.1}]\). We cancel \( I \) and get \( \cap_{i=1}^{k} P_i^{n_i} \subseteq P_{k+1} \), which is a contradiction. \( \square \)

Recall that a domain \( A \) has weak factorization, if every nonzero nondivisorial ideal \( I \) can be factored as the product of its divisorial closure \( I_0 \) and a finite product of maximal ideals; i.e., \( I = I_0M_1M_2 \cdots M_n \) where \( M_1, M_2, \ldots, M_n \) are maximal ideals, cf. \([5]\). By \([6, \text{Proposition 4.2.14}]\), an almost Dedekind domain \( A \) has weak factorization if and only if every nonzero element of \( A \) is contained in at most finitely many noninvertible maximal ideals.

Now let \( A \) be an almost Dedekind domain \( A \) which has weak factorization. Denote by \( Z \) the set of noninvertible maximal ideals of \( A \). We introduce an ad-hoc concept: call an ideal \( H \) of \( A \) a clean ideal, if \( H \) is invertible, \( V(H) \cap Z = \{ M \} \) and \( H \nsubseteq M^2 \). Let \( M \in Z \) and \( f \in M - \{ 0 \} \). By our hypothesis \( V(f) \cap Z \) is finite, say equal to \( \{ M, M_1, \ldots, M_n \} \). By Prime Avoidance Lemma (e.g. \([8, \text{Proposition 4.9}]\)), we can pick an element \( g \in M - (M^2 \cup M_1 \cup \cdots \cup M_n) \), so \((f, g)\) is clean. Hence every \( M \in Z \) contains a clean ideal. With terminology and notation above, we have:

**Theorem 13.** For an almost Dedekind domain \( A \) which has weak factorization, the following assertions are equivalent.

(a) \( A \) is an SP-domain.

(b) \( A \) is an ISP-domain.

(c) For every clean ideal \( H \), the set \( V_2(H) \) is finite.

(d) Every \( M \in Z \) contains a clean ideal \( H \) such that \( V_2(H) \) is finite.

**Proof.** We may assume that \( A \) is not a Dedekind domain. Set \( F = \text{Max}(A) - Z \).

(a) \( \Rightarrow \) (b) is obvious. (b) \( \Rightarrow \) (c) Assume, to the contrary, that \( H \) is a clean ideal and \( V_2(H) \) contains an infinite set \( \{ P_n \mid n \geq 1 \} \subseteq F \). Set \( V(H) \cap Z = \{ M \} \). Let \( I \) be the (integral) ideal \( \sum_{n \geq 0} HP_{2n+1}^{-1} \). Since \( H \nsubseteq I \) and \( V(H) \cap Z = \{ M \} \), we get \( V(I) \cap Z = \{ M \} \), because \( M \geq H = P_{2n+1}HP_{2n+1}^{-1} \) implies \( M \geq HP_{2n+1}^{-1} \).

As \( A \) is an ISP-domain, we can write \( I = JQ \) with \( J \) an invertible ideal and \( Q \neq A \) a product of radical ideals. Since \( M \in V(I) - V_2(I) \), we have one of the two cases below.
Case 1: $M \supseteq J$ and $M \not\supseteq Q$. Then $V(Q) \cap Z$ is empty, so $Q$ is invertible, cf. Lemma 12. So $I = JQ$ is invertible, hence finitely generated. Then $HP_{-1}^{2n+1} \subseteq HP_{-1}^1 + \cdots + HP_{-1}^{2n-1}$ for some $n \geq 1$. Since $H$ can be cancelled and the other ideals involved are invertible and comaximal, we get $P_{2n+1}^{-1} \subseteq (P_1 \cap \cdots \cap P_{2n-1})^{-1}$ (cf. [10, Lemma 5.1]), hence $P_{2n+1} \supseteq P_1 \cap \cdots \cap P_{2n-1}$, which is a contradiction.

Case 2: $M \not\supseteq J$ and $M \supseteq Q$. Since $H \subseteq Q$ and $H \not\subset M^2$, we have that $V_2(Q) \cap Z = \emptyset$. As $Q$ is a product of radical ideals, [1, Lemma 1.10] shows that $V_2(Q)$ is closed, so $V_2(Q)$ is finite, cf. Lemma 12. Note that $P_{2n} \in V_2(I)$ for every $n \geq 1$. Consequently, there exists some $m \geq 1$ such that $P_{2n} \in V(J)$ for each $n \geq m$. By Lemma 12 and the fact that $H \subseteq J$, we get $V(J) \cap Z = \{M\}$, which is a contradiction.

(c) $\Rightarrow$ (d) is clear.

(d) $\Rightarrow$ (a) By [12, Theorem 2.1], it suffices to show that each $M \in Z$ contains an invertible radical ideal. By (d), $M$ contains a clean ideal $H$ such that $V_2(H)$ is finite, say equal to $\{P_1, \ldots, P_n\}$. For each $i$ between 1 and $n$, we have $HA_{P_i} = P_{k_i}A_{P_i}$ for some $k_i \geq 2$. Then $HP_{-k_1} \cdots P_{-k_n}$ is an invertible radical ideal contained in $M$.

\[ \Box \]

The SP-domain $A$ constructed in [12, Example 4.3] has nonzero Jacobson radical and no $M \in \text{Max}(A)$ finitely generated. Thus $A$ does not have weak factorization.

**Corollary 14.** Let $A$ be almost Dedekind domain having only finitely many noninvertible maximal ideals. Then $A$ is an ISP-domain if and only if $A$ is an SP-domain.

**Corollary 15.** Let $A$ be an ISP-domain which has weak factorization and $B$ a one-dimensional overring of $A$. Then $B$ is an SP-domain.

**Proof.** By Theorem 5, $A$ is a strongly discrete Prüfer domain, so $B$ has weak factorization, cf. [6, Corollary 4.3.3]. Now apply Corollary 9 and Theorem 13.

\[ \square \]

The following question remains.

**Question 16.** Is every one-dimensional ISP-domain an SP-domain?

4. An example

In this final section we give an example of a two-dimensional ISP-domain $A$ which is not h-local. Hence $A$ is neither an SP-domain nor a ZPUI-domain.

**Proposition 17.** Let $C$ be an SP-domain but not Dedekind, $M = qC$ a maximal principal ideal of $C$ and $D$ a DVR with quotient field $C/M$. Assume there exists a unit $p$ of $C$ such that $\pi(p)$ generates the maximal ideal of $D$, where $\pi : C \to C/M$ is the canonical map. Then the pull-back domain $A = \pi^{-1}(D)$ is a two-dimensional ISP-domain which is not h-local.
Proof. As \( \pi(Mp^{-1}) = 0 \), it follows that \( M \subseteq pA \), so \( A/pA \) is the residue field of \( D \), because \( A/M = D \) and \( \pi(p) \) generates the maximal ideal of \( D \). Also, the only prime ideal of \( A \) strictly containing \( M \) is the maximal ideal \( pA \). By standard pull-back arguments (see for instance [7, Lemma 1.1.4]), the map \( P \mapsto P \cap A \) is a bijection from Spec\((C)\) to Spec\((A)\) and \( AP_{pA} = C_P \). By [7, Corollary 1.1.9], \( A \) is a two-dimensional Prüfer domain. Also, by [7, Lemma 1.1.6], we have \( A[p^{-1}] = C[p^{-1}] = C \). Roughly speaking, Spec\((A)\) is obtained from Spec\((C)\) by adding the maximal ideal \( pA \supseteq M \). Since \( C \) is an almost Dedekind domain which is not Dedekind, there exists a nonzero element \( z \in A \) belonging to infinitely many maximal ideals of \( A \), so \( A \) is not h-local. By [7, Proposition 5.3.3], \( B = A_{pA} \) is a two-dimensional strongly discrete valuation domain. It follows that \( \cap_{i \geq 1} p^iA = M \).

Let \( I \) be an ideal of \( A \). We observe that \( I = IB \cap IC \). Indeed, if \( N \in \text{Max}(A) - \{ pA \} \), then \( I \subseteq IC_{A-N} = IA_N \), so \( IB \cap IC \subseteq \cap_{Q \in \text{Max}(A)} IA_Q = I \).

In particular, we have \( A = B \cap C \). Since \( C \) is almost Dedekind and \( M = qC \), we can write \( IC = M^iJ \) where \( J \) is an ideal of \( C \) with \( M + J = C \) and \( i \geq 0 \), so \( IC = M^i \cap J \). We also see that \( H := J \cap A \not\subseteq M \). As \( \cap_{i \geq 1} p^iA = M \), we can write \( H = p^iL = p^iA \cap L \) where \( L \) is an ideal of \( A \) with \( L \not\subseteq pA \) and \( j \geq 0 \).

Consequently we get
\[
IC \cap A = M^i \cap J \cap A = M^i \cap H = M^i \cap p^jA \cap L
\]
which equals either \( M^i \cap L \) if \( i \geq 1 \) or \( p^jA \cap L \) if \( i = 0 \). Using basic facts on valuation domains (see [8, Section 17]), it suffices to consider the following three cases. Each time we use the equality \( I = (IB \cap A) \cap (IC \cap A) \).

Case 1: \( IB = p^nB \) for some \( n \geq 0 \). We have \( IB \cap A = p^nA \). If \( i \geq 1 \), we get \( I = p^nA \cap M^i \cap L = M^i \cap p^nA \cap L \) with \( k = \text{max}(n, i) \).

Case 2: \( IB = M^n \) for some \( n \geq 1 \). If \( i \geq 1 \), we get \( I = M^n \cap M^i \cap L = M^kL \) with \( k = \text{max}(n, i) \). If \( i = 0 \), we get \( I = M^n \cap p^kA \cap L = M^nL \).

Case 3: \( IB = p^nq^mA \) for some \( m \geq 1 \) and \( n \in \mathbb{Z} \). We have \( IB \cap A = p^nq^mA \), because \( pA \) is the only maximal ideal containing \( q \). If \( i > m \), we get \( I = p^nq^mA \cap M^i \cap L = M^iL \). If \( m \geq i \geq 1 \), we get \( I = p^nq^mA \cap M^i \cap L = p^nq^mA \cap L \). If \( i = 0 \), we get \( I = p^nq^mA \cap p^kA \cap L = p^nq^mA \cap L \).

Consequently, to complete our proof, it suffices to show that \( L \) is a product of radical ideals. Since \( C \) is an SP-domain, we can write \( LC = H_1 \cdots H_n \) with each \( H_i \) a radical ideal of \( C \). Then each \( J_i = H_i \cap A \) is a radical ideal of \( A \). Note that none of ideals \( J_i \) is contained in \( pA \), since \( L \not\subseteq pA \). Set \( R = J_1 \cdots J_n \). Then \( R + pA = A \) and \( L + pA = A \), so \( R : p = R \) and \( L : p = L \). Since \( RC = H_1 \cdots H_n = LC \), we get \( L = LC \cap A = RC \cap A = R \).

Finally, we construct a specific domain satisfying the hypothesis of Proposition 17. We modify appropriately [6, Example 3.4.1]. If \( A \) is a domain and \( P_1, \ldots, P_n \) are prime ideals of \( A \), we denote by \( A_{P_1 \cup \cdots \cup P_n} \) the fraction ring of \( A \) with denominators in \( A - (P_1 \cup \cdots \cup P_n) \). Let \( y \) and \( (x_n)_{n \geq 1} \) be indeterminates.
over the rational field \( \mathbb{Q} \). Consider the domain
\[
C = \bigcup_{n \geq 1} \mathbb{Q}[x_1, \ldots, x_n, y/(x_1 \cdots x_n)](x_1 \cup \cdots \cup (x_n) \cup (y/(x_1 \cdots x_n))).
\]
As \( C \) is a union of an ascending chain of (semi-local) PID's, it is a one-dimensional Bezout domain. Adapting the proof of [6, Example 3.4.1], we see that the maximal ideals of \( C \) are \( N = \sum_{n \geq 1} (y/(x_1 \cdots x_n))C \) and the principal ideals \((x_n)_{n \geq 1} \). As \( yC_M = MC_M \) for each \( M \in \text{Max}(C) \), it follows that \( yC \) is a maximal ideal, hence \( N \) is non-critical. By [12, Corollary 2.2], \( C \) is an SP-domain. The residue field \( C/(x_1) \) is isomorphic to \( K(y/x_1) \) where \( K = \mathbb{Q}(x_n; n \geq 2) \). Then \( D = K[y/x_1]/(y/x_1) \) is a DVR with quotient field \( C/(x_1) \). Note that \( x_1 + y/x_1 \) is a unit of \( \mathbb{Q}[x_1, y/x_1](x_1) \cup (y/x_1) \), hence a unit of \( C \). Moreover, the canonical map \( C \to C/(x_1) \) sends \( x_1 + y/x_1 \) to \( y/x_1 \) which is a generator of the maximal ideal of \( D \). Thus \( C \) satisfies the hypothesis of Proposition 17.

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References


MALIK TUSIF AHMED  
ABDUS SALAM SCHOOL OF MATHEMATICAL SCIENCES GCU LAHORE  
Pakistan  
Email address: tusif.ahmed@sms.edu.pk, tusif.ahmad92@gmail.com

TIBERIU DUMITRESCU  
FACULTATEA DE MATEMATICA SI INFORMATICA  
UNIVERSITY OF BUCHAREST  
14 ACADEMiei STR., BUCHAREST, RO 010014, ROMANIA  
Email address: tiberiu@fmi.unibuc.ro, tiberiu_dumitrescu2003@yahoo.com