MEROMORPHIC FUNCTIONS SHARING SOME FINITE SETS IM

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Abstract. We show that if two nonconstant meromorphic functions $f$ and $g$ on $\mathbb{C}$ sharing some finite sets IM, then there is a nonconstant rational function $R(z)$ such that $R(f) = R(g)$.

1. Introduction

For nonconstant meromorphic functions $f$ and $g$ on $\mathbb{C}$ and a finite set $S$ in $\mathbb{C} = \mathbb{C} \cup \{\infty\}$, we say that $f$ and $g$ share $S$ CM (counting multiplicities) if $f^{-1}(S) = g^{-1}(S)$ and if for each $z_0 \in f^{-1}(S)$ two functions $f - f(z_0)$ and $g - g(z_0)$ have the same multiplicity of zero at $z_0$, where the notations $f - \infty$ and $g - \infty$ mean $1/f$ and $1/g$, respectively. Also, if $f^{-1}(S) = g^{-1}(S)$, then we say that $f$ and $g$ share $S$ IM (ignoring multiplicities). In particular if $S$ is a one-point set $\{a\}$, then we say also that $f$ and $g$ share $a$ CM or IM.

In [3] and [4], R. Nevanlinna showed the following two theorems:

Theorem 1.1. Let $f$ and $g$ be two distinct nonconstant meromorphic functions on $\mathbb{C}$ and $a_1, \ldots, a_4$ four distinct points in $\mathbb{C}$. If $f$ and $g$ share $a_1, \ldots, a_4$ CM, then $f$ is a Möbius transform of $g$, i.e., $f = (aq + b)/(cq + d)$ for some complex numbers $a, b, c, d$ with $ad - bc \neq 0$, and there exists a permutation $\sigma$ of $\{1, 2, 3, 4\}$ such that $a_{\sigma(3)}, a_{\sigma(4)}$ are Picard exceptional values of $f$ and $g$ and the cross ratio $(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}) = -1$. Furthermore, the Möbius transformation fixes $a_{\sigma(1)}$ and $a_{\sigma(2)}$, and $a_{\sigma(3)}$ and $a_{\sigma(4)}$ interchanges under the Möbius transformation.

Theorem 1.2. Let $f$ and $g$ be two nonconstant meromorphic functions on $\mathbb{C}$ sharing distinct five points in $\mathbb{C}$ IM. Then $f = g$.

Remark 1.3. Let $T(z) = (az + b)/(cz + d)$ be a Möbius transformation of order 2, i.e., $T^2 = T \circ T$ is the identity. Then $d = -a$ and $a^2 + bc \neq 0$. This Möbius transformation has two distinct fixed points $\xi_1, \xi_2$ in $\mathbb{C}$. Let $T_0$ be a Möbius transformation such that $T_0(0) = \xi_1, T_0(\infty) = \xi_2$. Then the Möbius transformation fixes $\xi_1$ and $\xi_2$, and interchanges $\infty$ with $0$.
Let $f$ be a nonconstant meromorphic function on $\mathbb{C}$. If two distinct nonconstant meromorphic functions $f$ and $g$ on $\mathbb{C}$ share $S_1, S_2, S_3, S_4$ CM, then does there exist a nonconstant rational function $R(z)$ such that $R(f) = R(g)$?

These raise the following problems:

**Problem 1.** Let $q$ be an integer not less than 5. Let $S_1, \ldots, S_q$ be pairwise disjoint finite sets in $\mathbb{C}$. If two nonconstant meromorphic functions $f$ and $g$ share $S_1, \ldots, S_q$ IM, then does there exist a nonconstant rational function $R(z)$ such that $R(f) = R(g)$?

**Problem 2.** Let $q$ be an integer not less than 4. Let $S_1, \ldots, S_q$ be pairwise disjoint finite sets in $\mathbb{C}$. If two nonconstant meromorphic functions $f$ and $g$ share $S_1, \ldots, S_q$ CM, then does there exist a nonconstant rational function $R(z)$ such that $R(f) = R(g)$?

Both problems are affirmatively answered, as shown above, for the case that the all finite sets are one-point sets or two-points sets, and also we can find similar results for polynomials in [1] and [5]. In this paper, we give a partial solution for Problem 1.

**Theorem 1.5.** Let $p$ be a non-negative integer and let $q$ be an integer not less than 2. Let $S_1, \ldots, S_p$ be one-point sets in $\mathbb{C}$ and let $S_{p+1}, \ldots, S_{p+q}$ be $n$-point sets in $\mathbb{C}$, where $n$ is an integer not less than 2. Assume that $S_1, \ldots, S_{p+q}$ are pairwise disjoint and that $p + q \geq 5$. If two distinct nonconstant meromorphic functions $f$ and $g$ on $\mathbb{C}$ share $S_1, \ldots, S_{p+q}$ IM, then there exists distinct $j_1, j_2$ in $\{p+1, \ldots, p+q\}$ such that $P_{j_2}(f)/P_{j_1}(f) = P_{j_2}(g)/P_{j_1}(g)$, where $P_j(z)$ are defining polynomials of $S_j$.

By considering a suitable Möbius transformation, we have:

**Corollary 1.6.** Let $p$ be a non-negative integer and let $q$ be an integer not less than 2. Let $S_1, \ldots, S_p$ be one-point sets in $\mathbb{C}$ and let $S_{p+1}, \ldots, S_{p+q}$ be $n$-point
sets in \( \mathbb{C} \), where \( n \) is an integer not less than 2. Assume that \( S_1,\ldots,S_{p+q} \) are pairwise disjoint and that \( p+q \geq 5 \). If two nonconstant meromorphic functions \( f \) and \( g \) on \( \mathbb{C} \) share \( S_1,\ldots,S_{p+q} \) IM, then there exists a nonconstant rational function \( R(z) \) such \( R(f) = R(g) \).

We assume that the reader is familiar with the standard notations and results of the value distribution theory (see, for example, [2]). In particular, we express by \( S(r,f) \) quantities such that \( \lim_{r \to \infty, r \not\in E} S(r,f)/T(r,f) = 0 \), where \( E \) is a subset of \((0, \infty)\) with finite linear measure and it is variable in each cases.

2. Proof of Theorem 1.5

Now we start the proof of Theorem 1.5. We may assume that \( p \leq 4 \) by Theorem 1.2.

By the second main theorem and the first main theorem we have

\[
(p + nq - 2)T(r,f) \leq \sum_{j=1}^{p+q} \sum_{\xi \in S_j} N(r, \frac{1}{f - \xi}) + S(r,f)
= \sum_{j=1}^{p+q} \sum_{\xi \in S_j} N(r, \frac{1}{g - \xi}) + S(r,f)
\leq (p + nq)T(r,g) + S(r,f)
\]

and, by the same way,

\[
(p + nq - 2)T(r,g) \leq (p + nq)T(r,f) + S(r,g).
\]

Hence, by (1) and (2), there is no need to distinguish \( S(r,f) \) and \( S(r,g) \), and so we denote them by \( S(r) \).

By \( N_E(r, \frac{1}{f - \xi}) \) and \( N_N(r, \frac{1}{g - \xi}) \) we denote the counting functions which count the point \( z \) such that \( f(z) = \xi = g(z) \) and \( f(z) = \xi \neq g(z) \) counted once, respectively, and we define \( N_E(r, \frac{1}{g - \xi}) \) and \( N_N(r, \frac{1}{g - \xi}) \) by the same way. It is easy to see that \( N_N(r, \frac{1}{f - \xi}) = N_N(r, \frac{1}{g - \xi}) = 0 \) for \( \xi \in S_1 \cup \cdots \cup S_p \) and that

\[
\sum_{\xi \in S_j} N_E(r, \frac{1}{f - \xi}) = \sum_{\xi \in S_j} N_E(r, \frac{1}{g - \xi}),
\]

\[
\sum_{\xi \in S_j} N_N(r, \frac{1}{f - \xi}) = \sum_{\xi \in S_j} N_N(r, \frac{1}{g - \xi})
\]

for \( j = p + 1, \ldots, q \). Since \( f - g \not\equiv 0 \), we have

\[
\sum_{j=1}^{p+q} \sum_{\xi \in S_j} N_E(r, \frac{1}{f - \xi}) \leq N(r, \frac{1}{f - g}) \leq T(r,f) + T(r,g) + O(1),
\]
and hence
\[ \sum_{j=p+1}^{p+q} \sum_{\xi \in S_j} N_N(r, \frac{1}{f - \xi}) = \sum_{j=1}^{p+q} \sum_{\xi \in S_j} N(r, \frac{1}{f - \xi}) - \sum_{j=1}^{p+q} \sum_{\xi \in S_j} N_E(r, \frac{1}{f - \xi}) \geq (p + nq - 2)T(r, f) - T(r, f) - T(r, g) + S(r) = (p + nq - 3)T(r, f) - T(r, g) + S(r) \]
by using (1). By the same way and (3) we have
\[ \sum_{j=p+1}^{p+q} \sum_{\xi \in S_j} N_N(r, \frac{1}{f - \xi}) \geq (p + nq - 3)T(r, g) - T(r, f) + S(r). \]
Adding these two inequalities we obtain
\[ \sum_{j=p+1}^{p+q} \sum_{\xi \in S_j} N_N(r, \frac{1}{f - \xi}) \geq \frac{1}{2}(p + nq - 4)(T(r, f) + T(r, g)) + S(r). \]
Note that \( q \geq 2 \). From (4) we see that there exist distinct \( j_1 \) and \( j_2 \) in \( \{p + 1, \ldots, q\} \) and a subset \( I \) of \((0, +\infty)\) of infinite linear measure such that
\[ \frac{1}{q}(p + nq - 4)(T(r, f) + T(r, g)) + S(r) \leq \sum_{\xi \in S_{j_1} \cup S_{j_2}} N_N(r, \frac{1}{f - \xi}) \]
holds for \( r \in I \). Put \( Q(z, w) = (P_{j_1}(z)P_{j_2}(w) - P_{j_1}(w)P_{j_2}(z))/(z - w) \) and \( \Phi = Q(f, g) \). Assume that \( \Phi \neq 0 \). If \( f(z), g(z) \in S_{j_1} \cup S_{j_2} \) and \( f(z) \neq g(z) \), then \( \Phi(z) = 0 \). Therefore we have
\[ \sum_{\xi \in S_{j_1} \cup S_{j_2}} N_N(r, \frac{1}{f - \xi}) \leq N_0(r, \frac{1}{\Phi}) \]
holds for \( r \in I \), where \( N_0(r, \frac{1}{\Phi}) \) denotes the counting functions corresponding to the zeros of \( \Phi \) that are not the poles of \( f \) and \( g \). We see that \( Q(z, w) \) is a symmetric polynomial of \( z \) and \( w \) and it has degree at most \( n - 1 \) with respect to each of \( z \) and \( w \). By using the first fundamental theorem and the definition of counting function and that of proximity function, we have
\[ N_0(r, \frac{1}{\Phi}) \leq N(r, Q(f, g)) + m(r, Q(f, g)) \leq (n - 1)(N(r, f) + N(r, g) + m(r, f) + m(r, g)) + O(1) = (n - 1)(T(r, f) + T(r, g)) + O(1). \]
By connecting (5), (6) and this,
\[ \frac{1}{q}(p + nq - 4)(T(r, f) + T(r, g)) + S(r) \leq (n - 1)(T(r, f) + T(r, g)) + O(1) \]
holds for \( r \in I \). Here \( I \) may be different from that in (5). We obtain \( p + nq - 4 \leq q(n - 1) \), which contradicts hypothesis \( p + q \geq 5 \). Therefore we conclude that \( \Phi = 0 \), which induces that \( P_{j_2}(f)/P_{j_1}(f) = P_{j_2}(g)/P_{j_1}(g) \).
3. An application to the uniqueness

In this section, we apply the above results to the uniqueness of meromorphic functions. Let \( n \) be an integer not less than 2, and let \( S_1, \ldots, S_5 \) be pairwise disjoint \( n \)-point sets in \( \mathbb{C} \). For each \( j = 1, \ldots, 5 \), we take a defining polynomial \( P_j(z) \) of \( S_j \), and let \( \xi_1 \cdots \xi_n \) be the distinct elements of \( S_j \).

**Theorem 3.1.** Assume that
\[
\frac{P_j(\xi_\mu)}{P_k(\xi_\mu)} \neq \frac{P_j(\xi_\nu)}{P_k(\xi_\nu)}
\]
for distinct \( j, k, l \in \{1, \ldots, 5\} \) and \( 1 \leq \mu < \nu \leq n \). If two nonconstant meromorphic functions \( f \) and \( g \) on \( \mathbb{C} \) share \( S_1, \ldots, S_5 \) IM, then \( f = g \).

**Proof.** Assume that \( f \neq g \). From Theorem 1, we may assume that
\[
\frac{P_1(f)}{P_2(f)} = \frac{P_1(g)}{P_2(g)},
\]
by renumbering \( S_1, \ldots, S_5 \), if necessary. By (7), there is no \( z \) such that \( f(z), g(z) \) are distinct values in \( S_3 \cup S_4 \cup S_5 \). Therefore, \( f \) and \( g \) share each values in \( S_3 \cup S_4 \cup S_5 \). This fact yields, by Theorem 1.2, \( f = g \), which is a contradiction. Hence we conclude \( f = g \). \( \square \)

**Remark 3.2.** In the case of \( n = 2 \), the assumption (7) becomes to
\[
\begin{vmatrix}
1 & a_j & b_j \\
1 & a_k & b_k \\
1 & a_l & b_l
\end{vmatrix} \neq 0,
\]
where \( P_j(z) = z^2 + a_jz + b_j \) and so on. This is a necessary and sufficient condition for the absence of a Möbius transformation exchanging two elements of each \( S_j, S_k, S_l \).

For \( n \geq 3 \), we can weaken the assumption about (7). It is enough to hold (7) for distinct two \( l \), in the case of \( n = 3, 4 \), and for one \( l \), in the case of \( n \geq 5 \), different from any given \( 1 \leq j < k \leq 5 \).

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**References**


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