BIPROJECTIVITY OF MATRIX BANACH ALGEBRAS
WITH APPLICATION TO COMPACT GROUPS

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Abstract. In this paper, the necessary and sufficient conditions are considered for biprojectivity of Banach algebras $E_p(I)$. As an application, we investigate biprojectivity of convolution Banach algebras $A(G)$ and $L^2(G)$ on a compact group $G$.

1. Introduction

The Banach algebras $E_p(I)$, where $p \in [1, \infty] \cup \{0\}$, have been introduced and extensively studied in Section 28 of [4]. Recently, amenability, weak amenability and approximate weak amenability have been studied by H. Samea in [8] (see also [5]). The present paper is going to investigate biprojectivity of Banach algebras $E_p(I)$, together with their applications to a number of convolution Banach algebras on compact groups.

Let $H$ be an $n$-dimensional Hilbert space and suppose that $B(H)$ is the space of all linear operators on $H$. Clearly we can identify $B(H)$ with $M_n(\mathbb{C})$ (the space of all $n \times n$-matrices on $\mathbb{C}$). For $A \in M_n(\mathbb{C})$, let $A^* \in M_n(\mathbb{C})$ by $(A^*)_{ij} = \overline{A}_{ji}$ ($1 \leq i, j \leq n$), and let $|A|$ denote the unique positive-definite square root of $AA^*$. $A$ is called unitary, if $A^*A = AA^* = I$, where $I$ is the $n \times n$-identity matrix. For $E \in B(H)$, let $(\lambda_1, \ldots, \lambda_n)$ be the sequence of eigenvalues of the operator $|E|$, written in any order. Define $\|E\|_{\infty} = \max_{1 \leq i \leq n} |\lambda_i|$, and $\|E\|_p = (\sum_{i=1}^{n} |\lambda_i|^p)^{\frac{1}{p}}$ ($1 \leq p < \infty$). For more details see Definition D.37 and Theorem D.40 of [4].

Let $I$ be an arbitrary index set. For each $i \in I$, let $H_i$ be a finite dimensional Hilbert space of dimension $d_i$, and let $a_i \geq 1$ be a real number. The $*$-algebra $\prod_{i \in I} B(H_i)$ will be denoted by $E(I)$; scaler multiplication, addition, multiplication,
and the adjoint of an element are defined coordinate-wise. Let $E = (E_i)$ be an element of $\mathcal{E}(I)$. We define $\|E\|_p := (\sum_{i \in I} a_i \|E_i\|_p^p)^{\frac{1}{p}}$ $(1 \leq p < \infty)$, and $\|E\|_\infty = \sup_{i \in I} \|E_i\|_{\mathcal{E},\infty}$. For $1 \leq p \leq \infty$, $\mathcal{E}_p(I)$ is defined as the set of all $E \in \mathcal{E}(I)$ for which $\|E\|_p < \infty$.

For a locally compact group $G$ and a function $f : G \to \mathbb{C}$, $\hat{f}$ is defined by $\hat{f}(x) = f(x^{-1})$ ($x \in G$). Let $A(G)$ (or $\mathcal{A}(G)$, defined in 35.16 of [4]) consists of all functions $h$ in $C_0(G)$ that can be written in at least one way as $\sum_{n=1}^{\infty} f_n \ast \hat{g}_n$, where $f_n, g_n \in L^2(G)$, and $\sum_{n=1}^{\infty} \|f_n\|_2 \|g_n\|_2 < \infty$. For $h \in A(G)$, define

$$\|h\|_{A(G)} = \inf \left\{ \sum_{n=1}^{\infty} \|f_n\|_2 \|g_n\|_2 : h = \sum_{n=1}^{\infty} f_n \ast \hat{g}_n \right\}.$$ 

With this norm $A(G)$ is a Banach space. For more details see 35.16 of [4].

As [1], let $(A, \|\cdot\|)$ be a normed algebra, and let $I_1, \ldots, I_n$ be ideals in $A$, then $I_1 \ast \ldots \ast I_n$ is an ideal in $A$; we transfer the projective norm from $I_1 \otimes \ldots \otimes I_n$ into $I_1 \ast \ldots \ast I_n$. So that, for $A \in I_1 \ast \ldots \ast I_n$, we have

$$\|a\|_\pi = \inf \left\{ \sum_{j=1}^{m} \|a_{1,j} \ast \ldots \ast a_{n,j}\| : a = \sum_{j=1}^{m} a_{1,j} \ast \ldots \ast a_{n,j}, a_{i,j} \in I_i \right\}.$$ 

Clearly $\|\cdot\|_\pi$ is an algebra norm on $I_1 \ast \ldots \ast I_n$ with $\|a\| \leq \|a\|_\pi$ $(a \in I_1 \ast \ldots \ast I_n)$; the norm $\|\cdot\|_\pi$ is again called the projective norm. In particular, we may consider $\|\cdot\|_\pi$ on $A^2$. Let $A$ be a Banach algebra. Then the continuous linear map $\pi_A : A \otimes A \to A$ such that $\pi_A(x \otimes y) = ab$ $(a, b \in A)$ is the projective induced product map and $I_\pi = \ker \pi_A$. The quotient norm on the image $\pi_A(A \otimes A) \cong \frac{(A \otimes A)}{\ker \pi_A}$ is denoted by $\|\cdot\|_{\pi}$, so that

$$\|a\|_{\pi} = \inf \left\{ \sum_{j=1}^{\infty} \|a_j\| \|b_j\| : a = \sum_{j=1}^{\infty} a_j b_j \right\} (a \in \pi_A(A \otimes A)).$$

Note that by 2.1.15 of [1],

$$\|a\| \leq \|\|a\|\|_\pi \leq \|a\|_\pi (a \in A^2).$$

A normed algebra $A$ has $S$-property ($\pi$-property) if there is a constant $C > 0$ such that

$$\|a\|_\pi \leq C \|a\| (\|\|a\|\|_\pi \leq C \|a\|) (a \in A^2).$$
Clearly, if \( A \) has \( S \)-property, then \( A \) has \( \pi \)-property. A Banach algebra \( A \) is biprojective if \( \pi_A : A \hat{\otimes} A \longrightarrow A \) has a bounded right inverse as an \( A \)-bimodule homomorphism. By proposition 2.8.41 of [1], if \( A \) is biprojective then \( \pi_A(A \hat{\otimes} A) = A \) and \( A \) has \( \pi \)-property.

2. Main Results

In this section, among other results, we obtain the necessary and sufficient conditions such that \( \mathfrak{E}_p(I) \) for \( p \geq 1 \), has \( \pi \)-property and as a result we apply \( \pi \)-property of \( \mathfrak{E}_p(I) \) to find the necessary and sufficient conditions for biprojectivity of \( \mathfrak{E}_p(I) \).

**Theorem 2.1.** Suppose that \( p \geq 1 \) and \( A \in \mathfrak{E}^2_p(I) \). Then there are \( B, C \in \mathfrak{E}_p(I) \) such that \( A = B.C \) and \( \|B\|_p = \|C\|_p = \|A\|^{\frac{1}{2}}_p \).

**Proof.** First suppose \( p \neq \infty \). By Notation D.26 (i) of [4], for \( i \in I, |A_i| \) can be written uniquely in the form \( |A_i| = \sum^n_{j=1} b_j^i Q_j^i \), where the \( b_j^i \)'s are distinct positive numbers and \( Q_j^i \)'s are projections onto pairwise orthogonal nonzero subspaces of \( H_i \) and \( |A_i|^{\frac{1}{2}} = \sum^n_{j=1} (b_j^i)^{\frac{1}{2}} Q_j^i \). Therefore, \( |A_i| = |A_i|^{\frac{1}{2}} |A_i|^{\frac{1}{2}} \). For \( i \in I \), according to the polar decomposition, there is \( W_i \in \mathcal{U}(H_i) \) (the set of all unitary operators on \( H_i \)) such that

\[
A_i = |A_i| \cdot W_i = |A_i|^{\frac{1}{2}} \cdot |A_i|^{\frac{1}{2}} \cdot W_i.
\]

Let \( B_i = |A_i|^{\frac{1}{2}} \) and \( C_i = |A_i|^{\frac{1}{2}} \cdot W_i \). By Lemma 1.1 of [?] 

\[
\|B_i\|_p^p = \| |A_i|^{\frac{1}{2}} \|_p^p = \|A_i\|^{\frac{p}{2}}_p,
\]

therefore,

\[
\|B\|_p = \left( \sum_i a_i \|B_i\|_p^p \right)^{\frac{1}{p}} = \left( \sum_i a_i \|A_i\|_p^{\frac{p}{2}} \right)^{\frac{2}{p}} = \|A\|^{\frac{2}{p}}_p < \infty.
\]

So \( B \in \mathfrak{E}_p(I) \). The rest of the proof follows easily from Theorem D. 41 of [4] and Lemma 1.1 of [5]. For \( p = \infty \) the proof is similar.

**Corollary 2.2.** If \( p \geq 1 \), then \( \mathfrak{E}^2_p(I) \subseteq \mathfrak{E}_p(I) \mathfrak{E}_p(I) \).

Let \( A \) be a Banach algebra. We set \( A^{[2]} = A.A = \{ab : a, b \in A \} \) and \( A^2 = \text{lin} A^{[2]} = \text{lin} A.A = \{ \sum^n_{i=1} \alpha_i a_i b_i : \alpha_1, ..., \alpha_n \in \mathbb{C}, a_1, ..., a_n, b_1, ..., b_n \in A \} \).

**Theorem 2.3.** If \( p \geq 1 \), then \( \mathfrak{E}^2_p(I) = \mathfrak{E}^{[2]}_p(I) = \mathfrak{E}^2_p(I) \).
Proof. It is enough to show that if $E, F \in \mathcal{E}_p(I)$, then $EF \in \mathcal{E}_q(I)$. By using Theorem 2·3 of [5] for $p = q$, and applying Hölder inequality, we obtain
\[
\|EF\|_{\mathcal{P}_q}^p = \sum_i a_i \|(EF)_i\|_{\mathcal{P}_q}^p \\
\leq \sum_i a_i \|E_i\|_{\mathcal{P}_p}^p \|F_i\|_{\mathcal{P}_p}^p \\
\leq \sum_i a_i^\frac{1}{2} \|E_i\|_{\mathcal{P}_p} a_i^\frac{1}{2} \|F_i\|_{\mathcal{P}_p}^\frac{1}{2} \\
\leq \left( \sum_i a_i \|E_i\|_{\mathcal{P}_p}^p \right)^\frac{1}{2} \left( \sum_i a_i \|F_i\|_{\mathcal{P}_p}^p \right)^\frac{1}{2} \\
= \left( \|E\|_{\mathcal{P}_p}^p \right)^\frac{1}{2} \left( \|F\|_{\mathcal{P}_p}^p \right)^\frac{1}{2} < \infty.
\]
\[\square\]

Theorem 2.4. If $r > p \geq 1$, then $\mathcal{E}_p(I) \subseteq \mathcal{E}_r(I)$.

Proof. By Theorem 28.32 of [4], $\mathcal{E}_p(I) \subseteq \mathcal{E}_r(I)$. Let $A \in \mathcal{E}_r(I)$ and $B \in \mathcal{E}_p(I)$. For each $i \in I$, we denote the sequence of eigenvalues of $A_i$ by $s_j(A_i)$. Now, if $A_i, B_i \in B(H_i)$, then by 2.2 and 2.3 of [3],
\[
s_j(A_iB_i) \leq \|A_i\|_{\mathcal{P}_\infty} s_j(B_i), \\
s_j(B_iA_i) \leq \|A_i\|_{\mathcal{P}_\infty} s_j(B_i).
\]
Thus
\[
\|A_iB_i\|_{\mathcal{P}_p} = \left( \sum_j s_j(A_iB_i)^p \right)^\frac{1}{p} \leq \left( \sum_j \|A_i\|_{\mathcal{P}_\infty}^p s_j(B_i)^p \right)^\frac{1}{p} = \|A_i\|_{\mathcal{P}_\infty} \|B_i\|_{\mathcal{P}_p}.
\]
But $A \in \mathcal{E}_r(I)$, hence $A \in \mathcal{E}_\infty(I)$ and
\[
\|AB\|_{\mathcal{P}_p}^p = \sum_i a_i \|A_iB_i\|_{\mathcal{P}_p}^p \leq \sum_i a_i \|A_i\|_{\mathcal{P}_\infty}^p \|B_i\|_{\mathcal{P}_p}^p \leq \|A\|_{\mathcal{P}_\infty}^p \|B\|_{\mathcal{P}_p}^p < \infty.
\]
Therefore, $AB \in \mathcal{E}_p(I)$ and the proof is complete. \[\square\]

Corollary 2.5. If $p \geq 1$, then $\mathcal{E}_2(I) \subseteq \mathcal{E}_p(I)$.

Let $\|\cdot\|_{\mathcal{P}_p}$ and $\|\cdot\|_{\mathcal{P}_\infty}$ be the projective norms on $\mathcal{E}_p(I)\mathcal{E}_p(I)$ and the quotient norm from $\mathcal{E}_p(I)\mathcal{E}_p(I)$, respectively. Let
\[
\mathcal{U}(\mathcal{E}(I)) = \{ (E_i)_{i\in I} \in \mathcal{E}(I) : E_i \in \mathcal{U}(H_i) \},
\]
$U, V \in \mathcal{U}(\mathcal{E}(I))$ and $E \in \mathcal{E}_p(I)$. By Theorem D.41 of [4], we have
\[
\|VEU\|_p = \left( \sum_i a_i \|VE_iU_i\|_{\mathcal{P}_p}^p \right)^\frac{1}{p} = \left( \sum_i a_i \|E_i\|_{\mathcal{P}_p}^p \right)^\frac{1}{p} = \|E\|_p.
\]
By polar decomposition, for $i \in I$, there is a unitary operator $U_i$ such that $U_iE_i = |E_i|$. Let $U = (U_i)_{i \in I}$ then
\begin{equation}
\|E\|_{\pi,p} = \|UE\|_{\pi,p} = \|(|E_i|)_{i \in I}\|_{\pi,p}.
\end{equation}
Since the square root of a matrix is hermitian, then is diagonalizable, i.e. there is a unitary operator $V_i$ such that $V_i^{-1}|E_i|V_i = T_i$, where $T_i$ is a diagonal matrix. Let $V = (V_i)_{i \in I}$. Then
\begin{equation}
\|(|E_i|)_{i \in I}\|_{\pi,p} = \|V|E||_{\pi,p} = \|(|T_i|)_{i \in I}\|_{\pi,p}.
\end{equation}
By (2.1), $\|E\|_{\pi,p} = \|(|T_i|)_{i \in I}\|_{\pi,p}$. By the similar procedure, we can prove that $\|E\|_{\pi,p} = \|(|T_i|)_{i \in I}\|_{\pi,p}$. Consequently, for analyzing $\|.|_{\pi,p}$ and $\|.|_{\pi,p}$ it is enough to focus on $E = (E_i)_{i \in I}$ of $\mathcal{E}_p(I)$, where each $E_i$ is a diagonal matrix with positive diagonal entries.

For the rest of the section we set $\tilde{p} = \max\{1, \frac{p}{2}\}$.

**Theorem 2.6.** Let $2 \leq p < \infty$. Then for each $E \in \mathcal{E}_p(I)$,
\begin{equation}
\|E\|_{\pi,p} = \|E\|_{\pi,p} = \|E\|_{\tilde{p}}.
\end{equation}

**Proof.** Suppose $2 \leq p < \infty$ and $E \in \mathcal{E}_p(I)$. By Theorem 2.3, $E \in \mathcal{E}_{\frac{p}{2}}(I)$. Using Theorem 2.1, it follows that $\|E\|_{\pi,p} \leq \|E\|_{\tilde{p}}$. Also, if $E = \sum_{j=1}^{\infty} F(j)K(j)$ in $\mathcal{E}_p(I)$ with $\sum_{j=1}^{\infty} \|F(j)\|_p \|K(j)\|_p < \infty$, then by Theorem 28.3 of [4], we have
\begin{align*}
\|E\|_{\tilde{p}} &= \|E\|_{\frac{p}{2}} \leq \sum_{j=1}^{\infty} \|F(j)K(j)\|_{\frac{p}{2}} \leq \sum_{j=1}^{\infty} \|F(j)\|_p \|K(j)\|_p < \infty,
\end{align*}
which results $\|E\|_{\tilde{p}} \leq \|E\|_{\pi,p}$. Then the result follows from (1.1). \hfill \Box

**Theorem 2.7.** $\|.|_p$ and $\|.|_{\tilde{p}}$ are equivalent if and only if $p = 1$ or $I$ is finite.

**Proof.** The sufficient condition is evident. Let
\begin{equation}
K\|.|_p \leq \|.|_{\tilde{p}} \leq M\|.|_p,
\end{equation}
for some $K, M > 0$, and $p \neq 1$. If $1 < p < 2$, then $\tilde{p} = 1$ and by (2.2), $\|.|_1 \leq M\|.|_p$, that implies $\mathcal{E}_p(I) \subseteq \mathcal{E}_1(I)$ which is contradict with Theorem 28.32 of [4]. We can repeat the same argument for the case $p \geq 2$. \hfill \Box

For each $i \in I$, and $1 \leq m, n \leq d_i$, let $\varepsilon_{mn}$ be the elementary $d_i \times d_i$-matrix such that for $1 \leq k, l \leq d_i$,
\begin{equation}
(\varepsilon_{mn})_{kl} = \begin{cases}
1 & \text{if } k = m, l = n \\
0 & \text{otherwise}.
\end{cases}
\end{equation}
Theorem 2.8. Let \( 1 \leq p < 2 \). If \( M = \sup_{i \in I} a_i < \infty \), then for each \( E \in \mathcal{E}_p^2(I) \)

\[
\|E\|_p = \|E\|_1 \leq \|E\|_{\pi,p} \leq \|E\|_{\pi,p} \leq M\|E\|_p = M\|E\|_1
\]

Proof. Suppose that \( E = \sum_{j=1}^{\infty} F^{(j)} K^{(j)} \) in \( \mathcal{E}_p(I) \), where \( \sum_{j=1}^{\infty} \|F^{(j)}\|_p \|K^{(j)}\|_p < \infty \). Then Hölder inequality and Theorem 2.8·3 of [4], imply that

\[
\|E\|_1 \leq \sum_{j=1}^{\infty} \|F^{(j)} K^{(j)}\|_1 \leq \sum_{j=1}^{\infty} \|F^{(j)}\|_2 \|K^{(j)}\|_2 \leq \sum_{j=1}^{\infty} \|F^{(j)}\|_p \|K^{(j)}\|_p < \infty.
\]

Therefore, \( \|E\|_1 = \|E\|_p \leq \|E\|_{\pi,p} \). Let \( \delta_{i : I \to \mathbb{R}} \) be defined by \( \delta_{i}(j) = 1 \) if \( i = j \) and \( \delta_{i}(j) = 0 \) if \( i \neq j \). Then \((E_{i})_{i \in I} = \sum_{j \in I} E_{j} \delta_{j}\) and

\[
(E_{i})_{i \in I} \|_{\pi,p} \leq \sum_{j \in I} \|E_{j} \delta_{j}\|_{\pi,p} \leq \sum_{j} \|E_{j}\|_{\pi,p}
\]

where

\[
E_{j} = \begin{bmatrix}
\lambda_{1}^{j} & 0 & \ldots & 0 \\
0 & \lambda_{2}^{j} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_{d_{j}}^{j}
\end{bmatrix}.
\]

This gives

\[
E_{j} = \sum_{1 \leq k \leq d_{j}} \lambda_{k}^{j}_{e_{k}} e_{k},
\]

and

\[
\sum_{j} \|E_{j}\|_{\pi,p} \leq \sum_{j} \sum_{1 \leq k \leq d_{j}} \|\lambda_{k}^{j}_{e_{k}} e_{k}\|_{\pi,p}.
\]

In addition,

\[
\lambda_{k}^{j}_{e_{k}} e_{k} = \lambda_{k}^{j}_{e_{k}} e_{k} \cdot \lambda_{k}^{j}_{e_{k}} e_{k},
\]

so

\[
\|\lambda_{k}^{j}_{e_{k}} (e_{k})_{i}\|_{\pi,p} \leq \lambda_{k}^{j}_{e_{k}} a_{j}^{i} a_{j}^{i}.
\]

Combining the above two inequalities, we have

\[
\sum_{j} \|E_{j}\|_{\pi,p} \leq \sum_{j} \sum_{1 \leq k \leq d_{j}} \lambda_{k}^{j}_{e_{k}} a_{j}^{i} a_{j}^{i}.
\]

By using (2.3)

\[
\|(E_{i})_{i \in I}\|_{\pi,p} \leq \sum_{j} \sum_{1 \leq k \leq d_{j}} \lambda_{k}^{j}_{e_{k}} a_{j}^{i} a_{j}^{i},
\]

and moreover

\[
\|E\|_1 = \sum_{j} a_{j} \sum_{1 \leq k \leq d_{j}} \lambda_{k}^{j}.
\]
Now, since \(0 \leq \frac{2}{p} - 1 \leq 1 \Rightarrow a_j^{2 - 1} \leq a_j \leq M\),
we have
\[
\|(E_i)_{i \in I}\|_{\pi,p} \leq \sum_j \sum_{1 \leq k \leq d_j} \lambda_j^1 a_j^{2 - 1} \leq M \sum_j a_j \sum_{1 \leq k \leq d_j} \lambda_k^j = M\|E\|_1,
\]
and hence
\[
\|E\|_1 \leq \|\|E\|\|_{\pi,p} \leq \|E\|_{\pi,p} \leq M\|E\|_1.
\]
}\]

The following two corollaries follow from Theorem 2.6, Proposition 2.7 and Theorem 2.8.

**Corollary 2.9.** Let \(p \geq 2\). Then \(E_p(I)\) has \(S\)-property if and only if \(I\) is finite.

**Corollary 2.10.** Let \(1 \leq p < 2\) and \(\sup_{i \in I} a_i < \infty\). Then \(E_p(I)\) has \(S\)-property if and only if \(p = 1\).

**Remark 2.11.** The above two corollaries can be similarly proved for the case \(\pi\)-property.

### 3. Biprojectivity of \(E_p(I)\)

In the following proposition which the proof is straightforward, we use \(\oplus_1\) to denote the \(\ell_1\)-direct sum of Banach spaces.

**Theorem 3.1.** If \(E_\alpha\) (for \(\alpha \in A\)) and \(F_\beta\) (for \(\beta \in B\)) are Banach spaces, then
\[
(\oplus_1 E_\alpha) \hat{\otimes} (\oplus_1 F_\beta) = \oplus_1 (E_\alpha \hat{\otimes} F_\beta)
\]

From now on, we put \(a_i = d_i\) for each \(i \in I\). Let \(M_i\) stands for the algebra of \(d_i \times d_i\) matrices with \(\|T\| = d_i\|T\|_1 = d_i(\text{trace}(T^*T)^{\frac{1}{2}})\), and \(M_{ij}\) for the algebra of \(d_i d_j \times d_i d_j\) matrices with \(\|T\| = d_i d_j\|T\|_1\). It is easy to see that \(\oplus_1 M_i\) and \(E_1(I)\) are isometric. Similarly by Proposition 3.1, \(E_1(I) \hat{\otimes} E_1(I)\) and \(E_1(I \times I)\) are isometric with \(\oplus_1(M_i \hat{\otimes} M_j)\) and \(\oplus_1 M_{ij}\) respectively. The norm-decreasing maps \(\rho_{i,j} : M_i \hat{\otimes} M_j \to M_{ij}\) give a norm-decreasing map \(\rho : E_1(I) \hat{\otimes} E_1(I) \to E_1(I \times I)\).

**Theorem 3.2.** If \(\sup_{i \in I} d_i < \infty\), then \(E_1(I) \hat{\otimes} E_1(I) = E_1(I \times I)\).
Proof. Injectivity of $\rho$ follows from injectivity of the corresponding map between $\bigoplus_1(M_i \hat{\otimes} M_j)$ and $\bigoplus_1 M_{ij}$. But $M_{ij}$ may be realized, as a linear space, as $M_i \hat{\otimes} M_j$. Because these spaces are finite dimensional, the linear isomorphism between $M_{ij}$ and $M_i \hat{\otimes} M_j$ is bounded with both bounds dependant only on the dimensions. Hence if the dimensions are bounded, then the maps between the $\ell_1$-direct sums enjoy the same property. Therefore, $\rho^{-1}$ exists and is bounded. □

Theorem 3.3. The following assertions are equivalent.
(i) $\mathcal{E}_1(I)$ is biprojective.
(ii) $\mathcal{E}_1(I)$ is weakly amenable.
(iii) $\sup_{i \in I} d_i < \infty$.

Proof. By 5.3.13 of [7], (i) implies (ii) and if $\mathcal{E}_1(I)$ is weakly amenable, then by [8], $\sup_{i \in I} d_i < \infty$. Let $\sup_{i \in I} d_i < \infty$, then by Proposition 3.2, $\mathcal{E}_1(I) \hat{\otimes} \mathcal{E}_1(I) = \mathcal{E}_1(I \times I)$. Define $\varrho : \mathcal{E}_1(I) \longrightarrow \mathcal{E}_1(I \times I)$ by $\varrho((E_i)) = (E_i \delta_{i,i})$. It is easy to check that $\varrho$ is a bounded $\mathcal{E}_1(I)$-bimodule morphism which is the right inverse for $\pi : \mathcal{E}_1(I) \hat{\otimes} \mathcal{E}_1(I) \longrightarrow \mathcal{E}_1(I)$ and so $\mathcal{E}_1(I)$ is biprojective. □

Corollary 3.4. $\mathcal{E}_p(I)$ is biprojective if and only if $p = 1$ and $\sup_{i \in I} d_i < \infty$ or $I$ is finite.

Proof. The sufficient condition is evident. Let $p = 1$ and $\sup_{i \in I} d_i < \infty$, then by Proposition 3.3, $\mathcal{E}_1(I)$ is biprojective. Also it is evident that $\mathcal{E}_p(I)$ is biprojective if $I$ is finite. Now let $\mathcal{E}_p(I)$ is biprojective. Since $\mathcal{E}_p(I)$ has $\pi$-property, the result can be deduced from Corollary 2.9 and Corollary 2.10. □

4. Applications

Let $G$ be a compact group with dual $\hat{G}$ (the set of all irreducible representations of $G$). Let $H_\pi$ be the representation space of $\pi$ for each $\pi \in \hat{G}$. The algebras $\mathcal{E}(\hat{G})$ and $\mathcal{E}_p(\hat{G})$ for $p \in [1, \infty] \cup \{0\}$, are defined as mentioned above with each $a_\pi$ equals to the dimension $d_\pi$ of $\pi \in \hat{G}$ (c.f Definition 28.34 of [4]).

A unitary representation $\pi$ of $G$ is primary if the center $C(\pi)$, i.e., the space of interwining operators of the representations $\pi$ and $\pi$, is trivial. The group $G$ is said to be of type I if every primary representation of $G$ is a direct sum of copies of some irreducible representations (for complete discussion and proof of the following two theorem, see [2]).
Theorem 4.1. Every compact group is of type I.

Theorem 4.2. If either $G_1$ or $G_2$ is of type I, then there exists a bijection between $\hat{G}_1 \times \hat{G}_2$ and $G_1 \times G_2$.

The following proposition is a consequence of Proposition 3.2, Theorem 4.1 and Theorem 4.2.

Theorem 4.3. If $\sup_{\pi \in G} d_\pi < \infty$, then $\mathcal{E}_1(\hat{G}) \hat{\otimes} \mathcal{E}_1(\hat{G}) = \mathcal{E}_1(G \times G)$.

Corollary 4.4. If $\sup_{\pi \in G} d_\pi < \infty$, then $\mathcal{A}(G) \hat{\otimes} \mathcal{A}(G) = \mathcal{A}(G \times G)$.

Proof. By Theorem 34.32 of [4], the convolution Banach algebra $\mathcal{A}(G)$ is isometrically algebra isomorphic with $\mathcal{E}_1(\hat{G})$. □

Remark 4.5. By Theorem 1. of [6], there is an integer $M$ such that $d(\pi) \leq M$ for all $\pi \in \hat{G}$ if and only if there is an open abelian subgroup of finite index in $G$.

Corollary 4.6. If $G$ has an open abelian subgroup of finite index, then $\mathcal{A}(G) \hat{\otimes} \mathcal{A}(G) = \mathcal{A}(G \times G)$.

Theorem 4.7. Let $G$ be a compact group. Then,

(i) $(\mathcal{A}(G), \ast)$ is biprojective if and only if $\sup_{\pi \in \hat{G}} d_\pi < \infty$ and if and only if $(\mathcal{A}(G), \ast)$ is weakly amenable.

(ii) $(L^2(G), \ast)$ is biprojective if and only if $G$ is finite.

Proof. By above, $(\mathcal{A}(G), \ast)$ is isometrically algebra isomorphic with $\mathcal{E}_1(\hat{G})$, also by 28.43 of [4](Weyl-Peter Theorem) $(L^2(G), \ast)$ is isometrically algebra isomorphic with $\mathcal{E}_2(\hat{G})$. □

Corollary 4.8. Let $G$ be a compact group. Then $(\mathcal{A}(G), \ast)$ is biprojective if and only if there is an open abelian subgroup of finite index in $G$.

ACKNOWLEDGMENT

The referees have reviewed the paper very carefully. The authors express their deep thanks for the comments.
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