SOME OPERATOR INEQUALITIES INVOLVING IMPROVED YOUNG AND HEINZ INEQUALITIES

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ABSTRACT. In this work, by applying the binomial expansion, some refinements of the Young and Heinz inequalities are proved. As an application, a determinant inequality for positive definite matrices is obtained. Also, some operator inequalities around the Young's inequality for semidefinite invertible matrices are proved.

1. INTRODUCTION

The classical Young’s inequality for non-negative real numbers says that if $a$ and $b$ are non-negative and $0 \leq r \leq 1$, then

$$a^r b^{1-r} \leq ra + (1 - r)b$$

with equality if and only if $a = b$.

If $r = \frac{1}{2}$, we obtain the arithmetic-geometric mean inequality

$$\sqrt{ab} \leq \frac{a + b}{2}.$$

Here, by the following inequality a short proof for the Young’s inequality is given,

$$e^{x-1} \geq x \quad x \in \mathbb{R}.$$

The proof of the Young’s inequality:

Take $L = ra + (1 - r)b$ then,

$$e^L - 1 \geq \frac{a}{L} \geq 0,$$

$$e^L - 1 \geq \frac{b}{L} \geq 0.$$

So, we have

$$1 = e^{r(\frac{a}{L}) + (1-r)(\frac{b}{L})} \geq \left(\frac{a}{L}\right)^r \times \left(\frac{b}{L}\right)^{1-r}.$$
Hence,
\[ a^r b^{1-r} \leq ra + (1 - r)b. \]

Kittaneh and Manasrah obtained a refinement of Young’s inequality as the following\[5\]:
\[
(1.1) \quad a^r b^{1-r} \leq ra + (1 - r)b - r_0(\sqrt{a} - \sqrt{b})^2
\]
where, \( r_0 = \min\{r, 1-r\} \).

The Heinz’s means are defined as
\[
H_r(a, b) = \frac{a^r b^{1-r} + a^{1-r}b^r}{2}
\]
where \( a, b \) and \( r \) has the same conditions of the Young’s inequality.

The Heinz’s inequality asserts that
\[
\sqrt{ab} \leq H_r(a, b) \leq \frac{a + b}{2}.
\]

Kittaneh and Manasrah also obtained a refinement of the Heinz’s inequality as follows:
\[
H_r(a, b) \leq \frac{a + b}{2} - r_0(\sqrt{a} - \sqrt{b})^2
\]
where, \( r_0 = \min\{r, 1-r\} \).

In this work, by applying the binomial expansion we prove some refinements of the Young and Heinz inequalities and as an application, a determinant inequality is proved by these inequalities. Also, some operator inequalities around the Young’s inequality are obtained.

**Theorem 1.1** (Newton’s generalized binomial theorem). If \( x \) and \( y \) are real numbers with \( |x| > |y| \) and \( r \) is any complex number, then
\[
(x + y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k
= x^r + r x^{r-1} y + \frac{r(r-1)}{2!} x^{r-2} y^2 + \frac{r(r-1)(r-2)}{3!} x^{r-3} y^3 + \cdots .
\]

2. **Main Results**

In the following, at first a refinement of the Young’s inequality is given. Then a refinement of the Heinz’s inequality is proved by it.
Theorem 2.1. Let $a$ and $b$ be positive numbers and $0 \leq r \leq 1$.

(i) If $\frac{b}{2} < a < b$ then,

$$(2.1) \quad a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)(b-a)^2}{2a} + \frac{r(r-1)(r-2)(b-a)^3}{6a^2}.$$  

(ii) If $a < \frac{b}{2}$ then,

$$(2.2) \quad a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)(b-a)^2}{2b} - \frac{r(1-r^2)(b-a)^3}{6b^2}.$$  

Proof. By taking $y = \frac{b}{a}$, we have $y = 1 + x$. We consider two cases:

Case 1: $x < 1$.

By Theorem 1.1 we have

$$\left(\frac{b}{a}\right)^r = y^r = (1+x)^r = 1 + rx + \frac{r(r-1)}{2}x^2 + \frac{r(r-1)(r-2)}{6}x^3 + \cdots.$$  

For $0 < r < 1$, after the second term this series is alternative. So, we have

$$a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)(b-a)^2}{2a} + \frac{r(r-1)(r-2)(b-a)^3}{6a^2}.$$  

Note that the condition $\frac{b}{2} < a < b$ ensures that this inequality is a refinement of the Young’s inequality.

Case 2: $x > 1$.

In this case, by taking $y = \frac{a}{b} = 1 - t$, we have $\frac{1}{2} < t < 1$ and so

$$y^{1-r} = \left(\frac{a}{b}\right)^{1-r} = (1-t)^{1-r} = 1 + (r-1)t + \frac{r(r-1)}{2}t^2 - \frac{r(1-r^2)}{6}t^3 + \cdots.$$  

Since all terms of the series after the first element are negative we have

$$a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)(b-a)^2}{2b} - \frac{r(1-r^2)(b-a)^3}{6b^2}.$$  

As a conclusion a refinement of the Heinz inequality is given.

Corollary 2.2. Let $a$ and $b$ be positive numbers and $0 \leq r \leq 1$.

(i) If $\frac{b}{2} < a < b$ then,

$$H_r(a, b) \leq \frac{a+b}{2} + \frac{r(r-1)(b-a)^2}{2a} - \frac{r(r-1)(b-a)^3}{4a^2}.$$  

(ii) If $a < \frac{b}{2}$ then,
\[ H_r(a, b) \leq \frac{a + b}{2} + \frac{r(r-1)(b-a)^2}{2b} + \frac{r(r-1)(b-a)^3}{4b^2}. \]

Proof. In (2.1) and (2.2) applying $1-r$ instead of $r$, two inequalities are obtained. By adding and dividing by 2, these inequalities are proved. \qed

In the following, another refinement of the Young’s inequality is given. At first, this lemma is needed.

**Lemma 2.3.** If $r \in [0, 1]$ and $x \in [-1, 1]$ then,
\[ (1 + x)^r \leq 1 + rx + \frac{r(r-1)}{8}x^2. \]

Proof. By taking $f(x) = (1 + x)^r - 1 - rx - \frac{r(r-1)}{8}x^2$ we have
\[ f(0) = 0, \quad f'(x) = r(1 + x)^{r-1} - r - \frac{r(r-1)}{4}x \]
and
\[ f''(x) = \frac{r(r-1)}{4}[4(1 + x)^{r-2} - 1] < 0. \]
Consequently, $f'$ decreases on the interval $(-1, 1)$. Thus, $f'(x) > 0$ for $x \in (-1, 0)$ and $f'(x) < 0$ for $x \in (0, 1)$. It then follows that $f$ attains its minimum at zero. Since, $f(0) = 0$, we have $f(x) \leq 0$ for $x \in [-1, 1]$. \qed

**Theorem 2.4.** Let $a$ and $b$ be positive numbers and $0 \leq r \leq 1$.

(i) If $\frac{b}{2} < a < b$ then,
\[ a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)(b-a)^2}{8a}. \]

(ii) If $a < \frac{b}{2}$ then,
\[ a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)(b-a)^2}{8b}. \]

Proof. (i) If $\frac{b}{2} < a < b$ then, by taking $y = \frac{b}{a} = 1 + x$, we have $x \in (0, 1)$. Hence, applying Lemma 2.3, the following inequality is obtained.
\[ \left(\frac{b}{a}\right)^r = (1 + x)^r \leq 1 + r\left(\frac{b}{a} - 1\right) + \frac{r(r-1)}{8}\left(\frac{b}{a} - 1\right)^2. \]
So we have
\[ a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)(b-a)^2}{8a}. \]
(ii) If \( a < \frac{b}{2} \) then, by taking \( y = \frac{a}{b} = 1 - t \), we have \( t \in (\frac{1}{2}, 1) \). So, by Lemma 2.3

\[
y^{1-r} = \left( \frac{a}{b} \right)^{1-r} = (1 - t)^{1-r} \leq 1 + (r - 1)t + \frac{r(r - 1)}{8} t^2.
\]

This implies that

\[
a^{1-r}b^r \leq (1 - r)a + rb + \frac{r(r - 1)(b - a)^2}{8b}.
\]

\[\square\]

**Corollary 2.5.** Let \( a \) and \( b \) be positive numbers and \( 0 \leq r \leq 1 \).

(i) If \( \frac{b}{2} < a < b \) then,

\[
H_r(a, b) \leq \frac{a + b}{2} + \frac{r(r - 1)(b - a)^2}{8a}.
\]

(ii) If \( a < \frac{b}{2} \) then,

\[
H_r(a, b) \leq \frac{a + b}{2} + \frac{r(r - 1)(b - a)^2}{8b}.
\]

Young’s inequality in operator algebras has been considered in [2] and references therein. A determinant version of Young’s inequality is also known ([4], P. 467):

\[
det(AB^{1-r}) \leq det(rA + (1 - r)B).
\]

Let \( M_n(\mathbb{C}) \) be the space of \( n \times n \) complex matrices. Recently, Kittaneh and Manasrah, by inequality (1.1) for \( A, B \in M_n(\mathbb{C}) \) which are positive definite, prove that

\[
det(AB^{1-r}) + r_0^n \det(A + B - 2A\sharp B) \leq det(rA + (1 - r)B)
\]

where, \( r_0 = \min\{r, 1 - r\} \) and \( A\sharp B = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})B^{\frac{1}{2}} \) is the geometric mean of \( A \) and \( B \).

In the following, as an application of Theorem 2.4, we prove the inequality for positive definite matrices.

**Theorem 2.6.** Let \( A, B \in M_n(\mathbb{C}) \) be positive definite and \( 0 \leq r \leq 1 \), if \( \frac{1}{2}A < B < A \) then

\[
(2.5) \quad \det(AB^{1-r}) + \left( \frac{r(r - 1)}{8} \right)^n \det(AB^{-1}A + B - 2A) \leq \det(rA + (1 - r)B)
\]

Also, if \( B < \frac{A}{2} \) then

\[
(2.6) \quad \det(AB^{1-r}) + \left( \frac{r(r - 1)}{8} \right)^n \det(BA^{-1}B + A - 2B) \leq \det(rA + (1 - r)B)
\]
Proof. (i) If $\frac{1}{2} A < B < A$, then we have $\frac{1}{2} B^{-\frac{1}{2}} A B^{-\frac{1}{2}} < I < B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$. So by inequality (2.3)

$$r s_j(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) + (1 - r) \geq s_j'(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) + \frac{r(1 - r)}{8} (s_j(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) - 1)^2,$$

where $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the singular values of $A$. This implies that

$$\det \left( r B^{-\frac{1}{2}} A B^{-\frac{1}{2}} + (1 - r)I \right)$$

$$= \prod_{j=1}^{n} \left( r s_j(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) + (1 - r) \right)$$

$$\geq \prod_{j=1}^{n} s_j'(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) + \frac{r(1 - r)}{8} \left( s_j(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) - 1 \right)^2$$

$$\geq \prod_{j=1}^{n} s_j'(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) + \left( \frac{r(1 - r)}{8} \right)^n \prod_{j=1}^{n} \left( s_j(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) - 1 \right)^2$$

$$= \det(B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^r + \left( \frac{r(1 - r)}{8} \right)^n \det(B^{-\frac{1}{2}} A B^{-\frac{1}{2}} - I)^2.$$

By multiplying $\det(B)$, the proof of inequality (2.5) is completed.

(ii) If $B < \frac{1}{2}$, then we have $\frac{1}{2} > A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$. So by inequality (2.4) and similar to (i), the inequality (2.6) is proved.

The Hilbert-Schmidt norm of a matrix $A = [a_{ij}] \in M_n(\mathbb{C})$ is defined by

$$\|A\|_2 = \left( \sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}.$$

This norm is unitarily invariant, i.e.,

$$\|U A V\|_2 = \|A\|_2$$

for all unitary matrices $U, V \in M_n(\mathbb{C})$. By using the singular value decomposition of $A$, we have

$$\|A\|_2 = \left( \sum_{j=1}^{n} s_j^2(A) \right)^{\frac{1}{2}},$$

where, $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the singular values of $A$ (the eigenvalues of $|A| = (A^* A)^{\frac{1}{2}}$). In the following, the operator form of Theorem 2.4 will be given.

**Theorem 2.7.** Suppose that $A, B, X \in M_n(\mathbb{C})$ such that $A$ and $B$ are positive semidefinite invertible matrices and $r \in [0, 1]$. 
(i) If $B < 2A < 2B$, then
\[
\|A^{-r}XB^r\|_2^2 \leq (1-r)\|AX\|_2^2 + r\|XB\|_2^2 + \frac{r(r-1)}{8} (\|A^{-1}XB^2\|_2^2 + \|AX\|_2^2 - 2\|XB\|_2^2).
\]
(ii) If $2A < B$, then
\[
\|A^{-r}XB^r\|_2^2 \leq (1-r)\|AX\|_2^2 + r\|XB\|_2^2 + \frac{r(r-1)}{8} (\|B^{-1}XA^2\|_2^2 + \|XB\|_2^2 - 2\|AX\|_2^2).
\]

Proof. There are unitary matrices $U$ and $V$ such that $A = U\Lambda U^*$ and $B = VMV^*$, so that
\[
\lambda = \text{diag}(\lambda_1, \cdots, \lambda_n),
\]
\[
M = \text{diag}(m_1, \cdots, m_n)
\]
where, $\lambda_i, m_j \geq 0$. Let $Y = U^*XV = [y_{ij}]$. Then
\[
\|A^{-r}XB^r\|_2^2 = \|U[\lambda_1^{-r}y_{ij}m_j]V^*\|_2^2
\]
\[
= \sum_{i,j=1}^n (\lambda_i^2)^{1-r}(m_j^2)^r|y_{ij}|^2
\]
\[
\leq \sum_{i,j=1}^n (1-r)(\lambda_i^2)|y_{ij}|^2 + rm_j^2|y_{ij}|^2 + \frac{r(r-1)}{8} \frac{(m_j^2 - \lambda_i^2)^2}{\lambda_i^2} |y_{ij}|^2.
\]
On the other hand, since $AX = U\Lambda YV^*$ and $XB = UYMV^*$ and $A^{-1}XB^2 = U\Lambda^{-1}YM^2V^*$, we have
\[
\|AX\|_2^2 = \|U\Lambda YV^*\|_2^2 = \|AY\|_2^2 = \sum_{i,j=1}^n (\lambda_i^2)|y_{ij}|^2,
\]
\[
\|XB\|_2^2 = \|UYMV^*\|_2^2 = \|YM\|_2^2 = \sum_{i,j=1}^n m_j^2|y_{ij}|^2,
\]
\[
\|A^{-1}XB^2\| = \|U\Lambda^{-1}YM^2V^*\| = \sum_{i,j=1}^n \frac{m_j^2|y_{ij}|^2}{\lambda_i^2}.
\]
By using the above equalities, the proof of the first part of the theorem is completed. The second part is proved similarly. \qed
Let $A$ and $B$ be two positive operators, $r \in [0, 1]$. $r$-weighted arithmetic mean of $A$ and $B$, denoted by $A \nabla_r B$ is defined as

$$A \nabla_r B = (1 - r)A + rB.$$  

For an invertible operator $A$, $r$-geometric mean of $A$ and $B$, denoted by $A \sharp_r B$, is defined as

$$A \sharp_r B = (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r A^\frac{1}{2}.$$  

The power mean $A \sharp_r B$ is defined by F. Kubo and T. Ando [6]. In the special case $r = \frac{1}{2}$, the index is omitted.

There are several inequalities around $A \sharp_r B$ and $A \nabla_r B$. Recently, J. Zhao and J. Wu [7], for two positive invertible operators $A$ and $B$ of $B(H)$ and $r \in [0, 1]$, proved that:

If $0 < r < \frac{1}{2}$ then

$$r_0(A \sharp_r B - 2A \sharp_\frac{1}{2}B + A) + 2r(A \nabla_r B - A \sharp_r B) + A \sharp_r B \leq A \nabla_r B,$$

if $\frac{1}{2} < r < 1$, then

$$r_0(A \sharp_r B - 2A \sharp_\frac{1}{2}B + B) + 2(1 - r)(A \nabla_r B - A \sharp_r B) + A \sharp_r B \leq A \nabla_r B,$$

In the sequel, by the above refinements of the Young’s inequality and the following lemma, some inequalities concerning $A \sharp_r B$ and $A \nabla_r B$ will be obtained.

**Lemma 2.8** ([3]). Let $X \in B(H)$ be self-adjoint and let $f$ and $g$ be continuous functions such that $f(t) \geq g(t)$ for all $t \in \text{Sp}(X)$ (the spectrum of $X$). Then $f(X) \geq g(X)$.

**Theorem 2.9.** Let $A, B \in B(H)$ be two positive invertible operators and $r \in (0, 1)$.

(i) If $B < 2A < 2B$, then

$$A \sharp_r B \leq A \nabla_r B + \frac{r(r - 1)(r - 2)}{6} (BA^{-1})^2 B + \frac{r(r - 1)(3 - r)}{2} BA^{-1} B$$

$$+ \frac{r(r - 1)(r - 4)}{2} B + \frac{r(r - 1)(5 - r)}{6} I.$$  

(ii) If $2A < B$, then

$$A \sharp_r B \leq A \nabla_r B + \frac{r(1 - r^2)}{6} (AB^{-1})^2 A + \frac{r(r - 1)(r + 3)}{4} AB^{-1} A$$

$$- \frac{r(r - 1)(r + 3)}{2} B + \frac{r(r - 1)(5 - r)}{6} I.$$  

(2.8)
Proof. (i) By Theorem 2.1 if $1 < x < 2$ then,
\[
  x^r \leq (1 - r) + rx + \frac{r(r - 1)}{2}(x - 1)^2 + \frac{r(r - 1)(r - 2)}{6}(x - 1)^3.
\]
Taking $X = A^{-\frac{x}{2}}BA^{-\frac{x}{2}}$, we have $I < X < 2I$ and so $Sp(X) \subseteq (0, 1)$. Hence, by Lemma 2.8
\[
  X^r \leq (1 - r) + rX + \frac{r(r - 1)}{2}(X - 1)^2 + \frac{r(r - 1)(r - 2)}{6}(X - 1)^3.
\]
Multiplying both sides of the above inequality by $A^\frac{1}{2}$, inequality (2.7) is deduced.

(ii) If $y > 2$, by Theorem 2.1
\[
  y^{1-r} \leq (1 - r)y + r + \frac{r(r - 1)}{2}(y - 1)^2 + \frac{r(r - 1)(r - 2)}{6}(y - 1)^3.
\]
Taking $Y = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ we have $Y > 2I$ and so $Sp(Y) \subseteq (2, \infty)$. Hence, by Lemma 2.8
\[
  Y^{1-r} \leq (1 - r)Y + r + \frac{r(r - 1)}{2}(Y - 1)^2 + \frac{r(r - 1)(r - 2)}{6}(Y - 1)^3.
\]
Multiplying both sides of the above inequality by $B^\frac{1}{2}$ and since $A^\frac{1}{2}B = B^\frac{1}{2}A$, inequality (2.8) is deduced. \qed

Theorem 2.10. Let $A, B \in B(H)$ be two positive invertible operators and $r \in (0, 1)$.

(i) If $B < 2A < 2B$, then
\[
  (2.9) \quad \frac{A^\frac{1}{2}rB + B^\frac{1}{2}rA}{2} \leq A\nabla B + \frac{r(r - 1)}{8}(BA^{-1}B - 2B + I).
\]

(ii) If $2A < B$, then
\[
  (2.10) \quad \frac{A^\frac{1}{2}rB + B^\frac{1}{2}rA}{2} \leq A\nabla B + \frac{r(r - 1)}{8}(AB^{-1}A - 2A + I).
\]

Proof. (i) By Corollary 2.5 if $1 < x < 2$ then,
\[
  \frac{x^{1-r} + x^r}{2} \leq \frac{x + 1}{2} + \frac{r(r - 1)}{8}(x - 1)^2.
\]
If $A < B < 2A$ then, by taking $X = A^{-\frac{x}{2}}BA^{-\frac{x}{2}}$, we have $I < X < 2I$ and so $Sp(X) \subseteq (0, 1)$. Hence, by Lemma 2.8
\[
  \frac{X^{1-r} + X^r}{2} \leq \frac{X + 1}{2} + \frac{r(r - 1)}{8}(X - 1)^2.
\]
Multiplying both sides of the above inequality by $A^\frac{1}{2}$, inequality (2.9) is deduced.

(ii) If $y > 2$, by Corollary 2.5
\[
  \frac{y^{1-r} + y^r}{2} \leq \frac{y + 1}{2} + \frac{r(r - 1)}{8}(y - 1)^2.
\]
Taking \( Y = B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \) we have \( Y > 2I \) and so \( Sp(Y) \subseteq (2, \infty) \). Hence, by Lemma 2.8
\[
\frac{Y^r + Y^{1-r}}{2} \leq \frac{Y + 1}{2} + \frac{r(r-1)}{8}(Y-1)^2.
\]
Multiplying both sides of the above inequality by \( B^{\frac{1}{2}} \) and since \( A_{\#}rB = B_{\#1-r}A \), inequality (2.10) is deduced. \qed

REFERENCES


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